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Gravitational Brainwaves, Quantum Fluctuations and Stochastic Quantization Part II

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Abstract

In this part we expand our former research and find, using Stochastic Quantization methods, the equilibrium state of the quantum fluctuations related to the Lamb shift. As argued in the literature strong quantum fluctuations stand at the basis of Quantum Gravity which is believed to control the mechanism which operates also in the activity of human consciousness and thinking

Key Words: : quantum fluctuations, stochastic quantization, Lamb shift

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I. Introduction

In this part we continue and expand our former research and find the stationary state of the quantum fluctuations (Nelson, 1985) related to the known Lamb shift (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b). We use for that purpose the Stochastic Quantization (SQ) method (Namiki, 1992) and the Fokker-Plank equation (Risken,1984). As noted in the following the discussed electron-photon interaction does not conserve energy and, therefore, it should be physically forbidden (Haken, 1981). Nevertheless, this apparently improbable process was found, through Lamb's experiment (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; hansch et al, 1972b), to actually occur. This gives rise to a first demonstration of quantum fluctuations (Nelson, 1985) which actually enable even what seems to be unphysical and energy nonconserving processes. These fluctuations are present in all quantum phenomena and are believed to be very strong at the quantum gravity realm. And since it is believed by some leading scientists (Penrose, 1989; Penrose, 1994) that the human thinking activity and consciousness are based and determined by quantum gravity processes we find it appropriate to discuss quantum fluctuations in the context of Part I of this work which discusses gravitational brainwaves.

In Section II we discuss the electron-photon interaction, which results in the known Lamb shift (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b; Haken, 1981), in the framework of the SQ formalism (Namiki, 1992) and the Fokker-Plank equation (Risken, 1984). The Lamb shift is described as a stationary state of a stochastic process in the assumed extra dimension s. That is, the expressions related to the mentioned Lamb shift (as known from the quantum field literature (Haken, 1981)) are obtained as a stationary limit of some stochastic process in an extra dimension s.

The main body of the mathematical formalism which describes this electron-photon interaction in the framework of the SQ formalism is represented in Appendix C which is devoted to a detailed derivation of the important Eq (8). We summarize with a Concluding Remarks section

In Appendix *A* we outline a short review of the stochastic quantization theory (as,

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especially, represented in (Namiki, 1992)), and in Appendix *B* we detaily derive the correlation expression from Eq (39) of Part I.

II. The Lamb shift as a stationary state of stochastic processes in the extra dimension

The main lesson one learns from the discussion of Part I about the gravitational brainwaves is that introducing the cylindrical GW expression into the actions S of the path integrals related to the mentioned ensemble of variables (brains) results with the outcome that the probability to find them radiating this kind of waves is large. In this section we demonstrate this again regarding the quantum fluctuations which cause the shifting of the energy bands in the known Lamb shift experiment (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b). Here the ensemble of stochastic processes do not represent, as in the previous sections, any biological brain activity but the action of a two-state electron which emits a photon and then reabsorbs it where the total energy during this process is not conserved. This process, which is tracked to quantum fluctuations (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b), is regarded here in the framework of the SQ theory as obtained in the equilibrium limit of some stochastic process in an extra dimension s. That is, discussing this phenomenon as a stochastic process occuring in this extra dimension we show that taking the steady state limit of equating all the involved s values to each other and taking to infinity one obtains the known expressions of the Lamb shift experiment (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b; Haken, 1981).

As is customary in the SQ theory (see Appendix A) and exemplified in Part I we assume that there exists in an extra dimension a large ensemble of stochastic processes each of them may give rise in the stationary state to the Lamb shift phenomenon. Also, it is assumed that each of these stochastic processes is performed during finite s and t intervals $(s_{(0)},s)$, $(t_{(0)},t)$ and that each of these intervals is subdivided into an N subintervals $(s_{(0)},s_1)$, (s_1,s_2), ... (s_{N-1},s_N) and $(t_{(0)},t_1)$, (t_1,t_2), ..., (t_{N-1},t_N).

In the following we formulate the appropriate expression for the described electron-photon interaction over some representative subintervals $(t_{(k-1)},t_k)$ and $(s_{(k-1)},s_k)$ and calculate the probability to find the ensemble of stochastic processes giving rise to the same remarked electron-photon interaction. In contrast to the discussion in Part I where we use the stochastic Langevin formulation of the SQ theory we, now, find it better to discuss the equivalent Fokker-Plank version of it (Namiki, 1992; Risken, 1984). That is, we use the following Fokker Plank equation (Namiki, 1992; Risken, 1984)

$$\frac{\partial P(\psi^{(k)},t_{(k)},s_{(k)}\,|\,\psi^{(k-1)},t_{(k-1)},s_{(k-1)})}{\partial s} = F^{(k)}P(\psi^{(k)},t_{(k)},s_{(k)}\,|\,\psi^{(k-1)},t_{(k-1)},s_{(k-1)}), \quad \textbf{(1)}$$

where $P(\psi^{(k)},t_{(k)},s_{(k)}\,|\,\psi^{(k-1)},t_{(k-1)},s_{(k-1)})$ denotes the conditional probability to find the relevant ensemble of stochastic processes giving rise at t_k and s_k to the state $\psi^{(k)}$ if at the former $t_{(k-1)}$ and $s_{(k-1)}$ they give rise to the state $\psi^{(k-1)}$. In the context of this section the states $\psi^{(k)}$ and $\psi^{(k-1)}$ are in effect two total situations each of them includes all the particular photon-electron interaction states related to the ensemble of stochastic variables at the two different t and s values of $t_{(k)}$, $s_{(k)}$ and $t_{(k-1)}$ $s_{(k-1)}$. In this way the P 's here have similar meaning to the P 's of Part I which are related to cylindrical GW's. The $F^{(k)}$ in Eq (1) is (Namiki, 1992)

$$F^{(k)} = \frac{1}{2\alpha}H(\psi^{(k)},\pi^{(k)}), \quad \textbf{(2)}$$

where H, $\pi^{(k)}$ and $\psi^{(k)}$ are, respectively, the "stochastic" Hamiltonian, momentum and state for the subintervals $(s_{(k-1)},s_k)$, $(t_{(k-1)},t_k)$. The α, as mentioned after Eq (3) in Part I , is either

$$\alpha = \frac{k_\beta T}{f}$$

for classical phenomena or $\alpha = \hbar$ for quantum ones. The momentum $\pi^{(k)}$ is, as in quantum mechanics, a differential operator

defined by (Namiki, 1992)

$$\pi^{(k)} = -2\alpha\frac{\partial}{\partial\psi^{(k)}}$$

and satisfied the commutation relations (Namiki, 1992) $[\pi^{(m)},\psi^{(n)}]=2\alpha\delta^{mn}$. The operator F from Eq (2) is also a differential operator which may be written generally for the ensemble of n stochastic processes as (Namiki, 1992)

$$F=\sum_{i=1}^{i=n}(\alpha\frac{\partial^2}{\partial(\psi_{(i)})^2}-\frac{\partial K_{(i)}(\psi)}{\partial(\psi_{(i)})}) \qquad \textbf{(3)}$$

Noting that $K_{(i)}(\psi)$ has the same meaning as in the Langevin formalism of the SQ theory (see Eq (17) in Part I and Eq (A_1) in Appendix A) one may write the last equation (3) in a manner which emphasizes the underlying stochastic process η

$$\begin{aligned}F&=\frac{\partial}{\partial(\psi_{(i)})}\sum_{i=1}^{i=n}(\alpha\frac{\partial}{\partial(\psi_{(i)})}-K_{(i)}(\psi))\\&=\frac{\partial}{\partial(\psi_{(i)})}\{\sum_{i=1}^{i=n}(\alpha\frac{\partial}{\partial(\psi_{(i)})}-(\frac{\partial\psi_{(i)}}{\partial s}-\eta_{(i)}))\}\end{aligned} \qquad \textbf{(4)}$$

As emphasized in (Namiki, 1992) one may develop, using the former relations, a stochastic operator formalism which corresponds to the quantum one so that it is possible to formulate a ``Schroedinger'', ``Heisenberg'' and ``interaction'' pictures. Thus, assuming an ensemble of n stochastic processes, using the "interaction" picture and considering the whole intervals $(t_{(0)},t)$, and $(s_{(0)}.s)$ one may calculate the conditional probability to find at s and t these processes giving rise to the state ψ if at the initial $s_{(0)}$ and $t_{(0)}$ they give rise to the state $\psi^{(0)}$. This conditional probability is given by (Namiki, 1992)

$$\begin{aligned}&P^I(\psi,t,s\,|\,\psi^{(0)},t_{(0)},s_{(0)})=P^I(\psi^{(0)},t_{(0)},s_{(0)})+\\&+\int_{(0)}^{\psi}F^{(N)}P^I(\psi^{(N-1)},t_{(N-1)},s_{(N-1)}\,|\,\psi^{(0)},t_{(0)},s_{(0)})d\psi^{(N)},\end{aligned} \qquad \textbf{(5)}$$

where the superscript I reminds us that we use the "interaction" picture and $P^I(\psi^{(0)},t_{(0)},s_{(0)})$ is the probability that the ensemble of stochastic processes give rise at the initial $t_{(0)}$ and $s_{(0)}$ to the initial state $\psi^{(0)}$. The states ψ depends upon s and t and, therefore, the integration over ψ is, actually, a double one over s and t. Thus, substituting in a perturbative manner (Feynman, 1984; Feynman and Hibbs, 1965; Swanson, 1992) for $P^I(\psi^{(N-1)},t_{(N-1)},s_{(N-1)}\,|\,\psi^{(0)},t_{(0)},s_{(0)})$ one may write Eq (5) as

$$\begin{aligned}&P^I(\psi,t,s\,|\,\psi^{(0)},t_{(0)},s_{(0)})=P^I(\psi^{(0)},t_{(0)},s_{(0)})\\&+\sum_{k=1}^{k=N}\int_{\psi^{(0)}}^{\psi}F^{(N)}d\psi^{(N)}\int_{\psi^{(0)}}^{\psi^{(N)}}F^{(N-1)}d\psi^{(N-1)}\cdot\\&\cdot\int_{\psi^{(0)}}^{\psi^{(N-1)}}F^{(N-2)}d\psi^{(N-2)}\\&\ldots\int_{\psi^{(0)}}^{\psi^{(k)}}F^{(k-1)}d\psi^{(k-1)}\ldots\int_{\psi^{(0)}}^{\psi^{(3)}}F^{(2)}d\psi^{(2)}\cdot\\&\cdot\int_{\psi^{(0)}}^{\psi^{(2)}}F^{(1)}d\psi^{(1)}P^I(\psi^{(0)},t_{(0)},s_{(0)})\\&=P^I(\psi^{(0)},t_{(0)},s_{(0)})\\&+\int_{\psi^{(0)}}^{\psi}d\psi^{(1)}F^{(1)}P^I(\psi^{(0)},t_{(0)},s_{(0)})+\\&+\int_{\psi^{(0)}}^{\psi}d\psi^{(2)}\int_{\psi^{(0)}}^{\psi^{(2)}}d\psi^{(1)}F^{(1)}F^{(2)}P^I(\psi^{(0)},t_{(0)},s_{(0)})\\&+\int_{\psi^{(0)}}^{\psi}d\psi^{(N)}\int_{\psi^{(0)}}^{\psi^{(N)}}d\psi^{(N-1)}\ldots\\&\ldots\int_{\psi^{(0)}}^{\psi^{(3)}}d\psi^{(2)}\int_{\psi^{(0)}}^{\psi^{(2)}}d\psi^{(1)}F^{(1)}F^{(2)}\\&\ldots F^{(N-1)}F^{(N)}P^I(\psi^{(0)},t_{(0)},s_{(0)})\end{aligned} \qquad \textbf{(6)}$$

