# Haar Wavelet Collocation Technique for Solving Linear Volterra Integro Differential Equations

**Intisar Swedain Ali[1*]**

**Abstract**

Operational style is based on representing different integro-differential mathematical functions in terms of matrices. In this research, Haar wavelet collocation points and operational matrix are used for solving Volterra integro-differential equations. A modified computational method is elucidated to resolve Volterra integro-differential equations (VIDE). The integro-differential and integral equations are converted with initial conditions to a linear system of algebraic equations; where the interval expanded to, as noted in the examples. Illustrations are supplied with the help of three representation examples by appropriate comparisons with exact solutions. In addition, the simulation result indicate the accuracy can be enhanced by increasing the Haar wavelet resolution.

**Key Words:** Approximation Solutions, Collocation Points Method, Haar Wavelets, Volterra Integro-differential Equations.

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## Introduction

In general, some significant problems in field of science and engineering can usually be converted to integral and integro-differential equations. Integro-differential equations have charmed much interest and solving such equations has been one of the attention-grabbing missions for mathematicians (Wazwaz, 2011). Several approaches have been proposed for numerical solution of these equations.

One technique is collocation points technique for the numerical solution of integral and integro-differential equations (Khan, 2018). Wavelets are comparatively new instrument and have quite been prospering in domain mathematical research (Nievergelt, 1999). Numerical solutions of differential and integral equations demand evolution of accurate and quick algorithms based on wavelets (Khan, 2018). Haar wavelet presents a promising solution rules because of simple mathematical expressions and multi-resolution properties (Shahsavaran, 2011). Mundewadi and Bhaskar, (2018) solved nonlinear Fredholm Volterra integral and integro-differential equations by using Haar wavelet collocation points technique. Also, according to Khaireddine and Ellaggoune, (2016) the linear integro-differential equation was solved utilizing the modified Haar wavelet way on interval $[0,1)$.

While Sekar and Jaisankar, (2014) applied Single-Term Haar Wavelet Series technique to resolve linear Fredholm integro-differential equations of first and second order. Lepik, (2007) presented the Haar wavelet collocation method to solving nonlinear integral differential and evolution equations. Mishra *et. al* (2012), used operational matrix of Haar wavelet to find numerical solutions of problems regarding differential, integral and integro-differential on interval $[0,1)$.

In this research, A Haar wavelet based method to resolve Volterra integro-differential equations (VIDE) is presented.

**Corresponding author:** Intisar Swedain Ali
**Address:** [1*]Al Khwarizmi College of Engineering, University of Baghdad, Baghdad, Iraq.
[1*]E-mail: intisarswaidan@yahoo.com
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It is noticeable in most of the researches carried out that the interval $[0,1]$ was used for Haar wavelet collocation points (HWCP), while in our research the interval expanded to $[0,\delta]$, where $\delta \in \Re$ and as noted in the examples.

## Description of Problem

Consider the general form of linear Volterra integro-differential equations is (Wazwaz, 2011):

$$\begin{cases} u'(\mathbf{x}) = F(g(\mathbf{x}), u(\mathbf{x}), \int_{\Omega} \lambda(\mathbf{x},\tau) u(\tau)\, d\tau) \\ u(0) = u_0 \end{cases} \quad (1)$$

Where $g(\mathbf{x})$ and the kernel $\lambda(\mathbf{x},\tau)$ are known functions, $u(\tau)$ is the function to be determined, $u(\mathbf{x})'$ is differential operator of $u(\mathbf{x})$, $\Omega$ is a finite interval $a \le \tau \le \mathbf{x}$ and $u(0)=u_0$ is the initial condition.

## Haar Wavelets

The Haar wavelet is the function of square waves over the interval $[\delta_1, \delta_2)$ which is defined as follows:

$$\hat{h}_1(x) = \begin{cases} 1, & \delta_1 \le x < \frac{1}{2}(\delta_1+\delta_2), \\ -1, & \frac{1}{2}(\delta_1+\delta_2) \le x < \delta_2, \\ 0, & \textbf{otherwise}. \end{cases} \quad (2)$$

Also, the $\hat{h}_0(x)$ defined as:

$$\hat{h}_0(\mathbf{x}) = \begin{cases} 1, & \delta_1 \le \mathbf{x} < \delta_2, \\ 0, & \textbf{elsewhere.} \end{cases} \quad (3)$$

