A Class of Approximate Entanglement Networks

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ABSTRACT

One of the important concepts in logical studies is the self-reference, which often leads to semantic inconsistencies. It is reviewed that the cyclical time process, particularly in terms of computing machines, may be useful in analyzing the peculiar aspects of self-reference. On the other hand, due to Bell's inequalities, quantum entanglement is often considered a natural phenomenon that is distinguishable from classical theory. Moreover, entanglement has been shown to be an essential part of recent developments in quantum computing and quantum cryptography. In this paper, the entanglement swapping protocol is applied to four 2-level non-maximal states, and a new non-trivial class of non-maximal states that approximate the weakest link is examined numerically. The initial part of the article will review some foundational and philosophical issues of quantum theory in regard to consciousness, while the latter will use numerical methods to discuss a practical aspect of quantum information science.

Key Words: Numerical simulation, Entanglement, Network

Introduction

One of the important arguments often seen in the study of logic, mathematics, computer science and possibly physics, particularly in the 20th century, is called self-reference, an object that refers to itself. A well-known example that embodies the self-referential aspect is the liar's paradox, i.e.,

“This statement is false.” (1)

Indeed, a strange situation happens when "this statement" refers to the whole statement (Fig. 1), such that it leads to inconsistency in its logic. A similar reasoning may be applied to a computing machine; however, the difficulty again immediately arises as it is unclear how a physical computer could compute on itself, i.e., when the machine has an input that is the machine itself. Similar to the liar's paradox, this problem may be envisioned as the machine referring to itself infinitely (Fig. 2).

This discussion can be applied to quantum theory involving the subject and the object. In general, quantum theory involves an observation, or an interaction, between the subject and the object (Fig. 3 (A)) (Song, 2017b). For instance, Max Planck, who many consider the founder of the quantum revolution, has remarked (Planck, 1932),

"The first and most important quality of all scientific ways of thinking must be the clear distinction between the outer object of observation and the subjective nature of the observer."

However, self-referencing occurs when the object is the subject itself. In such cases, a self-reference similar to the liar's paradox may happen, as seen in Fig. 3 (B). It has been shown that, similar to with Russell's paradox in logic, Turing's halting problem in computer science, and Gödel's incompleteness theorem in mathematics, quantum theory breaks down because the symmetry between the Schrödinger and Heisenberg pictures no longer exists. (Song, 2007).

Another important issue involving the self-reference is time. In order to see this problem more clearly, let us consider a self-referential machine. Not
only is the machine computing the machine itself, i.e., as an input, the machine that is being computed has to be in the present state, rather than in the past one, as is the machine that is computing. This time problem may be analyzed by introducing two elements into the paradox. As shown in Fig. 4, let us assume time flows from left to right. The first element involves the sentence

"The following statement is true"      (2)

that is moving forward in time. The second element, with time going backwards, corresponds to the sentence

"The previous sentence was false."      (3)

In fact, cyclical time refers to a process where time evolves from $t_0$ to $t_1$, $t_1$ to $t_2$, and at certain $t_n$, it evolves back to $t_0$. Cyclical time is different from the usual linear time model where time only moves forward as from $t_0$ to $t_1$, $t_1$ to $t_2$, $t_2$ to $t_3$, etc (see (Reynolds, 1994; Song, 2017a) for a review). With the two elements in cyclical time, i.e., (2) and (3), the argument is equivalent to the single sentence case in (1) (Kaufmann 1987; Hofstadter, 2007). The cyclical process may also be applied to consciousness as in a subject and its object. The subject observes the object in a time forward manner; however, the object was the subject in a time backward manner. This may establish the same process of the subject observing itself as seen in consciousness (Song, 2017b; 2017c).

**Entanglement Network**

Quantum theory indeed yields some striking features that are not seen in its classical counterpart (Peres, 1997). For example, a photon may pass through two different slits at the same time, as shown in Young’s experiment (Carnal et al., 1991). In particular, one of the arguably most distinguishable features of quantum theory in comparison to classical theory is the phenomenon of entanglement (Einstein et al., 1935; Bell, 1964; Aspect et al., 1982). Entanglement, which is present in quantum theory, exhibits the property that an action at one place may influence a phenomenon at another place, regardless of how far they are apart (Tittel et al., 1998; Yin et al., 2012, Ma et al., 2012; Yin et al., 2017). This instantaneous influencing has puzzled some of the brightest minds of the last hundred years or so.

In recent years, philosophical discussion, which has often centered around entanglement, has shifted towards the study of practical applications of this spooky action at a distance, namely, computation (Deutsch, 1985; Ladd et al., 2010), cryptography (Ekert, 1991), and communication (Cleve et al., 1997). Entanglement swapping is a protocol that converts multiple short entanglements into a long-distance correlation (Zukowski et al., 1993; Bouland et al., 2003; Kaltenbaek et al., 2009). Because many useful applications of entanglement use a particular type of correlation, namely, maximal states, it is important to have methods of converting non-maximal correlations into maximal correlations.
In this section, a numerical method is used to examine the approximate class of states that may be useful in practical, i.e., non-ideal, situations of long-distance entanglement network. Given a non-maximal state with ordered Schmidt coefficients, i.e., non-negative real numbers,