Note that in the last equation we have obtained in each term the same factor of $P^I(\psi^{(0)},t_{(0)},s_{(0)})$. Now, since the Lamb shift results from quantum fluctuations (Nelson, 1985) and since the states in quantum mechanics as well as in SQ (Namiki, 1992) have a probabilistic interpretation we may assume that the probabilities P^I denote states. We should, however, emphasize (again) that these P 's from Eq (6) as those of Part I (see, for example, Eqs (34)-(35) in Part I), refer to the states of the whole ensemble of stochastic variables in the sense of the conditional probability to find them at a later s and t in some situation ψ if, for example, at the initial $s_{(0)}$ and $t_{(0)}$ they were at the situation $\psi^{(0)}$. We later in Appendix C denote the respective particular states of the interacting electron and photon by ϕ and u.

Thus, following the last discussion one may use the quantum rules and terms as in (Haken, 1981), except for the introduction of the extra variable s, for representing the electron and photon before and after the interaction between them as well as the general state of the whole ensemble of stochastic variables. The variable s is introduced into the relevant quantities so that in the limit of equating all the different s values to each other and taking to infinity, as required in the SQ theory (Namiki, 1992; Parisi, 1981), the known expressions (Haken, 1981) which represent the electron and photon and the correlation between them are obtained. Thus, one may assign to the initial $s_{(0)}$ and $t_{(0)}$ the value of zero and refer to $P^{I}(\psi^{(0)}, t_{(0)}=0, s_{(0)}=0)$ as the initial state of the ensemble system.

In Appendix C we have calculated in a detailed manner, using stochastic quantization methods, the whole process of emitting and reabsorbing a photon by a two-state electron during the time and s intervalls $(t_{(0)}, t)$, and $(s_{(0)}.s)$. This process during these intervals may be represented by the first two terms at the right hand side of Eq (6). That is,

$$P^{I}(\psi,t,s\,|\,\psi^{(0)},t_{(0)},s_{(0)}) = P^{I}(\psi^{(0)},t_{(0)},s_{(0)}) + \int_{\psi^{(0)}}^{\psi} d\psi^{(1)} F^{(1)} P^{I}(\psi^{(0)},t_{(0)},s_{(0)}) \qquad \textbf{(7)}$$

The detaily calculated expression for this process is given in Eq (C_18) in Appendix C as

$$P^{I}(\psi,t,s\,|\,\psi^{(0)},0,0) = P^{I}(\psi^{(0)},0,0)(1+G(t,s)) = = P^{I}(\psi^{(0)},0,0)(1+it\Delta\varepsilon_{\lambda}+is\Delta\varepsilon_{\lambda_s}), \qquad \textbf{(8)}$$

where – as mentioned before Eq (7) – the initial t *and* s are assigned the zero value, that is, $t_{(0)}=0, s_{(0)}=0$. $G(t,s)$ is given by Eq (C_17) in Appendix C and $\Delta\varepsilon_{\lambda}, \Delta\varepsilon_{\lambda_s}$ are given by Eqs (C_19) in that Appendix. The result in Eq (8) is only for the first-order term in Eq (6) which involves one emission and one reabsorption done over the intervals $(s_{(0)}, s)$, $(t_{(0)}, t)$. If these emission and reabsorption are repeated for each one of the many subintervals into which the former finite s and t intervals were subdivided so that all the higher order terms of this process $(N \to \infty)$ are taken into account one obtains, analogously to the quantum analog (Haken, 1981) (in which the variable s is absent), the result

$$\begin{aligned} &P^{I}(\psi,t,s\,|\,\psi^{(0)},0,0) \\ &= P^{I}(\psi^{(0)},0,0)(1+G_{\lim N\to\infty}(t,s)) \\ &= P^{I}(\psi^{(0)},0,0)\{1+(it\Delta\varepsilon_{\lambda}+ \\ &+\frac{1}{2!}(it\Delta\varepsilon_{\lambda})^{2}+\ldots+\frac{1}{k!}(it\Delta\varepsilon_{\lambda})^{k}+\ldots) \\ &+(is\Delta\varepsilon_{\lambda_s}+\frac{1}{2!}(is\Delta\varepsilon_{\lambda_s})^{2}+\ldots \\ &\ldots+\frac{1}{k!}(is\Delta\varepsilon_{\lambda_s})^{k}+\ldots)\} = \\ &P^{I}(\psi^{(0)},0,0)(e^{it\Delta\varepsilon_{\lambda}}+e^{is\Delta\varepsilon_{\lambda_s}}-1) \end{aligned} \qquad \textbf{(9)}$$

Note that in contrast to the $G(t,s)$ of Eq (8) which involves only one emission and one reabsorption done over the entire intervals $(s_{(0)}, s)$, $(t_{(0)}, t)$ here in Eq (9) it involves all the higher order terms of them and, therefore, it is denoted $G_{\lim N\to\infty}(t,s)$. That is, $G_{\lim N\to\infty}(t,s) = G_1(t,s)+G_2(t,s)+\ldots +G_N(t,s)+\ldots,$

Where the first term $G_1(t,s)$ at the right hand side of the last equation denotes the first order term in which one emission and one reabsorption are done during the whole intervals $(s_{(0)}, s)$, $(t_{(0)}, t)$. $G_1(t,s)$ is, actually, given by Eq (8). The second order term $G_2(t,s)$ denotes the case where the acts of emission and reabsorption are repeated twice during the intervals $(s_{(0)}, s)$, $(t_{(0)}, t)$ and likewise for the other higher order terms.

The left hand side of Figure 1 shows a Feynman diagram (Feynman and Hibbs,1965; Haken,1981; Mattuck, 1976) of the emission and reabsorption process performed once over the relevant t interval whereas the right hand side of it shows a Feynman diagram of the fourth order term of this process over the same t interval. Now, as required by the SQ theory, the stationary situations are obtained in the limit of eliminating the extra variable s which is done by equating all the s values to each other and taking to infinity. Thus, since, as remarked, we have equated the initial $s_{(0)}$ to zero we must

equate all the other s values to zero. That is, the stationary state is

$$\begin{aligned}&\lim_{s\to 0} P^{I}(\psi,t,s\,|\,\psi^{(0)},0,0)\\&=\lim_{s\to 0} P^{I}(\psi^{(0)},0,0)(e^{it\Delta\varepsilon_{\lambda}}+e^{is\Delta\varepsilon_{\lambda}s}-1)\\&=P^{I}(\psi^{(0)},0,0)e^{it\Delta\varepsilon_{\lambda}}\end{aligned}\qquad \textbf{(10)}$$

The last result is the one obtained in quantum field theory (Haken, 1981) for the same interaction (without any extra variable). The quantity $\Delta\varepsilon_{\lambda}$, given by the first of Eqs (C_19) in Appendix C, has the same form also in the quantum version (Haken, 1981), where it is termed the energy shift. This shift have been experimentally demonstrated in the quantum field theory for the case of a real many-state particle in the famous Lamb shift of the Hydrogen atom (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b; Haken, 1981).

Note that, as for the gravitational brainwaves case, introducing the expression of the detailed electron-photon interaction for all the subintervals of s and t of all the stochastic processes yields a correlation among them which truly represents, in the stationary situation, the corelation of the real interaction. That is, when all the values of s are equated to each other and eliminated the equilibrium stage is obtained. One may, also, note that the elimination of the s variable is fulfilled by only equating all its values to each other without having to take the infinity limit (see the discussion before Eq (A_14) in Appendix A).

Concluding Remarks

In this part we continue our former line of reasoning from Part I and show, using SQ methods (Namiki, 1992), that the phenomena of quantum fluctuations may be regarded as stationary states of an assumed stochastic processes which occur in an extra dimension. That is, these quantum fluctuations are obtained as the stationary limit of these hypothetical stochastic processes. We have concentrated our discussion on the Lamb shift (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b) which may be regarded as one of the first actual demonstrations of quantum fluctuations.

Naturally, quantum fluctuations, as observed in nature, are very tiny and infinitesimal but are believed to be very much strong and violent at the quantum gravity regime. This quantum gravity regime is supposed (Penrose, 1989; Penrose, 1994) to reign and control the mechanism which stands at the basis of coherent human thinking and consciousness.

It is shown that the known expressions And formulas which characterize the Lamb shift (Haken, 81) are obtained as stationary limit of the mentioned stochastic processes in an assumed extra dimension.

A large part of the mathematical apparatus which describes the Lamb shift in the context of SQ is represented in Appendix C which is devoted to deriving Eq (8). Appendix A outlines a short review of the stochastic quantization method as, especially, represented in (Namiki, 1992). Appendix B detaily derives Eq (39) from Part I of this work.

References

Abramowitz M, Stegun IA. (eds). Handbook of mathematical functions. Dover, New York, 1970.

Coffey W. The Langevin Equation. World Scientific, Singapore, 1996.

Doob JL. Stochastic Processes. Wiley, New York 1953.

Feynman RP. Spacetime approach to non relativistic quantum mechanics. Rev Mod Phys 1984; 20: 367-387

Feynman RP, Hibbs AR. Quantum Mechanics and Path Integrals. McGraw-Hill, 1965.