And the other wavelets can be obtained as:

$$\hat{h}_n(\mathbf{x}) = \hat{h}_1(2^n \mathbf{x} - k) \quad (4)$$

where $n = 2^j + k$, $n, j$ belong to $Z^+$, $j = 0,1,2,\ldots,\log_2(m-1)$, $m$ is the Haar wavelet resolution and $0 \le k < 2^j$ that satisfies:

$$\int_{\delta_1}^{\delta_2} \hat{h}_n(\mathbf{x})\,\hat{h}_u(\mathbf{x})dx = \begin{cases} 2^{-j}(\delta_2-\delta_1), & n = u \\ 0, & n \ne u \end{cases} \quad (5)$$

Any analytic function $\varphi(\mathbf{x}) \in L^2([\delta_1, \delta_2))$ can be written to a finite of Haar series:

$$\varphi_{\mathbf{m}}(\mathbf{x}) = \sum_{n=0}^{\mathbf{m}-1} \alpha_n \hat{h}_n(\mathbf{x}) \quad (6)$$

Where $\varphi(\mathbf{x})$ is a piecewise constants, which can be written into compacted form as:

$$\varphi(\mathbf{x}) = \alpha_{\mathbf{m}}^{\mathbf{T}}\, \hat{\mathbf{h}}_{\mathbf{m}}(\mathbf{x}) \quad (7)$$

where,

$\hat{\mathbf{h}}_m(\mathbf{x}) = [\,\hat{h}_0(\mathbf{x}) \;\; \hat{h}_1(\mathbf{x}) \;\; \hat{h}_2(\mathbf{x}) \;\ldots\; \hat{h}_{\mathbf{m}-1}(\mathbf{x})\,]^{\mathbf{T}}$ is the Haar function vector and $\alpha_{\mathbf{m}} = [\alpha_0 \;\; \alpha_1 \ldots \alpha_{\mathbf{m}-1}]^{\mathbf{T}}$ is the coefficient vector which can be determined from

$$\alpha_n = \frac{2^j}{(\delta_2-\delta_1)} \int_{\delta_1}^{\delta_2} \varphi(\mathbf{x})\,\hat{h}_i(\mathbf{x})\,\mathbf{dx} \quad (8)$$

At collocation points (Swaidan and Hussin, 2013).

$$\mathbf{x}_s = (\delta_1 + \frac{\delta_2 - \delta_1}{2\mathbf{m}}(2s-1)),$$
$$s = 1, 2, 3, \ldots, \mathbf{m}-1. \quad (9)$$

The Haar function vector $\hat{\mathbf{h}}_m(\mathbf{x})$ which can be represented into the matrix form $\hat{\mathbf{H}}_{\mathbf{m}}$ and the elements of the matrix are given via

$$(\hat{\mathbf{H}}_{\mathbf{m}})_{n,s} = \hat{h}_n(\mathbf{x}_s). \quad (10)$$

For example, the Haar wavelet matrix of fourth-order $\hat{\mathbf{H}}_4$ can be articulated into matrix formula in the interval of $[0,1)$ with the collocation points from equation (9) as follows:

$$\hat{\mathbf{H}}_4 = \begin{bmatrix} \hat{h}_0(\frac{1}{8}) & \hat{h}_0(\frac{3}{8}) & \hat{h}_0(\frac{5}{8}) & \hat{h}_0(\frac{7}{8}) \\ \hat{h}_1(\frac{1}{8}) & \hat{h}_1(\frac{3}{8}) & \hat{h}_1(\frac{5}{8}) & \hat{h}_1(\frac{7}{8}) \\ \hat{h}_2(\frac{1}{8}) & \hat{h}_2(\frac{3}{8}) & \hat{h}_2(\frac{5}{8}) & \hat{h}_2(\frac{7}{8}) \\ \hat{h}_3(\frac{1}{8}) & \hat{h}_3(\frac{3}{8}) & \hat{h}_3(\frac{5}{8}) & \hat{h}_3(\frac{7}{8}) \end{bmatrix}, \quad (11)$$

Which given by the following recurrence relation (Chen and Hsiao, 1997).

