A Conjecture on the Origin of Gravity

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ABSTRACT
In this paper, we argue that a certain kind of discretness of space-time may imply the existence of gravity as a geometric property of space-time described by general relativity (GR). In particular, the dynamical relationship between matter and space-time holds true not only for macroscopic objects, but also for microscopic particles. This argument provides a possible basis for Einstein’s theory of gravity. Moreover, the Einstein gravitational constant in GR can be determined by the minimum intervals of discrete space-time. Lastly, we note that this new analysis may have some further implications for a complete theory of quantum gravity.

Key Words: spacetime, gravity, general relativity, minimum length, uncertainty principle

Introduction
The origin of gravity is still a controversial issue. The solution to this problem may have important implications for a complete theory of quantum gravity. On the other hand, independent of the nature of gravity, the existence of a minimum interval of space and time has been widely argued and acknowledged as a model-independent result of the proper combination of quantum mechanics (QM) and general relativity (GR) (see, Garay, 1995 for a review). Moreover, the arguments indicate that the minimum time interval and the minimum length are of the order of Planck time ($T_p$) and Planck length ($L_p$), respectively. The model-independence of the arguments strongly suggests that discreteness is probably a more basic feature of space-time, and it may have a firmer basis beyond QM and GR, which are still based on continuous space-time. Therefore, it may be appropriate to re-examine the relationship between the discreteness of space-time and the existing fundamental theories from the opposite direction. In this essay, we will analyze the possible implications of space-time discreteness for the nature of gravity. Since the formulations and meanings of discrete space-time are different in existing theories and arguments, we only resort to its minimum explanation here, namely that a space-time interval shorter than the minimum interval of space-time (i.e., the Planck scale) is physically meaningless. For example, the localization length of a point-like particle should be not shorter than the minimum length.

A conjecture on the origin of gravity
According to the Heisenberg uncertainty principle in QM we have

$$\Delta x \geq \frac{\hbar}{2\Delta p}$$

(1)

The momentum uncertainty of a particle, $\Delta p$, will result in the uncertainty of its position, $\Delta x$. This poses a limitation on the localization of a point-like particle.

For instance, according to the holographic principle (Bekenstein, 1981; 't Hooft, 1993; Susskind, 1995), the information inside any finite spatial region is finite.
particle in nonrelativistic domain. There is a more strict limitation on $\Delta x$ in relativistic QM. A particle at rest can only be localized within a distance of the order of its reduced Compton wavelength, namely

$$\Delta x \geq \frac{\hbar}{2m_0c} \quad (2)$$

where $m_0$ is the rest mass of the particle. The reason is that when the momentum uncertainty $\Delta p$ is greater than $2m_0c$, the energy uncertainty $\Delta E$ will exceed $2m_0c^2$, but this will create a particle anti-particle pair from the vacuum and make the position of the original particle invalid. It then follows that the minimum localization length of a particle at rest can only be the order of its reduced Compton wavelength as denoted by Eq. (2). Using Lorentz transformation, the minimum localization length of a particle moving with (average) velocity $v$ is

$$\Delta x \geq \frac{\hbar}{2mc} \quad \text{or} \quad \Delta x \geq \frac{\hbar c}{2E} \quad (3)$$

where $m = m_0/\sqrt{1-v^2/c^2}$ is the relativistic mass of the particle, and $E = mc^2$ is the total energy of the particle. This means that when the energy uncertainty of a particle is of the order of its (average) energy, it has the minimum localization length. Note that Eq. (3) also holds true for particles with zero rest mass such as photons.

The above limitation is valid in continuous space-time; when the energy and energy uncertainty of a particle becomes arbitrarily large, its localization length $\Delta x$ can still be arbitrarily small. However, the discreteness of space-time will demand that the localization of any particle should have a minimum value $L_u$, namely $\Delta x$ should satisfy the limiting relation

$$\Delta x \geq L_u \quad (4)$$

In order to satisfy this relation, the r.h.s of Eq. (3) should at least contain another term proportional to the (average) energy of the particle$^3$, namely in the first order of $E$ it should be

$$\Delta x \geq \frac{\hbar c}{2E} + \frac{L_u^2E}{2hc} \quad (5)$$

This new inequality, which may be regarded as one form of generalized uncertainty principle$^4$, can satisfy the limitation relation imposed by the discreteness of space. It means that the localization length of a point-like particle has a minimum value $L_u$.