Kannan D. An Introduction to Stochastic Processes. Elsevier, 1979.

Haken H. Light. Vol 1. Elsevier, 1981.

Hansch TW, Shahin IS, Schawlow AL. Optical resolution of the Lamb shift in atomic Hydrogen by laser saturation spectroscopy. Nature (London) Phys Sci 1972; 235: 63-65

Hansch TW, Schawlow AK, Toschek PE. Ultrasensitive response of a CW dye laser to selective extinction. IEEE J Quant Electr 1972; QE-8:802-804.

Kuchar K. Canonical quantization of cylindrical gravitational waves. Phys Rev D 1971; 4: 955-986.

Lamb WE Jr. The Interpretation of Quantum Mechanics. Rinton Press, 2001.

Lamb WE Jr and Sargent M. Laser Physics. Addison-Wesley, 1974.

Mattuck RD. A Guide to feynman Diagrams in the Many Body Problem. 2nd edition. McGraw-Hill, 1976.

Merzbacher E. Quantum Mechanics. Second edition. Wiley, 1961.

Namiki M. Stochastic Quantization. Springer-Verlag, 1992.
Nelson E. Quantum Fluctuation. Princeton University Press, New Jersey, 1985.
Parisi G, Wu Y. Perturbation theory without gauge fixing. Sci Sin 1981; 24:483-596
Penrose R. The Emperor's New Mind. Oxford University Press, 1989.
Penrose R. Shadows of the mind. Oxford University Press, 1994.
Pipes AL. Applied Mathematics for Engineers and Physicists. 2nd edition, McGraw-Hill, 1958.
Risken H. The Fokker-Plank Equation. Springer-Verlag, 1984.
Roepstorff G. Path Integral Approach to Quantum Physics. Springer-Verkag, 1994.
Rogers LC, Williams D. Diffusions, Markov Processes and Martingales. 2nd edition. Wiley, 1987.
Schiff LI. Quantum Mechanics. 3-rd Edition. McGraw-Hill, 1968.
Swanson M. Path Integrals and Quantum Processes. Academic Press, 1992.
Tannoudji CC, Diu B, Lalo F. Quantum Mechanics. Wiley, 1977.

APPENDIX A

Representation of the Parisi-Wu-Namiki Stochastic Quantization

The Parisi-Wu-Namiki SQ theory (Namiki, 1992; Parisi and Wu, 1981) for any stochastic process (Kannan, 1979; Doob, 1953; Rogers and Williams, 1987) may use either the Langevin equation (Coffey, 1996) or the Fokker-Plank one (Risken, 1984) as its basic starting point. For the following introductory representation of the SQ theory and in Part I of this work we find it convenient to use the Langevin equation whereas in this part II we discuss the electron-photon interaction which results in the known Lamb shift (Lamb, 2001; Lamb and Sargent, 1974; Hansch et al, 1972a; Hansch et al, 1972b) from the point of view of the Fokker-Plank equation. The stochastic process, which is assumed in the SQ theory to occur in some extra dimension s, is generally considered to be of the Wienner-Markoff type (Kannan, 1979; Doob, 1953; Rogers and Williams, 1987) and to be described by the n variables $\psi(s,t)=(\psi_1(s,t),\psi_2(s,t),\dots\psi_{(n-1)}(s,t),\psi_n(s,t))$. This stochastic process is also characterized by the n random forces $\eta(s,t)=(\eta_1(s,t),\eta_2(s,t),\dots\eta_{(n-1)}(s,t),\eta_n(s,t))$ which are Gaussian white noise (Kannan, 1979; Doob, 1953; Rogers and Williams, 1987). Thus, denoting the process related to the i variable by ψ_i, where $1\le i\le n$, one may analyze it by taking its rate of change with respect to s according to the generalized Langevin equation (Coffey, 1996; Namiki, 1992)

$$\frac{\partial\psi_i(s,t,r)}{\partial s}=K_i(\psi(s,t,r))+\eta_i(s,t,r),\qquad i=1,2,\dots n, \tag{A_1}$$

where n denotes the remarked n-member ensemble of variables and η_i denotes stochastic process related to the variable ψ_i. The variables ψ_i depends upon s and upon the spatial variable r and the time t. The K_i are given in the SQ theory by (Parisi and Wu, 1981; Namiki, 1992)

$$K_i(\psi(s,t,r))=-(\frac{\delta S_i[\psi]}{\delta\psi})_{\psi=\psi(s,t,r)}, \tag{A_2}$$

where S_i are the actions $S_i=\int\int drdtL_i(\psi,\dot{\psi})$ and L_i are the Lagrangians. For properly discussing the ``evolution'' of the related process ψ_i one, generally, subdivides the t and s intervals $(t_{(0)},t)$, $(s_{(0)},s)$ into N subintervals $(t_{(0)},t_1)$, (t_1,t_2), ... (t_{N-1},t) and $(s_{(0)},s_1)$, (s_1,s_2), ... (s_{N-1},s). We assume that the Langevin Eq (A_1) is satisfied for each member of the ensemble of variables at each subinterval with the following Gaussian constraints (Namiki, 1992)

$$<\eta_i^{(k)}(t_k,s_k)>=0,\quad <\eta_i^{(k)}(t_k,s_k)\eta_j^{(k)}(t'_k,s'_k)>=2\alpha\delta_{ij}\delta(t_k-t'_k)\delta(s_k-s'_k), \tag{A_3}$$

where the angular brackets denote an ensemble average with the Gaussian distribution, the k

superscript denotes the k-th subinterval from the N available and the i, j refer to the mentioned n variables where $n \geq i, j \geq 1$. Note that both intervals $(t_{(0)}, t)$, $(s_{(0)}, s)$ of each one of the n variables are subdivided, as mentioned, into N subintervals. The α from Eq (A_3) have different meanings which depend upon the involved process and the context in which Eqs (A_1) and (A_3) are used. Thus, in the classical regime α is (Namiki, 1992)

$\alpha = \frac{k_\beta T}{f}$ where k_β, T, and f are respectively the Boltzman constant, the temperature in Kelvin units and the relevant friction coefficient. In the quantum regime α is identified (Namiki, 1992) with the Plank constant $\hbar$. We note that using Eqs (A_1)-(A_3) enables one (Namiki, 1992) to discuss a large number of different classical and quantum phenomena. It has been shown (Namiki, 1992) that the right hand side of Eq (A_3) may be derived from the following Gaussian distribution law (Namiki, 1992)

$$P_i(y)dy_i = \frac{1}{\sqrt{2\pi(<\eta_i^{(k)}>)^2}} \exp(-\frac{(y_i^{(k)})^2}{2(<\eta_i^{(k)}>)^2})dy_i, \quad \textbf{(A_4)}$$

which is the probability density for the variable ψ_i and for the subintervals $(s_{(k-1)}, s_k)$, $(t_{(k-1)}, t_k)$ to have a value of $\eta_i^{(k)}$ in $(y_i^{(k)}, y_i^{(k)} + dy_i)$ (Namiki, 1992), where

$$y_i^{(k)} = \frac{\partial \psi_i^{(k)}(s,t,x)}{\partial s} - K_i(\psi_i^{(k)}(s,t,x)) \quad \textbf{(A_5)}$$

For the n variables one may write Eq (A_4) for the subintervals $(s_{(k-1)}, s_k)$, $(t_{(k-1)}, t_k)$ as

$$P_{i....}(y)dy = \exp(-\sum_{i=1}^{n} \frac{(y_i^{(k)})^2}{2(<\eta_i^{(k)}>)^2}) \prod_{i=1}^{n} \frac{dy_i}{\sqrt{2\pi(<\eta_i^{(k)}>)^2}}, \quad \textbf{(A_6)}$$

which is the probability density for the n variables ψ_i $1 \leq i \leq n$ to have a value of $\eta_i^{(k)}$ in $(y_i^{(k)}, y_i^{(k)} + dy_i)$ where dy at the left of (A_6) is $dy = \prod_i dy_i$. The angular brackets are product over any two variables as given in Eq (A_3). We note in this context that the general correlation $<\eta_i \eta_j \ldots \eta_m \eta_n>$ is expressed in terms of $<\eta_i \eta_j>$ by (Namiki, 1992)

$$<\eta_i \eta_j \ldots \eta_m \eta_n> = \begin{cases} 0 & \text{for odd number of } \eta's \\ \sum <\eta_i \eta_j><\eta_m \eta_n> \ldots & \text{for even number of } \eta's \end{cases} \quad \textbf{(A_7)}$$

where the sum is taken over every possible pair of $\eta's$. For the whole intervals $(s_{(0)}, s)$, $(t_{(0)}, t)$, which, as mentioned, were each subdivided into N subintervals, one may generalize Eq (A_6) as

$$P_{i...}(y)dy = \exp(-\sum_{i=1}^{n}\sum_{k=1}^{N} \frac{(y_i^{(k)})^2}{2(<\eta_i^{(k)}>)^2}) \prod_{i=1}^{n}\prod_{k=1}^{N} \frac{dy_i^{(k)}}{\sqrt{2\pi(<\eta_i^{(k)}>)^2}}, \quad \textbf{(A_8)}$$

where now the dy at the left is $dy = \prod_i \prod_k dy_i^{(k)}$. Note that Eqs (A_4), (A_6) and (A_8) denote probability densities as realized from the dy at the left hand sides of these equations. In order to find the probabilities themselves one have to integrate the right hand sides of these equations over the appropriate variables. Thus, using Eqs (A_1), (A_3) and (A_5) one may write Eq (A_4) in a more informative way as

$$P_i(\psi_i^{(k)},t_k,s_k\,|\,\psi_i^{(k-1)},t_{(k-1)},s_{(k-1)})=$$
$$=\int d\psi_i^{(k)}\frac{1}{\sqrt{2\pi(2\alpha)}}\exp\{-\frac{(\frac{\psi_i^{(k)}-\psi_i^{(k-1)}}{(s_k-s_{(k-1)})}-K_i(\psi_i^{(k-1)}))^2}{2(2\alpha)}\}dy, \qquad \textbf{(A\_9)}$$

where we have approximated

$$\frac{\partial\psi_i^{(k)}(s,t,x)}{\partial s}\approx\frac{\psi_i^{(k)}-\psi_i^{(k-1)}}{(s_k-s_{(k-1)})}.$$