$$\hat{\mathbf{H}}_{\mathbf{m}\times\mathbf{m}} = \begin{bmatrix} \hat{\mathbf{H}}_{\frac{\mathbf{m}}{2}\times\frac{\mathbf{m}}{2}} \otimes [1 \;\; 1] \\ \mathbf{I}_{\frac{\mathbf{m}}{2}\times\frac{\mathbf{m}}{2}} \otimes [1 \;\; -1] \end{bmatrix}, \quad \hat{\mathbf{H}}_{1\times 1} = [1] \quad (12)$$

Where $\mathbf{I}_{\frac{\mathbf{m}}{2}\times\frac{\mathbf{m}}{2}}$ is the identity matrix of size $\frac{\mathbf{m}}{2}$ and $\otimes$ is the Kronecker product (Brewer, 1978).

For example, the second Haar wavelet matrix can be expressed as:

$$\hat{\mathbf{H}}_{2\times 2} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

So the coefficient $\alpha_{\mathbf{m}}^{\mathbf{T}}$ in equations (6) and (7) can be readily obtained as

$$\alpha_{\mathbf{m}}^{\mathbf{T}} = \varphi_{\mathbf{m}} \;\; \hat{\mathbf{H}}_{\mathbf{m}}^{-1}, \quad (13)$$

where

$$\varphi_{\mathbf{m}} = [\varphi(x_1)\ \ \varphi(x_2)\ \ \varphi(x_3)\ \ \cdots\ \ \varphi(x_{\mathbf{m}})] \quad (14)$$

In the specific interval of $[\mathbf{0}, \delta)$, $\hat{\mathbf{h}}_{\mathbf{m}}(\mathbf{x})$ can be extended in a Haar series by the integration as in (Chen and Hsiao, 1997):

$$\int_0^{\mathbf{x}} \hat{\mathbf{h}}_{\mathbf{m}}(\mathbf{x})\,\mathbf{dx} \cong \mathbf{P}_{\mathbf{m}}\,\hat{\mathbf{h}}_{\mathbf{m}}(\mathbf{x}), \quad (15)$$

where $\mathbf{P}_{\mathbf{m}}$ is an $\mathbf{m} \times \mathbf{m}$ the operational matrix of integration, which is acquired recursively by Swaidan and Hussin, (2013):

$$\mathbf{P}_m = \frac{1}{2\mathbf{m}}\begin{bmatrix} 2\mathbf{m}\,\mathbf{P}_{\mathbf{m}/2} & -\delta\,\hat{\mathrm{H}}_{\mathbf{m}/2} \\ -\delta\,\hat{\mathrm{H}}^{-1}_{\mathbf{m}/2} & O_{\mathbf{m}/2} \end{bmatrix}, \quad \mathbf{P}_1 = \left[\frac{\delta}{2}\right]. \quad (16)$$

## Numerical Solution of VIDE Using HWCM

In this segment, the linear Volterra integro-differential equation given by:

$$\begin{cases} u'(\mathbf{x}) = g(\mathbf{x}) + \int_a^{\mathbf{x}} \lambda(\mathbf{x},\tau)u(\tau)d\tau \\ u(0) = u_0 \end{cases} \quad (17)$$

Without loss of generality, the field of benefit has been selected as $[0, \delta]$ and $\lambda(\mathbf{x},\tau) = 1$ for the sake of convenience. At first, approximate the differential function of $u'(x)$ into terms of Haar wavelet basis function by using equation (6) as follow:

$$u'(\mathbf{x}) = \sum_{n=0}^{\mathbf{m}-1} b_n \hat{h}_n(\mathbf{x}), \quad (18)$$

by utilizing the compacted form

$$u'(\mathbf{x}) = \mathbf{b}^{\mathbf{T}}\,\hat{\mathbf{h}}\,(\mathbf{x}) \quad (19)$$

Where $\mathbf{b}^{\mathbf{T}} = [b_0\ \ b_1\ \ b_2\ \ b_3\ \ \cdots\ \ b_{\mathbf{m}-1}]$ is an $1 \times \mathbf{m}$ unknown Haar wavelet coefficient vector and $\hat{\mathbf{h}}(x)$ is the vector of known Haar wavelet function with dimension of $\mathbf{m} \times 1$, where $\hat{\mathbf{h}}(\mathbf{x}) = [\hat{h}_0(\mathbf{x})\ \ \hat{h}_1(\mathbf{x})\ \ \hat{h}_3(\mathbf{x})\ \ \ldots\ \hat{h}_{\mathbf{m}-1}(\mathbf{x})]^{\mathbf{T}}$ and $\mathbf{T}$ denoted to the transpose.