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{\mu_i} |i\rangle |i\rangle$$  \hspace{1cm} (4)

where \( \sum_{i=1}^{n} \mu_i = 1 \) and the maximum average entanglement - after conversion into maximal states of equal coefficients - correspond to

$$E_{\text{max}} = \sum_{i=1}^{n} \left( \mu_i \right) / \log_2 l$$  \hspace{1cm} (5)

with \( \mu_0 = 0 \) (Jonathan et al., 1999; Hardy, 1999). It has been shown that when there are two 2-level non-maximal states, entanglement swapping yields an optimal outcome, namely, the weaker link between the two initial states (Bose et al., 1998; Shi et al., 2000) (also see (Song, 2018a; Song, 2018b)). Let us consider the following entangled states:

$$|\phi^{(k)}\rangle = \sum_{i=0}^{1} \sqrt{V_i^{(k)}} |i\rangle |i\rangle$$  \hspace{1cm} (6)

where \( k=1,2,3,4 \) and \( |\phi^{(0)}\rangle \) will be assumed to have the weakest entanglement. Unlike the two 2-level states, the entanglement swapping protocol on the four 2-level correlations does not yield the optimal value (Fig. 5). When Bell-type measurements, i.e., with the following basis

$$|\psi\rangle = \frac{1}{4} \left( |00\rangle + |01\rangle + |10\rangle + |11\rangle \right)$$
$$|\phi\rangle = \frac{1}{\sqrt{2}} \left( |01\rangle - |10\rangle \right)$$

![Figure 5. Comparison of the average maximal entanglement between the weakest link (straight line), that is, the optimal case, and the actual value (dashed line) when \( V_j^{(k)} = V_k^{(i)} \) for \( k, k' = 12,3,4 \) and \( i = 0.1 \).](image)

are made on qubits 2 and 3, 4 and 5, and 6 and 7, the new correlation between states 1 and 8 may be considered:

$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (7)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (8)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (9)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (10)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (11)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (12)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (13)
$$V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)} \geq V_1^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)}$$  \hspace{1cm} (14)

(Hardy et al., 2000) discussed that when the coefficients satisfy (7-14), then the outcome of maximum average entanglement corresponds to 2\( V_0^{(0)} \), that is, the weakest link in the chain of four 2-level states. This result is optimal since, if it were not true, it would violate the principle that the average entanglement cannot increase under local operation and classical communications, namely, the entanglement version of the second law. In Fig. 6, the maximum average entanglement is shown with respect to \( V_0^{(0)} \) and \( V_0^{(2)} \) in (i), \( V_0^{(0)} \) and \( V_1^{(2)} \) in (ii), \( V_0^{(3)} \) and \( V_0^{(4)} \) in (iii), and \( V_1^{(3)} \) and \( V_1^{(4)} \) in (iv).

It can be seen that while not all non-maximal states yield the weakest link, as shown in Fig. 5, a relatively wide class of coefficients does in fact approximate the optimal value. Although not the optimal outcome, we wish to examine a new class of non-maximal states that yield a value close to that of the weakest link with the following condition:

$$V_0^{(0)} V_1^{(2)} V_1^{(3)} V_1^{(4)} < V_0^{(0)} V_0^{(2)} V_0^{(3)} V_0^{(4)}$$  \hspace{1cm} (15)

The existence of a new class of coefficients that satisfy (7-13) and (15) yet yield an outcome that is close to the weakest link, \( E_{\text{max}} \) with margin <0.01 may be numerically examined. The new class of non-maximal states that approximate the optimal value is indicated with X’s in Fig. 7. In particular, (i) indicates the comparison of coefficients \( V_0^{(0)} \) and \( V_0^{(2)} \) between the conditions (7-14) and (7-13,15), while (ii),(iii),(v) show the comparison for \( V_0^{(3)} \) and \( V_0^{(4)} \) with \( V_0^{(3)} \) and \( V_0^{(4)} \), respectively. Table 1 shows
Figure 6. The average maximal entanglement with respect to coefficients of non-maximal states that satisfy (7-14). (i) shows entanglement as a function of $\nu^{(1)}_0$ and $\nu^{(2)}_0$, while (ii),(iii),(iv) indicate average maximal entanglement with $\nu^{(1)}_1$ and $\nu^{(2)}_1$, $\nu^{(3)}_0$ and $\nu^{(4)}_0$, $\nu^{(3)}_1$ and $\nu^{(4)}_1$, respectively.

Figure 7. A new class of coefficients (satisfying (7-13,15)) indicated by $X$'s is shown compared to the previous non-maximal states with (7-14) (shown with $O$'s). (i) shows the comparison of coefficients $\nu^{(1)}_0$ and $\nu^{(2)}_0$, while (ii),(iii),(iv) indicate the distribution of coefficients for $\nu^{(1)}_1$ and $\nu^{(2)}_1$, $\nu^{(3)}_0$ and $\nu^{(4)}_0$, $\nu^{(3)}_1$ and $\nu^{(4)}_1$, respectively.
some examples of numerical values of maximum average entanglement satisfying the conditions (7-13) and (15).

**Remarks**

In this paper, we have numerically examined a new non-trivial class of non-maximal states that yield an outcome close to the weakest link in a chain of four 2-level general correlations. The numerical approach may help to establish long-distance entanglement, which is important not only in verifying the nonlocal nature of quantum theory but also in practical issues such as establishing secret keys for distant parties.

**References**