The $P_i(\psi_i^{(k)},t_k,s_k\,|\,\psi_i^{(k-1)},t_{(k-1)},s_{(k-1)})$ of Eq (A_9) is the conditional probability to find the variable ψ_i at t_k and s_k with the configuration $\psi_i^{(k)}$ if at $t_{(k-1)}$ and $s_{(k-1)}$ it has the configuration $\psi_i^{(k-1)}$. Since it involves the same variable it may be termed autocorrelation of $\psi_{(i)}$ over the subintervals $(s_{(k-1)},s_{(k)})$, $(t_{(k-1)},t_{(k)})$. In a similar manner one may write Eq (A_9) for the whole ensemble of n variables in the subintervals $(s_{(k-1)},s_k)$ and $(t_{(k-1)},t_k)$ as

$$P_{i....}(\psi_{(n)}^{(k)},t_k,s_k\,|\,\psi_{(0)}^{(k-1)},t_{(k-1)},s_{(k-1)})=\int\cdots\int\exp\{-\sum_i\frac{(\frac{\psi_i^{(k)}-\psi_i^{(k-1)}}{(s_k-s_{(k-1)})}-K_i(\psi_i^{(k-1)}))^2}{2(2\alpha)}\}\cdot \qquad \textbf{(A\_10)}$$
$$\cdot\prod_{(i=1)}^{(i=n)}\frac{d\psi_{(i)}^{(k)}}{\sqrt{2\pi(2\alpha)}}$$

And the conditional probability over the whole intervals $(s_{(0)},s)$ and $(t_{(0)},t)$ may similarly be obtained by adding other factors and sums over the remaining $(N-1)$ subintervals. If one assume N to be very large, and therefore the length of each subinterval to be very small, one may use Feynman path integral (Feynman, 1984; Feynman and Hibbs, 1965; Swanson, 1992) as follows

$$P_{i....}(\psi,t,s\,|\,\psi_{(0)},t_{(0)},s_{(0)})=\lim_{N\to\infty}C\int\cdots\int\exp\{-\sum_{i=1}^{n}\sum_{k=1}^{N}\frac{(\frac{\psi_i^{(k)}-\psi_i^{(k-1)}}{(s_k-s_{(k-1)})}-K_i(\psi_i^{(k-1)}))^2}{2(2\alpha)}\}\cdot \qquad \textbf{(A\_11)}$$
$$\cdot\prod_{i=1}^{n}\prod_{k=1}^{N}(\frac{d\psi_{(i)}^{(k)}}{\sqrt{2\pi(2\alpha)}}),$$

where C is a normalization constant. The former formula may equivalently be written as (Namiki, 1992)

$$P(\psi,t,s\,|\,\psi_{(0)},t_{(0)},s_0)=C\int\cdots\int\cdots\int P(\psi_{(n)}^{N},t_N,s_N\,|\,\psi_{(0)}^{(N-1)},t_{(N-1)},s_{(N-1)})\cdots \qquad \textbf{(A\_12)}$$
$$\cdots P(\psi_{(n)}^{k},t_k,s_k\,|\,\psi_{(0)}^{k-1},t_{(k-1)},s_{(k-1)})\cdots P(\psi_{(n)}^{1},t_1,s_1\,|\,\psi_{(0)}^{0},t_{(0)},s_0)d\psi^N\cdots d\psi^k\cdots d\psi^1,$$

where each P at the right is essentially of the form of Eq (A_10) and the integrals are over the N subintervals. The last equation, which is the conditional probability to find the ensemble of n variables at t and s with the configuration ψ if at $t_{(0)}$ and $s_{(0)}$ they have the configuration $\psi_{(0)}$, is also equivalent (Namiki, 1992) to the Green's functions $\Delta_{ij...}(t_{(0)},s_{(0)},t_1,s_1,\ldots)$ which determine the correlation among the members of the ensemble (Namiki, 1992). This function, as defined in (Namiki, 1992), is

$$\Delta_{ij\ldots}(t_{(0)},s_{(0)},t_1,s_1,\ldots)=<\psi_i(t_{(0)},s_{(0)})\psi_j(t_1,s_1)\ldots>= \\ =C\int D\psi(t,s)\psi_i(t_{(0)},s_{(0)})\psi_j(t_1,s_1)\ldots\exp(-\frac{S_i(\psi(t,s))}{\alpha}), \qquad \textbf{(A\_13)}$$

where S_i are the actions

$S_i=\int dsL_i(\psi,\dot{\psi})$,

C is a normalization constant, and

$D\psi(t,s)=\prod\prod_{i=1}^{i=n}d\psi_i(t,s)$.

As seen from the last equation the

$\Delta_{ij\ldots}(t_{(0)},s_{(0)},t_1,s_1\ldots)$

were expressed as path integrals (Feynman, 1984; Feynman and Hibbs, 1965; Swanson, 1992; Roepstorff, 1994) where the quantum feynman measure $e^{\frac{iS(\psi)}{\hbar}}$ is replaced in Eq (A_13) and in the following Eq (A_14) by $e^{-\frac{S(q)}{\alpha}}$ as required for the classical path integrals (Namiki, 1992; Roepstorff, 1994).

It can be seen that when the s's are different for the members of the ensemble so that each have its specific $S_i(\psi(s_i,t))$, $K_i(\psi(s_i,t))$, and $\eta_i(s_i,t)$ the correlation in (A_13) is obviously zero. Thus, in order to have a nonzero value for the probability to find a large part of the ensemble of variables having the same or similar forms we have to consider the stationary configuration where, as remarked, all the s values are equated to each other and taken to infinity. For that matter we take account the fact that the dependence upon s and t is through ψ so this ensures (Namiki, 1992) that this dependence is expressed through the s and t differences. For example, referring to the members i and j the correlation between them is $\Delta_{ij}(t_i-t_j,s_i-s_j)$, so that for eliminating the s variable from the correlation function one equates all these different s's to each other. We, thus, obtain the following stationary equilibrium correlation (Namiki, 1992)

$$\Delta_{ij\ldots}(t_{(0)},s_{(0)},\ldots)_{st}=<\psi_i(t_{(0)},s_{(0)})\psi_j(t_1,s_1)\ldots>_{st}=C\int D\psi(t)\psi_i(t_{(0)})\psi_j(t_1)\ldots\exp(-\frac{S(\psi)}{\alpha}), \qquad \textbf{(A\_14)}$$

where the suffix of st denotes the stationary configuration. In other words, the equilibrium correlation in our case is obtained when all the different s values are equated to each other and taken to infinity in which case one remains with the known stationary result.

Thus, if all the members of the ensemble of variables have similar actions S (in which case the s values are equated to each other) one finds with a large probability these members, in the later equilibrium stage, with the same configuration. That is, introducing the similar actions into the corresponding path integrals one finds, with a large probability, all the members of the ensemble in the same stationary configuration. This has been expicitly shown in Section IV of Part I for the cylindrical gravitational wave and in Section II here for the Lamb shift case.

APPENDIX B

Derivation of The Correlation Expression From Eq (39) Of Part I

We, now, derive the expression for the correlation from Eq (39) of Part I. For that we may use Eq (A_12) of Appendix A in which we substistute for the P 's from Eqs (A_9)-(A_10) of that Appendix. As noted in Appendix A the correlation is calculated not only among the ensemble of n variables but also for each of the N subintervals into which the finite t and s intervals are divided. Thus, assuming, as noted in Appendix A, that N is very large we may use the Feynman path integral of Eq (A_11) and write this correlation as

$$P_{i....}(\psi,t,s\,|\,\psi_{(0)},t_{(0)},s_{(0)}) = C\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{-\sum_{k=1}^{N}\sum_{i=1}^{n}\frac{1}{4\alpha(s_k-s_{(k-1)})^2}(\psi_i^{(k)}- \\ -\psi_{(i-1)}^{(k)}-K_i(\psi_{(i-1)}^{(k)})(s_k-s_{(k-1)}))^2\}\prod_{k=1}^{k=N}\prod_{i=1}^{i=(n-1)}\frac{d\psi_i^k}{\sqrt{2\pi(2\alpha)}}, \quad \textbf{(B\_1)}$$

where C is a normalization constant to be determined later from $\int P_{i....}(\psi,t,s\,|\,\psi^0,t_{(0)},s_{(0)})d\psi = 1$. Note that in the exponent of Eq (B_1), in contrast to that of Eq (A_11) in Appendix A, the sum over i precedes that over k and, therefore, the squared expression involves the variables $\psi_{(i)}^{(k)}$, $\psi_{(i-1)}^{(k)}$ etc (instead of $\psi_{(i)}^{(k)}$, $\psi_{(i)}^{(k-1)}$ of (A_11) in Appendix A). Note also that the number of integrals are $N\times(n-1)$ over the N subintervals and $(n-1)$ variables which is related to the fact that the suffix i in the exponent is summed from $i=1$ to $i=n$ whereas the i in the differentials outside the exponent is summed up to $i=n-1$ (compare with equation (4.4) in (Namiki, 1992)). The reason for this is that each $\psi_{(i)}^{(k)}$, except for $i=0$ and $i=n$, with superscript k and suffix i appears in two consecutive squared expressions of the sum over i so for calculating the correlation for the observer i over the subinterval $(s_k-s_{(k-1)})$ one has to solve the following integral which is related to $\psi_{(i)}^{(k)}$.

$$P_i(\psi_i^{(k)},t_{(k)},s_{(k)}\,|\,\psi_{(i-1)}^{(k)},t_{(k-1)},s_{(k-1)}) = \\ =\int_{-\infty}^{\infty}\exp\{-[\frac{(\psi_i^{(k)}-\psi_{(i-1)}^{(k)}-K_i(\psi_{(i-1)}^{(k)})(s_k-s_{(k-1)}))^2}{2(2\alpha)(s_k-s_{(k-1)})^2}+\frac{(\psi_{(i+1)}^{(k)}-\psi_i^{(k)}-K_i(\psi_i^{(k)})(s_{(k)}-s_{(k-1)}))^2}{2(2\alpha)(s_{(k)}-s_{(k-1)})^2}]\}\frac{d\psi_i^{(k)}}{\sqrt{2\pi(2\alpha)}} \quad \textbf{(B\_2)}$$