The integration of $\hat{\mathbf{h}}_{\mathbf{m}}(\mathbf{x})$ in the specific interval $[0, \delta)$ for equation (19) with respect to $\mathbf{x}$ besides applying equation (15), we find $u(\mathbf{x})$, which is represented into terms of Haar operational matrix with the Haar wavelet function by

$$u(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{b}^{\mathbf{T}}\hat{\mathbf{h}}(\mathbf{x})\,dx + u_0 \quad (20)$$

Thus

$$u(x) = \mathbf{b}^{\mathbf{T}}\,\mathbf{P}\,\hat{\mathbf{h}}(\mathbf{x}) + u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (21)$$

where $u_0$ is the initial condition and $\theta = [1\ 0\ 0\ \cdots\ 0]^{\mathbf{T}}$ is an $\mathbf{m} \times 1$ vector.

According to equations (7), function approximation for $g(\mathbf{x})$ can be formed as

$$g(\mathbf{x}) = \mu^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (22)$$

where the coefficient vectors $\mu^{\mathbf{T}}$ can be calculate form equation (11).

Substituting the equations (19), (21) and (22) into equation (17), we obtain

$$\mathbf{b}^{\mathbf{T}}\,\hat{\mathbf{h}}\,(\mathbf{x}) = \mu^{\mathbf{T}}\hat{\mathbf{h}}(\mathbf{x}) + \int_0^x \{\,\mathbf{b}^{\mathbf{T}}\mathbf{P}\,\hat{\mathbf{h}}(\mathbf{x}) + u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x})\,\}dx \quad (23)$$

By integrating the right side of equation (23) as indicated in an equation (15) on interval $[0, \mathbf{x}]$ obtain:

$$\mathbf{b}^{\mathbf{T}}\,\hat{\mathbf{h}}\,(\mathbf{x}) = \mu^{\mathbf{T}}\hat{\mathbf{h}}(\mathbf{x}) + \mathbf{b}^{\mathbf{T}}\mathbf{P}^2\,\hat{\mathbf{h}}(\mathbf{x}) + \mathbf{x}\,u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (24)$$

where the verable $\mathbf{x}$ in term $\mathbf{x}\,u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x})$ must be convert to Haar wavelet approximation function as:

$$\mathbf{x}\,u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) = \omega^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (25)$$

Then equation (24) became after rearangment

$$\mathbf{b}^{\mathbf{T}}\,\hat{\mathbf{h}}\,(\mathbf{x}) - \mathbf{b}^{\mathbf{T}}\mathbf{P}.\mathbf{P}\,\hat{\mathbf{h}}(\mathbf{x}) = \mu^{\mathbf{T}}\hat{\mathbf{h}}(\mathbf{x}) + \omega^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (26)$$

Multipling equation (26) by the inverse of vector $\mathbf{h}^{\mathbf{T}}(\mathbf{x})$ at the collocation points as defined in equation (9) to remove the term of $\mathbf{h}^{\mathbf{T}}(\mathbf{x})$. Thus, we obtain

$$\mathbf{b}^{\mathbf{T}} - \mathbf{b}^{\mathbf{T}}\mathbf{P}.\mathbf{P} = \mu^{\mathbf{T}} + \omega^{\mathbf{T}} \quad (27)$$

Now equation (27) with the simplification by taking the transport for the both side is transformed into a standard linear system of algebric equations as defined as follows:

$$\left[\mathbf{I}_{m\times m} - (\mathbf{P}.\mathbf{P})^{\mathbf{T}}\right][\mathbf{b}] = [\mu + \omega] \quad (28)$$

Equation (28) is a linear system equations with $\mathbf{m}$ unknown variables and $\mathbf{m}$ equations that can be solved for the unknown vector $\mathbf{b}$ such in MATLAB (Xue and Chen, 2011). As soon as, determined the results to the unknown variables, then replace these variables in equation (21) to identify the solution $u(\mathbf{x})$ as:

$$u(\mathbf{x}) = \mathbf{b}^{\mathbf{T}}\mathbf{P}\,\hat{\mathbf{h}}(\mathbf{x}) + u_0\,\theta^{\mathbf{T}}\,\hat{\mathbf{h}}(\mathbf{x}) \quad (29)$$

## Numerical Examples

In this segment, the proposed technique HWCP is carried out to solve linear Volterra integro-differential equations. Numerical solutions of these examples are compared with exact solutions in formed of collocation points in order to show the efficiency of the method.

*Example 1*

Consider the following linear VID equation of the second kind with initial condition (Wazwaz, 2011):

$$u'(x) = 1 - \int_0^x u(t)dt \qquad u(0) = 0\,,\ \ x \in [0,5)$$

and with the exact solution $u(x) = \sin(x)$

According to the proposed method. The numerical results obtained from this example for diverse Haar wavelet resolution $m = 4,\ 8,\ 16,\ 32,\ 64$ and exact solution are presented in Figure 1. Also, by increasing Haar wavelet resolution, the HWCM will be capable of yielding more accurate results. Although we have considered $m = 64$ in computation of figure1, but Table 1 displays that the use of $m = 32$ is enough to approximate the linear Volterra integro-differential equation to the exact solution and the absolute errors illustrated that.

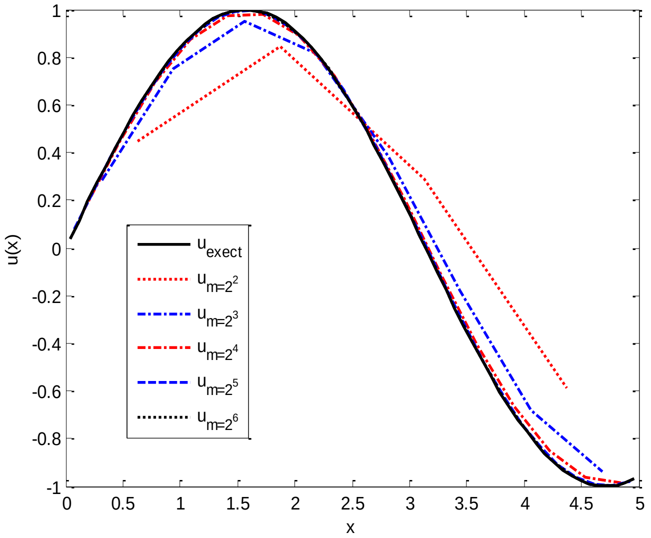


**Figure 1.** Exact and Approximate solution for Example 1 with Haar wavelet resolutions $m = 4\,,8\,,16\,,\,32\,,64$ and $x \in [0,5)$

**Tab1e 1.** Comparison between the exact and numerical solution using Haar wavelets method for Example **1** with $m = 32$ and $x \in [0,5)$

| $x$ | **Exact Solution** $u_{Exact}(x)$ | **Approxmation Solution** $u_{Approx.}(x)$ | **Absolute Error** $\lvert u_{Exact} - u_{Approx.}\rvert$ |
|---|---|---|---|
| 0.078 | 0.07805 | 0.07765 | 0.00039 |
| 0.234 | 0.23224 | 0.23107 | 0.00117 |
| 0.391 | 0.38077 | 0.37888 | 0.00189 |
| 0.547 | 0.52002 | 0.51750 | 0.00252 |
| 0.703 | 0.64660 | 0.64356 | 0.00305 |
| 0.859 | 0.75743 | 0.75400 | 0.00344 |
| 1.016 | 0.84981 | 0.84614 | 0.00367 |
| 1.172 | 0.92148 | 0.91776 | 0.00372 |
| 1.328 | 0.97070 | 0.96710 | 0.00360 |
| 1.484 | 0.99627 | 0.99298 | 0.00329 |
| 1.641 | 0.99756 | 0.99476 | 0.00280 |
| 1.797 | 0.97455 | 0.97240 | 0.00215 |
| 1.953 | 0.92780 | 0.92645 | 0.00135 |
| 2.109 | 0.85844 | 0.85801 | 0.00043 |
| 2.266 | 0.76816 | 0.76875 | 0.00059 |
| 2.422 | 0.65917 | 0.66084 | 0.00167 |
| 2.578 | 0.53412 | 0.53690 | 0.00277 |
| 2.734 | 0.39606 | 0.39992 | 0.00386 |
| 2.891 | 0.24834 | 0.25324 | 0.00490 |
| 3.047 | 0.09458 | 0.10042 | 0.00584 |
| 3.203 | -0.06149 | -0.05484 | 0.00665 |
| 3.359 | -0.21606 | -0.20877 | 0.00729 |
| 3.516 | -0.36537 | -0.35764 | 0.00773 |
| 3.672 | -0.50578 | -0.49783 | 0.00795 |
| 3.828 | -0.63386 | -0.62593 | 0.00793 |
| 3.984 | -0.74650 | -0.73885 | 0.00765 |
| 4.141 | -0.84095 | -0.83384 | 0.00711 |
| 4.297 | -0.91491 | -0.90859 | 0.00632 |
| 4.453 | -0.96658 | -0.96130 | 0.00528 |
| 4.609 | -0.99470 | -0.99068 | 0.00402 |
| 4.766 | -0.99858 | -0.99602 | 0.00257 |
| 4.922 | -0.97814 | -0.97719 | 0.00095 |