The solution of this integral involves the substitution for $K_i(\psi_{(i-1)}^{(k)})$ and $K_i(\psi_i^{(k)})$ from Eqs (32) and (36) of Part I so that one may write the two squared expressions of Eq (B_2) as

$$\frac{(\psi_i^{(k)}-\psi_{(i-1)}^{(k)}-K_i(\psi_{(i-1)}^{(k)})(s_k-s_{(k-1)}))^2}{2(2\alpha)(s_{(k)}-s_{(k-1)})^2} = \frac{1}{2(2\alpha)(s_{(k)}-s_{(k-1)})^2}[\psi_i^{(k)}- \\ -\psi_{(i-1)}^{(k)}-2\pi(B_1(R,t)\psi_{(i-1)}^{(k)}-B_2(R,t)-iB_3(R,t))(s_k-s_{(k-1)})]^2 \\ \frac{(\psi_{(i+1)}^{(k)}-\psi_i^{(k)}-K_i(\psi_i^{(k)})(s_{(k)}-s_{(k-1)}))^2}{2(2\alpha)(s_{(k)}-s_{(k-1)})^2} = \frac{1}{2(2\alpha)(s_{(k)}-s_{(k-1)})^2}[\psi_{(i+1)}^{(k)}- \\ -\psi_i^{(k)}-2\pi(B_1(R,t)\psi_i^{(k)}-B_2(R,t)-iB_3(R,t))(s_{(k)}-s_{(k-1)})]^2 \quad \textbf{(B\_3)}$$

In order to deal with manageable expressions we first assume that in the limit of large N and n the subintervals over t and s are equal so that one may write for any integral k

$$\Delta s_k = (s_k - s_{(k-1)}) = \Delta s_{(k+1)} = (s_{(k+1)} - s_{(k)}) = \Delta s$$
$$\Delta t_k = (t_k - t_{(k-1)}) = \Delta t_{(k+1)} = (t_{(k+1)} - t_{(k)}) = \Delta t \qquad \textbf{(B\_4)}$$

We, now, define the following expressions

$$a_1 = (1 + 2\pi B_1 \Delta s)^2, \quad a_2 = 2\pi \Delta s (B_2 + iB_3) \qquad \textbf{(B\_5)}$$

Using Eqs (B_3)-(B_5) one may write the two squared terms of Eq (B_2) as

$$\frac{(\psi_i^{(k)} - \psi_{(i-1)}^{(k)} - K_i(\psi_{(i-1)}^{(k)})(s_k - s_{(k-1)}))^2}{2(2\alpha)(s_{(k)} - s_{(k-1)})^2} + \frac{(\psi_{(i+1)}^{(k)} - \psi_i^{(k)} - K_i(\psi_i^{(k)})(s_{(k)} - s_{(k-1)}))^2}{2(2\alpha)(s_{(k)} - s_{(k-1)})^2} =$$
$$= \frac{1}{4\alpha(\Delta s)^2}\{(\psi_i^{(k)} - \sqrt{a_1}\psi_{(i-1)}^{(k)} + a_2)^2 + (\psi_{(i+1)}^{(k)} - \sqrt{a_1}\psi_i^{(k)} + a_2)^2\} = \qquad \textbf{(B\_6)}$$
$$= \frac{1}{4\alpha(\Delta s)^2}\{(\psi_i^k)^2 + a_1(\psi_{(i-1)}^{(k)})^2 - 2\sqrt{a_1}\psi_i^{(k)}\psi_{(i-1)}^{(k)} + 2a_2\psi_i^{(k)} - 2a_2\sqrt{a_1}\psi_{(i-1)}^{(k)} +$$
$$+ (\psi_{(i+1)}^{(k)})^2 + a_1(\psi_i^{(k)})^2 - 2\sqrt{a_1}\psi_{(i+1)}^{(k)}\psi_i^{(k)} + 2a_2\psi_{(i+1)}^{(k)} - 2a_2\sqrt{a_1}\psi_i^{(k)} + 2a_2^2\}$$

The last result is now substituted for the two squared terms of Eq (B_2) and the integral over ψ_i^k may be solved by using the following integral (Abramowitz and Steegun, 1970)

$$\int_{-\infty}^{\infty} dx \exp(-(ax^2 + bx + c)) = \sqrt{\frac{\pi}{a}} \exp(\frac{(b^2 - 4ac)}{4a}) \qquad \textbf{(B\_7)}$$

Thus, using Eq (B_6), one may find the appropriate coefficients $a_{\psi_i^k}$, $b_{\psi_i^k}$ and $c_{\psi_i^k}$, related to ψ_i^k, to be substituted in the integral (B_2) as follows

$$a_{\psi_i^k} = \frac{(1 + a_1)}{4\alpha(\Delta s)^2}, \quad b_{\psi_i^k} = \frac{(2a_2(1 - \sqrt{a_1}) - 2\sqrt{a_1}(\psi_{(i-1)}^{(k)} + \psi_{(i+1)}^{(k)}))}{4\alpha(\Delta s)^2}$$
$$c_{\psi_i^k} = \frac{[(\psi_{(i+1)}^{(k)})^2 + a_1(\psi_{(i-1)}^{(k)})^2 + 2a_2(a_2 + \psi_{(i+1)}^{(k)} - \sqrt{a_1}\psi_{(i-1)}^{(k)})]}{4\alpha(\Delta s)^2} \qquad \textbf{(B\_8)}$$

Using the last expressions for the coefficients $a_{\psi_i^k}$, $b_{\psi_i^k}$ and $c_{\psi_i^k}$ one may realize, after some calculations, that they satisfy the following relation

$$\frac{b^2_{\psi_i^k} - 4a_{\psi_i^k}c_{\psi_i^k}}{4a_{\psi_i^k}} = -\frac{1}{4(1 + a_1)\alpha(\Delta s)^2}((\psi_{(i+1)}^{(k)} - a_1\psi_{(i-1)}^{(k)}) + a_2(1 + \sqrt{a_1}))^2 \qquad \textbf{(B\_9)}$$

Thus, using the former discussion and, especially, the integral (B_7) one is able to solve the integral from Eq (B_2) and write it as

$$P_i(\psi_{(i+1)}^{(k)}, t_{(k)}, s_{(k)} \mid \psi_{(i-1)}^{(k)}, t_{(k-1)}, s_{(k-1)}) =$$
$$= \int_{-\infty}^{\infty} \frac{d\psi_i^{(k)}}{\sqrt{2\pi(2\alpha)}} \exp[-(a_{\psi_i^{(k)}}(\psi_i^{(k)})^2 + b_{\psi_i^{(k)}}\psi_i^{(k)} + c_{\psi_i^{(k)}})] = \frac{1}{\sqrt{4\alpha a_{\psi_i^k}}} \cdot$$
$$\cdot \exp(\frac{b_{\psi_i^k}^2 - 4a_{\psi_i^k}c_{\psi_i^k}}{4a_{\psi_i^k}}) = \frac{\Delta s}{\sqrt{(1+a_1)}} \exp\{-[\frac{1}{4(1+a_1)\alpha(\Delta s)^2}((\psi_{(i+1)}^{(k)} - \quad \textbf{(B\_10)}$$
$$- a_1\psi_{(i-1)}^{(k)}) + a_2(1+\sqrt{a_1}))^2]\}$$

The last result is the correlation for the variable $\psi_{(i)}$ over the subinterval $(s_k - s_{(k-1)})$ and it means the conditional probability to find this variable at $s = s_{(k)}$ and $t = t_{(k)}$ at the state $\psi_{(i+1)}^{(k)}$ if at $s = s_{(k-1)}$ and $t = t_{(k-1)}$ it was at the state $\psi_{(i-1)}^{(k)}$. Note that the superscript of the variable $\psi_{(i-1)}$ at the beginning of the subintervals $s_{(k-1)}$ and $t_{(k-1)}$ is the same as that at the end of it, i.e., k. If one wish to find the correlation of the two variables $\psi_{(i)}$ and $\psi_{(i+1)}$ for the same subinterval Δs then he has to add to the last result another squared term from the general relation (B_1) and perform the required integration over $\psi_{(i+1)}^{(k)}$ as follows

$$P_{i,(i+1)}(\psi_{(i+2)}^{(k)}, t_{(k)}, s_{(k)} \mid \psi_{(i-1)}^{(k)}, t_{(k-1)}, s_{(k-1)}) =$$
$$= \frac{\Delta s}{\sqrt{(1+a_1)}} \int_{-\infty}^{\infty} \exp\{-[\frac{1}{4(1+a_1)\alpha(\Delta s)^2}\{((\psi_{(i+1)}^{(k)} - a_1\psi_{(i-1)}^{(k)}) + a_2(1+\sqrt{a_1}))^2 + \quad \textbf{(B\_11)}$$
$$+ (\psi_{(i+2)}^{(k)} - \psi_{(i+1)}^{(k)} - K_i(\psi_{(i+1)}^{(k)})\Delta s)^2(1+a_1)\}]\}\frac{d\psi_{(i+1)}^{(k)}}{\sqrt{2\pi(2\alpha)}}$$

In this case the corresponding $a_{\psi_{(i+1)}^{(k)}}$, $b_{\psi_{(i+1)}^{(k)}}$ and $c_{\psi_{(i+1)}^{(k)}}$ are

$$a_{\psi_{(i+1)}^{(k)}} = \frac{(1+a_1+a_1^2)}{2(2\alpha)(\Delta s)^2},$$
$$b_{\psi_{(i+1)}^{(k)}} = \frac{2a_2(1+\sqrt{a_1}) - 2a_1\psi_{(i-1)}^{(k)} - (1+a_1)(2\sqrt{a_1}\psi_{(i+2)}^{(k)} + 2a_2\sqrt{a_1})}{2(2\alpha)(\Delta s)^2} \quad \textbf{(B\_12)}$$
$$c_{\psi_{(i+1)}^{(k)}} = \frac{(a_1^2(\psi_{(i-1)}^{(k)})^2 + a_2^2(1+\sqrt{a_1})^2 - 2a_1a_2(1+\sqrt{a_1})\psi_{(i-1)}^{(k)} + (1+a_1)(\psi_{(i+2)}^{(k)} + a_2)^2)}{2(2\alpha)(\Delta s)^2}$$

Thus, using the last equations and the integral from Eq (B_7) one may write the correlation from Eq (B_11) as

$$P_i(\psi_{(i+2)}^{(k)}, t_{(k)}, s_{(k)} \mid \psi_{(i-1)}^{(k)}, t_{(k-1)}, s_{(k-1)}) =$$
$$= \frac{(\Delta s)^2}{\sqrt{(1+a_1)(1+a_1+a_1^2)}} \exp\{-[\frac{(\psi_{(i+2)}^{(k)} - a_1\sqrt{a_1}\psi_{(i-1)}^{(k)} + a_2(1+\sqrt{a_1}+(\sqrt{a_1})^2))^2}{4\alpha(\Delta s)^2(1+a_1+a_1^2)}]\} \quad \textbf{(B\_13)}$$

Using the results of Eq (B_10) for the observer i one may realize that the correlation from Eq (B_13) means the conditional probability to find at $s = s_{(k)}$ and $t = t_{(k)}$ the two variables $\psi_{(i)}$ and $\psi_{(i+1)}$ at the respective states of $\psi_{(i+1)}^{(k)}$ and $\psi_{(i+2)}^{(k)}$ if at $s = s_{(k-1)}$ and $t = t_{(k-1)}$ they were at the states $\psi_{(i-1)}^{(k)}$, $\psi_{(i)}^{(k)}$. As remarked after Eq (B_10) the superscripts of the variables $\psi_{(i-1)}^{(k)}$, $\psi_{(i)}^{(k)}$ at the beginning of the subintervals $s_{(k-1)}$ and $t_{(k-1)}$ are the same as that at the end of it, i.e., k. One may, now, realize that the correlation of the n observers $i, j, l...$ over the subinterval $(s_{(k-1)}, s_{(k)})$ may be obtained from the

results of Eqs (B_10), (B_13) and from Eq (B_1) as

$$P_{i,j,l,\ldots}(\psi_{(n)}^{(k)}, t_{(k)}, s_{(k)} \mid \psi_0^k, t_{(k-1)}, s_{(k-1)}) = \frac{(\Delta s)^{(n-1)}}{\sqrt{\prod_{j=1}^{j=(n-1)} (\sum_{m=0}^{m=j} a_1^m)}} \cdot$$
$$\cdot \exp\{-\frac{1}{4\alpha(\Delta s)^2 \sum_{p=0}^{p=(n-1)} a_1^p} (\psi_n^{(k)} - (\sqrt{a_1})^{n+1} \psi_0^{(k)} + a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r)^2\} \quad \textbf{(B_14)}$$

The last correlation means the conditional probability to find at $s = s_{(k)}$ and $t = t_{(k)}$ the variables $\psi_{(n-1)}$, $\psi_{(n-2)}, \ldots \psi_{(1)}$ at the respective states of $\psi_{(n)}^{(k)}$, $\psi_{(n-1)}^{(k)}, \ldots \psi_{(2)}^{(k)}$ if at $s = s_{(k-1)}$ and $t = t_{(k-1)}$ they were at $\psi_{(n-2)}^{(k)}$, $\psi_{(n-3)}^{(k)}, \ldots \psi_{(0)}^{(k)}$. Note again, as remarked after Eqs (B_10) and (B_13), that the superscripts of each of the $(n-1)$ variables at the beginning of the subintervals $s_{(k-1)}$ and $t_{(k-1)}$ are the same as that at the end of it, i.e., k. In a similar manner one may calculate, through the double sum

$$\sum_{k=1}^{N} \sum_{i=1}^{n} \frac{1}{4\alpha(s_k - s_{(k-1)})^2} (\psi_i^{(k)} - \psi_{(i-1)}^{(k)} - K_i(\psi_{(i-1)}^{(k)})(s_k - s_{(k-1)}))^2$$

in the exponent of Eq (B_1), the correlation for each of the other $(N-1)$ subintervals. Taking into account that all these subintervals are, as realized from Eq (B_4), identical it is obvious that the result of calculating the correlation for each of them is, except for change of the superscripts k of ψ, the same as that of Eq (B_14). Thus, the correlation of the ensemble of the n observers over all the N subintervals $(s_{(0)}, s_1), \ldots (s_{(N-1)}, s_{(N)})$ is obtained by multiplying together N expressions of the kind of Eq (B_14). That is,

$$P_{i,j,l,\ldots.}(\psi_{(n)}^{(N)}, t_{(N)}, s_{(N)} \mid \psi_0^{(1)}, t_{(0)}, s_{(0)}) = \frac{C(\Delta s)^{N(n-1)}}{(\prod_{j=1}^{j=(n-1)} (\sum_{m=0}^{m=j} a_1^m))^{\frac{N}{2}}} \cdot$$
$$\cdot \exp\{-\frac{N}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k} (\psi_n^{(N)} - (\sqrt{a_1})^{n+1} \psi_0^{(N)} + a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r)^2\}, \quad \textbf{(B_15)}$$

where C is the normalizing constant which is, as mentioned after Eq (B_1), calculated from the normalizing condition (Namiki, 1992) $\int P_{l\ldots.}(\psi_{(n-1)}, t_{(N)}, s_{(N)} \mid \psi_0, t_{(0)}, s_{(0)}) d\psi = 1$. Using the results of Eqs (B_10), (B_13)-(B_14) one may realize that the correlation from Eq (B_15) means the conditional probability to find at $s = s_{(N)}$ and $t = t_{(N)}$ the variables $\psi_{(n-1)}$, $\psi_{(n-2)}, \ldots \psi_{(1)}$ at the respective states of $\psi_{(n)}^{(N)}$, $\psi_{(n-1)}^{(N)}, \ldots \psi_{(2)}^{(N)}$ if at $s = s_{(N-1)}$ and $t = t_{(N-1)}$ they were found at $\psi_{(n-2)}^{(N)}$, $\psi_{(n-3)}^{(N)}, \ldots \psi_{(0)}^{(N)}$ …. and at $s = s_{(N-3)}$ and $t = t_{(N-3)}$ they were found at $\psi_{(n-2)}^{(N-2)}$, $\psi_{(n-3)}^{(N-2)}, \ldots \psi_{(0)}^{(N-2)}$ …… and at $s = s_{(0)}$ and $t = t_{(0)}$ they were at $\psi_{(n-2)}^{(1)}$, $\psi_{(n-3)}^{(1)}, \ldots \psi_{(0)}^{(1)}$. That is, the conditional probability here involves N conditions at the beginnings of the N subintervals so that, as remarked for the specific cases of Eqs (B_10), (B_13) and (B_14), the superscript of each of the $(n-1)$ variables $\psi_{(n-1)}$, $\psi_{(n-2)}, \ldots \psi_{(1)}$ at the beginning of each of the N subintervals $(s_{(N-1)}, s_{(N)}), (s_{(N-3)}, s_{(N-2)}), \ldots (s_{(0)}, s_{(1)})$ is as same as that at end of it. Thus, substituting from Eq (B_15) into this normalizing equation one obtains

$$\int_{-\infty}^{\infty} P_{i....}(\psi_{(n)}, t_{(N)}, s_{(N)} \mid \psi_0, t_{(0)}, s_{(0)}) d\psi_{(n)}^{(N)} = \frac{C(\Delta s)^{N(n-1)}}{(\prod_{j=1}^{j=(n-1)} (\sum_{m=0}^{m=j} a_1^m))^{\frac{N}{2}}} \cdot$$

$$\int_{-\infty}^{\infty} \exp\{-\frac{N}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k} (\psi_n^{(N)} - (\sqrt{a_1})^{n+1} \psi_0^{(N)} + a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r)^2\} \cdot \qquad \textbf{(B\_16)}$$

$$\cdot d\psi_n^N = 1$$

Note that the value of $\psi_{(0)}^{(N)}$ is generally given so the variable is $\psi_{(n)}^{(N)}$ as denoted in the last expression. Now, expanding the squared expression in the last equation and using the integral from Eq (B_7) one may note that the coefficients $a_{\psi_n^k}$, $b_{\psi_n^k}$, $c_{\psi_n^k}$ are

$$a_{\psi_{(n)}^{(N)}} = \frac{N}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k}$$

$$b_{\psi_{(n)}^{(N)}} = \frac{N(2a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r - 2(\sqrt{a_1})^{n+1} \psi_0^{(k)})}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k} \qquad \textbf{(B\_17)}$$

$$c_{\psi_{(n)}^{(N)}} = \frac{N(((\sqrt{a_1})^{n+1} \psi_0^{(k)})^2 + (a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r)^2 - 2a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r (\sqrt{a_1})^{(n+1)} \psi_0^{(N)})}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k}$$

Thus, substituting from the last equations into Eq (B_7) and noting that $(b_{\psi_{(n)}^{(N)}})^2 - 4a_{\psi_{(n)}^{(N)}} c_{\psi_{(n)}^{(N)}} = 0$ one may calculate the integral from Eq (B_16) over $\psi_{(n)}^{(N)}$ as

$$\int_{-\infty}^{\infty} d\psi_{(n)}^{(N)} \exp\{-\frac{N}{4\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k} (\psi_{(n)}^{(N)} - (\sqrt{a_1})^{n+1} \psi_0^{(N)} +$$

$$+ a_2 \sum_{r=0}^{r=n+1} (\sqrt{a_1})^r)^2\} = (\frac{4\pi\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k}{N})^{\frac{1}{2}} \qquad \textbf{(B\_18)}$$

Substituting the last result into Eq (B_16) and solving for C one obtains

$$C = \frac{N^{\frac{1}{2}} (\prod_{j=1}^{j=(n-1)} (\sum_{m=0}^{m=j} a_1^m))^{\frac{N}{2}}}{(\Delta s)^{N(n-1)} (4\pi\alpha(\Delta s)^2 \sum_{k=0}^{k=(n-1)} a_1^k)^{\frac{1}{2}}} \qquad \textbf{(B\_19)}$$

Substituting this value of C in Eq (B_15) one obtains the complete expression for the correlation of the n observers over the N subintervals as written in Eq (39) of Part I

$$P_{i,j,l,.....}(\psi_{(n)},t_N,s_N\,|\,\psi_0,t_{(0)},s_{(0)})=\left(\frac{N}{4\pi\alpha(\Delta s)^2\sum_{k=0}^{k=(n-1)}a_1^k}\right)^{\frac{1}{2}}\cdot$$
$$\cdot\exp\{-\frac{N}{4\alpha(\Delta s)^2\sum_{k=0}^{k=(n-1)}a_1^k}(\psi_n^{(N)}-(\sqrt{a_1})^{n+1}\psi_0^{(N)}+a_2\sum_{r=0}^{r=n+1}(\sqrt{a_1})^r)^2\} \qquad \textbf{(B\_20)}$$

APPENDIX C

Derivation of Eq (8)

We derive here in a detailed manner Eq (8) which denotes the first order term in Eq (6) that involves one emission and one reabsorption of the photon by the electron during the given finite time and s Intervals $(t_{(0)},t),(s_{(0)},s)$.