*Example 2*

Consider the equation with initial condition and a finite interval as following (Khan, 2018):

$$u'(x) = 1 + \int_0^x u(t)dt\,,\ u(0) = 1\ ,\ x \in [0, 2)$$

with the exact solution $u(x) = e^x$

According to the proposed method. The numerical results obtained from the example 2 for diverse Haar wavelet resolution $m = 4, 8, 16, 32$ and exact solution are presented in Figure 2. Also Table 2 shows the $m = 16$ is enough converges the linear Volterra integro-differential equation to the exact results as the absolute errors illustration that.

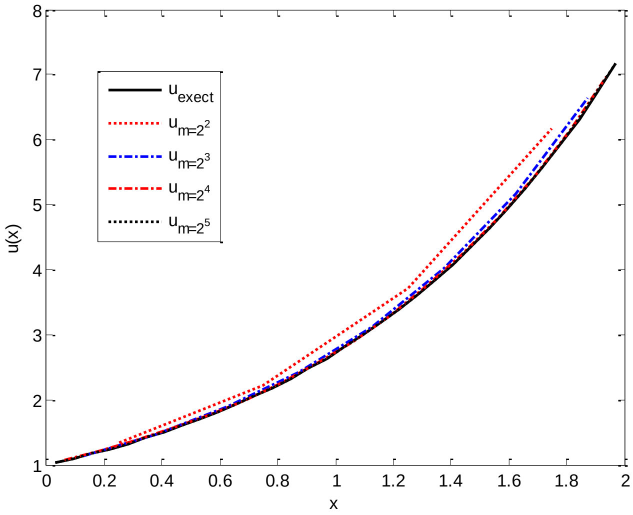


**Figure 2.** Exact and Approximate solution for Example 2 with Haar wavelet resolutions $m = 4, 8, 16, 32$ and $x \in [0, 2)$

**Tab1e 2.** Comparison between the exact and numerical solution using Haar wavelets method for Example 2 with $m = 16$ and $[0, 2)$

| $x$ | **Exact Solution** $u_{Exact}(x)$ | **Approximation Solution** $u_{Approx.}(x)$ | **Absolute Error** $\lvert u_{Exact} - u_{Approx.}\rvert$ |
|---|---|---|---|
| 0.0625 | 1.06449446 | 1.06666667 | 0.00217221 |
| 0.1875 | 1.20623025 | 1.20888889 | 0.00265864 |
| 0.3125 | 1.36683794 | 1.37007407 | 0.00323613 |
| 0.4375 | 1.54883030 | 1.55275062 | 0.00392032 |
| 0.5625 | 1.75505466 | 1.75978403 | 0.00472938 |
| 0.6875 | 1.98873747 | 1.99442190 | 0.00568443 |
| 0.8125 | 2.25353479 | 2.26034482 | 0.00681004 |
| 0.9375 | 2.55358946 | 2.56172413 | 0.00813468 |
| 1.0625 | 2.89359594 | 2.90328735 | 0.00969141 |
| 1.1875 | 3.27887377 | 3.29039233 | 0.01151856 |
| 1.3125 | 3.71545074 | 3.72911131 | 0.01366057 |
| 1.4375 | 4.21015726 | 4.22632615 | 0.01616890 |
| 1.5625 | 4.77073318 | 4.78983631 | 0.01910312 |
| 1.6875 | 5.40594893 | 5.42848115 | 0.02253222 |
| 1.8125 | 6.12574266 | 6.15227863 | 0.02653597 |
| 1.9375 | 6.94137582 | 6.97258245 | 0.03120663 |