As remarked, the electron is assumed to have two different states so that at t_1 and s_1 it was at the higher state 2 from which it descends to the lower state 1 through emitting a photon. Then at t_2 and s_2 it reabsorbs the photon and returns to state 2 as schematically shown at the left hand side of Figure 1. In the following we denote the higher and lower energies of the electron by ε_2 and ε_1 respectively and that of the photon by w_λ where, due to the nonconserved energy character of the interaction, $\varepsilon_2-\varepsilon_1\neq w_\lambda$. We wish to represent the s dependence of the electron and photon in the extra dimension in a similar manner as their t dependence. The conventional t dependence (see, for example, Chapter 7 in (Haken, 1981)) of an incoming electron with energy ε_2 at time t_1 (before any interaction of it) is $e^{-i\varepsilon_2 t_1}$ and that of an outgoing electron with energy ε_1 at time t_2 (after any interaction of it) is $e^{i\varepsilon_1 t_2}$. The t dependence of the emitted photon at t_1 is (Haken, 1981) $e^{iw_\lambda t_1}$ and that of the reabsorbed photon at t_2 by $e^{-iw_\lambda t_2}$. Thus, according to the former discussion the (s,t) dependence of the incoming electron $\phi(s,t)$ and the emitted photon $u(s,t)$ at t_1 and s_1 may be represented by

$$\phi(s_1,t_1)_{before\ emission}=e^{-i\varepsilon_2 t_1}+e^{-i\varepsilon_2 s_1}(1-i\delta)$$
$$u(s_1,t_1)_{after\ emission}=e^{iw_\lambda t_1}+e^{iw_\lambda s_1}(1+i\delta), \qquad \textbf{(C\_1)}$$

where δ is an infinitesimal satisfying $\delta\cdot\infty=\infty$, and $\delta\cdot c=0$ (c is a constant, see P. 40 in (Mattuck, 1976)). This is done so that for finite values of s the dependence upon s, for both the electron and photon, is similar, as remarked, to the dependence upon t and when $s\to\infty$, which is the equilibrium situation in the SQ theory, the terms in s vanish as required. That is

$$\phi(s_1<\infty,t_1)_{before\ emission}=e^{-i\varepsilon_2 t_1}+e^{-i\varepsilon_2 s_1}$$
$$\lim_{s\to\infty}\phi(s,t_1)_{before\ emission}=e^{-i\varepsilon_2 t_1} \qquad \textbf{(C\_2)}$$
$$u(s_1<\infty,t_1)_{after\ emission}=e^{iw_\lambda t_1}+e^{iw_\lambda s_1},\quad \lim_{s\to\infty}u(s,t_1)_{after\ emission}=e^{iw_\lambda t_1}$$

The expression for the outgoing electron at t_1 and s_1 with the lower energy ε_1 (after emitting the photon) and its reduction for finite and infinite s are

$$\phi(s_1,t_1)_{after\ emision}=e^{i\varepsilon_1 t_1}+e^{i\varepsilon_1 s_1}(1+i\delta),$$
$$\phi(s_1<\infty,t_1)_{after\ emission}=e^{i\varepsilon_1 t_1}+e^{i\varepsilon_1 s_1},\quad \lim_{s\to\infty}\phi(s,t_1)_{after\ emission}=e^{i\varepsilon_1 t_1}, \tag{C_3}$$

where the δ has the same meaning as before. Just before the reabsorption stage at t_2 and s_2 the electron and photon are represented by

$$\phi(s_2,t_2)_{before\ reabsorption}=e^{-i\varepsilon_1 t_2}+e^{-i\varepsilon_1 s_2}(1-i\delta)$$
$$u(s_2,t_2)_{before\ reabsorption}=e^{-iw_\lambda t_2}+e^{-iw_\lambda s_2}(1-i\delta) \tag{C_4}$$

Needless to remark that, according to our discussion, the former expressions reduce, for finite and infinite s, to

$$\phi(s_2<\infty,t_2)_{before\ reabsorption}=e^{-i\varepsilon_1 t_2}+e^{-i\varepsilon_1 s_2}$$
$$\lim_{s\to\infty}\phi(s,t_2)_{before\ reabsorption}=e^{-i\varepsilon_1 t_2} \tag{C_5}$$
$$u(s_2<\infty,t_2)_{before\ reabsorption}=e^{-iw_\lambda t_2}+e^{-iw_\lambda s_2},\quad \lim_{s\to\infty}u(s,t_2)_{before\ reabsorption}=e^{-iw_\lambda t_2}$$

Just after the reabsorption at s_2 and t_2 the expression for the electron and its reduction for finite and infinite s are

$$\phi(s_2,t_2)_{after\ reabsorption}=e^{i\varepsilon_2 t_2}+e^{i\varepsilon_2 s_2}(1+i\delta)$$
$$\phi(s_2<\infty,t_2)_{after\ reabsorption}=e^{i\varepsilon_2 t_2}+e^{i\varepsilon_2 s_2} \tag{C_6}$$
$$\lim_{s\to\infty}\phi(s,t_2)_{after\ reabsorption}=e^{i\varepsilon_2 t_2}$$

Beside the former expressions for the separate electron and photon we should take into acount also the interaction between them. That is, the acts of the emission and reabsorption of the photon by the electron. This interaction for the emission part in the extra dimension s, denoted $g^{em}_{\lambda_s}$, may be written as

$$g^{em}_{\lambda_s}=-\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int\phi_1(s,t)u(s,t)p\phi_2(s,t)dV, \tag{C_7}$$

where $\phi_2(s,t)$, $\phi_1(s,t)$, denote the two energy states of the electron as given by Eqs (C_1)-(C_6) and $u(s,t)$ is the expression for the photon given by Eqs (C_1)-(C_2) and (C_4)-(C_5). The w_λ and ε_0 are respectively the energy of the emitted photon and the dielectric constant in vacuum. The integration is over the volume which includes also the s dimension and the p is the momentum operator which is represented by

$$p=\frac{\hbar}{i}\Delta .$$

The former expression for the emission interaction is suggested so that in the limit of $s\to\infty$ it reduces to the known emission interaction which does not involve the s variable (see Eq (7.112) in (Haken, 1981)). That is,

$$\lim_{s\to\infty} g_{\lambda_s}^{em} = \lim_{s\to\infty}\{-\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int \phi_1(s,t)u(s,t)p\phi_2(s,t)dV\} =$$
$$= -\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int \lim_{s\to\infty}(\phi_1(s,t))\lim_{s\to\infty}(u(s,t))p\lim_{s\to\infty}(\phi_2(s,t))dV = \quad \textbf{(C\_8)}$$
$$= -\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int \phi_1(t)u(t)p\phi_2(t)dV = g_\lambda^{em},$$

where the last result is obtained by noting from Eqs (C_1)-(C_6) that in the limit $s\to\infty$ the expressions for the electron and photon reduce to their known forms (Haken, 1981). The interaction for the reabsorption part may be obtained by noting that the expressions for the electron and photon participating in the reabsorption interaction are obtained by taking the hermitian adjoints of the expressions for the electron and photon participating in the emission process. Thus, using the rule (Schiff, 1968; Merzbacher, 1961; Tannoudji, 1977) that the hermitian adjoint of the product of some expressions is the product of their adjoints in the reverse order, one may obtain the interaction for the reabsorption part, denoted $g_{\lambda_s}^{re}$, from that of the emission part $g_{\lambda_s}^{em}$ as follows

$$g_{\lambda_s}^{re} = (g_{\lambda_s}^{em})\dagger = (-\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int \phi_1(s,t)u(s,t)p\phi_2(s,t)dV)\dagger =$$
$$= -\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int (\phi_2(s,t))\dagger(p)\dagger(u(s,t))\dagger(\phi_1(s,t))\dagger dV \quad \textbf{(C\_9)}$$

The reabsorption interaction reduces at the limit of $s\to\infty$, just like the emission process in Eq (C_8), to the known reabsorption interaction (Haken, 1981) which does not involve the extra s variable. That is,

$$\lim_{s\to\infty} g_{\lambda_s}^{re} = \lim_{s\to\infty}\{-\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int (\phi_2(s,t))\dagger(p)\dagger(u(s,t))\dagger(\phi_1(s,t))\dagger dV\} =$$
$$= -\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int \lim_{s\to\infty}(\phi_2(s,t))\dagger p\lim_{s\to\infty}(u(s,t))\dagger\lim_{s\to\infty}(\phi_1(s,t))\dagger dV = \quad \textbf{(C\_10)}$$
$$= -\sqrt{\frac{e^2}{2m^2\hbar w_\lambda \varepsilon_0}}\int (\phi_2(t))\dagger p(u(t))\dagger(\phi_1(t))\dagger dV = g_\lambda^{re}$$

Note that the whole processes of emission and reabsorption may, respectively, be read directly from Eqs (C_8) and (C_10) if one realizes that the operator p in each of these equations denotes the interaction undergone by the expressions (denoting electron or (and) photon) at its right hand side which result with the expressions (also denoting electron or (and) photon) at its left hand side. Thus, in Eq (C_8), which describes the emission process, the $\phi_2(s,t)$ at the right of p denotes the initial electron with the higher energy state 2 and the $\phi_1(s,t)u(s,t)$ at the left of p are the electron with the lower energy state 1 (after emission) and the emitted photon. Likewise, in Eq (C_10), which describes the reabsorption process, the $(u(s,t))\dagger(\phi_1(s,t))\dagger$ at the right of p denotes the initial lower energy electron and the photon, before the reabsorption, and the $(\phi_2(s,t))\dagger$ at the left of p is the electron with the higher energy state 2 after the reabsorption.

Now, we must realize that the final state at t and s after the reabsorption of the photon, where we remain with one electron with the higher energy state 2, is the same as the initial state at $t_{(0)}$ and $s_{(0)}$ before the emission of the photon from the higher energy electron. Thus, we may write for the relevant P^l at the end of the whole process of emission and reabsorption (Haken, 1981)

$$P^{I}(\psi,t,s\,|\,\psi^{(0)},t_{(0)},s_{(0)})=P^{I}(\psi^{(0)},t_{(0)},s_{(0)})+G(t,s)P^{I}(\psi^{(0)},t_{(0)},s_{(0)}), \qquad \textbf{(C\_11)}$$

where the coefficient $G(t,s)$ denotes the mentioned evolution during the $(t_{(0)},t)$ and $(s_{(0)},s)$ intervals from the initial state $P^{I}(\psi^{(0)},t_{(0)},s_{(0)})$ back to the same state. We first note that as the (s,t) dependence of the states of the electron and photon were represented as sums of two terms, one involves only the t term and the second only the s term, so the (s,t) dependence of the entire mentioned interaction of (emission + reabsorption) $G(t,s)$ may also be written as a sum of two separate terms, denoted $G(t)$ and $G(s)$ each of them involves only one variable. This is done, as will just be realized, so that at the equilibrium limit the s term vanishes and remains only the t term as is the case regarding the mentioned (s,t) representation of the states of the electron and photon (see Eqs (C_1)-(C_6)).