*Example 3*

Consider the equation obtained from equation (17) for $\lambda(x,\tau) = 1$, $a = 0$ given by (Khan, 2018):

$$u'(x) = \frac{1}{4} + \frac{3}{4}x + \sin x + \int_0^x u(t)dt\quad ,u(0) = -1\ ,$$
$$x \in [0, 1.1]$$

with the exact solution $u(x) = \frac{1}{4}e^x - \frac{3}{4} - \frac{1}{2}\cos(x)$.

According to the proposed method. The numerical results obtained from the example 3 for diverse Haar wavelet resolution $m = 4, 8, 16, 32$ and exact solution are presented in Figure 3. Also Table 2 shows the $m = 16$ is enough converges the linear Volterra integro-differential equation to the exact results as the absolute errors illustration that.

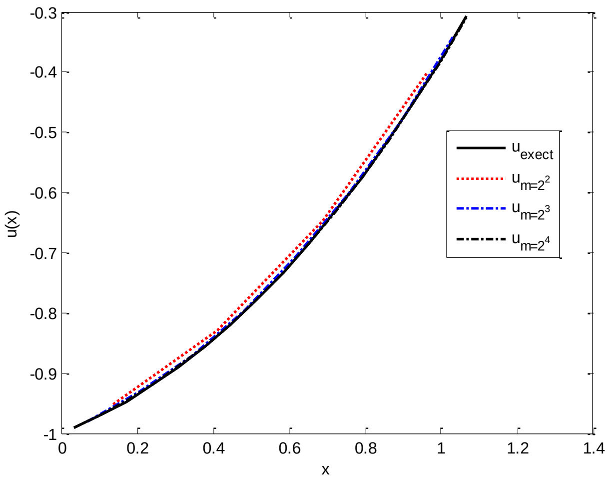


**Figure 3.** Exact and Approximate solution for Example 3 with Haar wavelet resolutions $m = 4, 8, 16$ and $x \in [0, 1.1]$

**Tab1e 3.** Comparison between the exact and numerical solution using Haar wavelets method for Example 3 with $m = 16$ and $[0,1.1]$

| $x$ | Exact Solution $u_{Exact}(x)$ | Approximation Solution $u_{Approx.}(x)$ | Absolute Error $\lvert u_{Exact} - u_{Approx.} \rvert$ |
|---|---|---|---|
| 0.0344 | -0.99101272 | -0.99061653 | 0.000396 |
| 0.1031 | -0.97064685 | -0.97025027 | 0.000397 |
| 0.1719 | -0.94702655 | -0.94664599 | 0.000381 |
| 0.2406 | -0.92007375 | -0.91972549 | 0.000348 |
| 0.3094 | -0.88971131 | -0.88941144 | 0.000300 |
| 0.3781 | -0.85586239 | -0.85562687 | 0.000236 |
| 0.4469 | -0.81844986 | -0.81829450 | 0.000155 |
| 0.5156 | -0.77739564 | -0.77733606 | 0.000060 |
| 0.5844 | -0.73261987 | -0.73267156 | 0.000052 |
| 0.6531 | -0.68404018 | -0.68421843 | 0.000178 |
| 0.7219 | -0.63157071 | -0.63189066 | 0.000320 |
| 0.7906 | -0.57512119 | -0.57559782 | 0.000477 |
| 0.8594 | 0.51459590 | 0.51524400 | 0.000648 |
| 0.9281 | -0.44989251 | -0.45072674 | 0.000834 |
| 0.9969 | -0.38090092 | -0.38193577 | 0.001035 |
| 1.0656 | -0.30750195 | -0.30875176 | 0.001250 |

## Conclusion

A numerical approach based on Haar wavelet operation matrix with collocation points technique is used to determine the solution of linear VIDEs. The integral and integro-differential equations are converted with initial conditions to a linear system of algebraic equations that can be readily resolved by computer program such as MATLAB. The proposed method is simple and easy to apply, then the required less computational complexity and supply further quantitatively reliable solutions. The obtained numerical solutions are very accurate, in comparison with the exact solutions. In addition, numerical results are converges to exact solution with lower Haar wavelet resolutions.

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