Thus, for the t dependence of the emission process one should take into account that: (1) the emission process is executed during the interval $0<t_1<t_2$, (2) the electron before and after emission at $t_{(1)}$ is, respectively, represented by $e^{-i\varepsilon_2 t_1}$ and $e^{i\varepsilon_1 t_1}$, (3) the emitted photon at $t_{(1)}$ is given by $e^{iw_\lambda t_1}$ and (4) the emission itself is described by the interaction g_λ^{em}. And for the t dependence of the reabsorption process one should take into account that: (1) the reabsorption process is executed during the interval $0<t_2<t$, (2) the electron before and after reabsorption at $t_{(2)}$ is, respectively, represented by $e^{-i\varepsilon_1 t_2}$ and $e^{i\varepsilon_2 t_2}$, (3) the reabsorbed photon at $t_{(2)}$ is given by $e^{-iw_\lambda t_2}$ and (4) the reabsorption itself is described by the interaction $g_\lambda^{re}=(g_\lambda^{em})\dagger$. Thus, one may write the t dependence of the (emission + reabsorption) process $G(t)$ as

$$G(t)=g_\lambda^{em}\cdot(g_\lambda^{em})\dagger\cdot\int_0^{t_2}\exp(i(\varepsilon_1+w_\lambda-\varepsilon_2)t_1)dt_1\cdot\int_0^{t}\exp(i(\varepsilon_2-w_\lambda-\varepsilon_1)t_2)dt_2 \qquad \textbf{(C\_12)}$$

Simiarly, for the s dependence of the emission process one should take into account that: (1) the emission process is executed during the interval $0<s_1<s_2$, (2) the electron before and after emission is, respectively, represented by $e^{-i\varepsilon_2 s_1(1-i\delta)}$ and $e^{i\varepsilon_1 s_1(1+i\delta)}$, (3) the emitted photon is given by $e^{iw_\lambda s_1(1+i\delta)}$ and (4) the emission itself is described by the interaction $g_{\lambda_s}^{em}$. And for the s dependence of the reabsorption process one should take into account that: (1) the reabsorption process is executed during the interval $0<s_2<s$, (2) the electron before and after reabsorption is, respectively, represented by $e^{-i\varepsilon_1 s_2(1-i\delta)}$ and $e^{i\varepsilon_2 s_2(1+i\delta)}$, (3) the reabsorbed photon is given by $e^{-iw_\lambda s_2(1-i\delta)}$ and (4) the reabsorption itself is described by the interaction $g_{\lambda_s}^{re}=(g_{\lambda_s}^{em})\dagger$. Thus, one may write the s dependence of the (emission + reabsorption) process $G(s)$ as

$$\begin{aligned}G(s)&=g_{\lambda_s}^{em}\cdot(g_{\lambda_s}^{em})\dagger\cdot\int_0^{s_2}\exp[i(\varepsilon_1+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda)+w_\lambda-\varepsilon_2)s_1]ds_1\\&\cdot\int_0^{s}\exp[i(\varepsilon_2+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda)-w_\lambda-\varepsilon_1)s_2]ds_2,\end{aligned} \qquad \textbf{(C\_13)}$$

where we have set, as remarked, $s_{(0)}=t_{(0)}=0$ for both $G(t)$ and $G(s)$. The coefficient $G(t,s)$ from Eq (C_11) is given, as remarked, by the sum $G(t)+G(s)$ so that at the equilibrium state - obtained in the limit in which all the values of s are equated to each other and taken to infinity - the term $G(s)$ vanishes and remains only the term $G(t)$ as it should be (Haken, 1981). The term $G(s)$

vanishes in the stationary state because we have already equated the initial $s_{(0)}$ to zero so for equating all the s's to each other one have to set also the other values of s equal to zero which obviously causes $G(s)$ from Eq (C_13) to vanish. Note that thus far we have discussed a single mode λ for the emitted and reabsorbed photon which makes sense in a cavity whose closed walls are of the same order as the wavelength of the photon. But for an infinite space or a cavity with open sides one should consider a continuum of modes $\sum_{\lambda}$. Thus, considering this continuum of modes and performing the integration over t_1 and s_1 from Eqs (C_12)-(C_13) one obtains

$$G(t,s)=G(t)+G(s)=\sum_{\lambda} g_{\lambda}^{em}\cdot(g_{\lambda}^{em})\dagger\cdot\int_0^t dt_2\,\frac{\{\exp[i(\varepsilon_1+w_\lambda-\varepsilon_2)t_2]-1\}}{i(\varepsilon_1+w_\lambda-\varepsilon_2)}\cdot$$
$$\cdot\exp[i(\varepsilon_2-w_\lambda-\varepsilon_1)t_2]+\sum_{\lambda_s} g_{\lambda_s}^{em}\cdot(g_{\lambda_s}^{em})\dagger\cdot$$
$$\cdot\int_0^s ds_2\,\frac{\{\exp[i(\varepsilon_1-\varepsilon_2+w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))s_2]-1\}}{i(\varepsilon_1-\varepsilon_2+w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))}\cdot$$
$$\cdot\exp[i(\varepsilon_2-\varepsilon_1-w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))s_2] \qquad \textbf{(C_14)}$$

Now, performing the integration over s_2 and t_2 we obtain from Eq (C_14)

$$G(t,s)=G(t)+G(s)=\sum_{\lambda}\frac{g_{\lambda}^{em}\cdot(g_{\lambda}^{em})\dagger}{i(\varepsilon_1+w_\lambda-\varepsilon_2)}\{t-\frac{\exp(i(\varepsilon_2-\varepsilon_1-w_\lambda)t)-1}{i(\varepsilon_2-\varepsilon_1-w_\lambda)}\}+$$
$$+\sum_{\lambda_s}\frac{g_{\lambda_s}^{em}\cdot(g_{\lambda_s}^{em})\dagger}{i(\varepsilon_1-\varepsilon_2+w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))}\{-\frac{\exp(-2\delta(\varepsilon_2+\varepsilon_1+w_\lambda)s)-1}{2\delta(\varepsilon_2+\varepsilon_1+w_\lambda)}- \qquad \textbf{(C_15)}$$
$$-\frac{\exp[i(\varepsilon_2-\varepsilon_1-w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))s]-1}{i(\varepsilon_2-\varepsilon_1-w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))}\}$$

One may realize that, because of the δ (see its definition after Eq (C_1)), the quotient $-\frac{\exp(-2\delta(\varepsilon_2+\varepsilon_1+w_\lambda)s)-1}{2\delta(\varepsilon_2+\varepsilon_1+w_\lambda)}$ in the second sum, which is of the kind $\frac{0}{0}$, may be evaluated, using L'hospital theorem (Pipes, 1958), to obtain for it the result of s so that Eq (C_15) becomes

$$G(t,s)=G(t)+G(s)=\sum_{\lambda}\frac{g_{\lambda}^{em}\cdot(g_{\lambda}^{em})\dagger}{i(\varepsilon_1+w_\lambda-\varepsilon_2)}\{t-\frac{\exp(i(\varepsilon_2-\varepsilon_1-w_\lambda)t)-1}{i(\varepsilon_2-\varepsilon_1-w_\lambda)}\}+$$
$$+\sum_{\lambda_s}\frac{g_{\lambda_s}^{em}\cdot(g_{\lambda_s}^{em})\dagger}{i(\varepsilon_1-\varepsilon_2+w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))}\{s- \qquad \textbf{(C_16)}$$
$$-\frac{\exp[i(\varepsilon_2-\varepsilon_1-w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))s]-1}{i(\varepsilon_2-\varepsilon_1-w_\lambda+i\delta(\varepsilon_2+\varepsilon_1+w_\lambda))}\}$$

The last expression for $G(t,s)$ contains terms which are proportional to t and s, others which are oscillatory in these variables, and also constant terms. Thus, for large t and s the oscillatory as well as the constant terms may be neglected compared to t and s as in the analogous quantum discussion of the same process (Haken, 1981) (without the extra variable). That is, one may obtain for $G(t,s)$

$$G(t,s)=G(t)+G(s)=\sum_{\lambda}\frac{g_{\lambda}^{em}\cdot(g_{\lambda}^{em})\dagger\cdot t}{i(\varepsilon_1+w_{\lambda}-\varepsilon_2)}+\sum_{\lambda_s}\frac{g_{\lambda_s}^{em}\cdot(g_{\lambda_s}^{em})\dagger\cdot s}{i(\varepsilon_1-\varepsilon_2+w_{\lambda}+i\delta(\varepsilon_2+\varepsilon_1+w_{\lambda}))} \quad \textbf{(C\_17)}$$

Substituting from the last equation in Eq (C_11) one obtains Eq (8)

$$P^{I}(\psi,t,s\,|\,\psi^{(0)},0,0)=P^{I}(\psi^{(0)},0,0)(1+G(s,t))==P^{I}(\psi^{(0)},0,0)(1+it\Delta\varepsilon_{\lambda}+is\Delta\varepsilon_{\lambda_s}), \quad \textbf{(C\_18)}$$

where $\Delta\varepsilon_{\lambda}$ and $\Delta\varepsilon_{\lambda_s}$ are

$$\Delta\varepsilon_{\lambda}=\sum_{\lambda_s}\frac{g_{\lambda_s}^{em}\cdot g_{\lambda_s}^{em}}{\varepsilon_2-\varepsilon_1-w_{\lambda}},\quad \Delta\varepsilon_{\lambda_s}=\sum_{\lambda_s}\frac{g_{\lambda_s}^{em}\cdot g_{\lambda_s}^{em}}{\varepsilon_2-\varepsilon_1-w_{\lambda}-i\delta(\varepsilon_2+\varepsilon_1+w_{\lambda})} \quad \textbf{(C\_19)}$$