How to understand the new term demanded by the discreteness of space then? Obviously it indicates that the (average) energy of a particle increases the size of its localized state, and the increase is proportional to the energy. Since there is only one particle here, the increase of its localization length cannot result from any interaction between it and other particles such as electromagnetic interaction. Besides, since the increased part, which is proportional to the energy, is very distinct from the original quantum part, which is inverse proportional to the energy, it is a reasonable assumption that the increased localization length does not come from the quantum motion of the particle either. As a result, it seems that there is only one possibility left, namely that the (average) energy of the particle influences the geometry of its background space-time and further results in the increase of its localization length. We can also give an estimate of the strength of this influence in terms of the new term $\frac{L_u^2E}{2hc}$. This term shows that the energy $E$ will lead to an length increase $\Delta L \approx \frac{L_u^2T_1E}{2h}$. In other words, the energy $E$ contained in a region with size $L$ will change the proper size of the region to

$$L' \approx L + \frac{L_u^2T_1E}{2h} \quad (6)$$

When the energy is equal to zero or there are no particles, the background space-time will not be changed. Since what changes space-time here is the average energy, this relation

$^3$ Note that if a constant term such as $L_u$ is added to the r.h.s of the inequality, it may also satisfy the limitation relation imposed by the discreteness of space. However, it seems difficult to explain the origin of the constant term. The reason is that the Heisenberg uncertainty principle in QM may have a deeper basis in flat space-time, and if energy does not influence the background space-time, then no additional term will appear in the inequality.

$^4$ The argument given here might be regarded as a reverse application of the generalized uncertainty principle (see, e.g., Garay, 1995; Adler and Santiago, 1999). But it should be stressed that the existing arguments for the principle are based on the analysis of measurement process, and their conclusion is that it is impossible to measure positions to better precision than a fundamental limit. On the other hand, in the above argument, the uncertainty of position is objective, and the discreteness of space-time means that the objective localization length of a particle has a minimum value, which is independent of measurement.
between energy and proper size increase change is irrelevant to the quantum fluctuations.

The above argument might provide a deeper basis for Einstein’s theory of gravity. The theory is usually argued with the help of classical mechanics and Newton’s law of gravity, along with the experimental evidence of the equivalence of gravitational and inertial mass. The drawback of such an argument is that it may obscure the physical meaning of GR. For example, it does not exclude the possibility that gravity is merely emergent at the macroscopic level. By comparison, the above argument based on QM and the discreteness of space-time implies that gravity is essentially a geometric property of space-time, which is determined by the energy-momentum contained in that space-time, not only at the macroscopic level but also at the microscopic level.

On the basis of the above argument, there are some common steps to “derive” the Einstein field equations, the concrete relation between the geometry of space-time and the energy-momentum contained in that space-time, in terms of Riemann geometry and tensor analysis as well as the conservation of energy and momentum etc. For example, it can be shown that there is only one symmetric second-rank tensor that will satisfy the following conditions: (1) Constructed solely from the space-time metric and its derivatives; (2) Linear in the second derivatives; (3) The four-divergence of which is vanishes identically (this condition guarantees the conservation of energy and momentum); (4) Is zero when space-time is flat (i.e. without cosmological constant). These conditions will yield a tensor capturing the dynamics of the curvature of space-time, which is proportional to the stress-energy density, and we can then obtain the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \]  

(7)

where \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the scalar curvature, \( g_{\mu\nu} \) is the metric tensor, \( \kappa \) is the Einstein gravitational constant, and \( T_{\mu\nu} \) the stress-energy tensor.

The left thing is to determine the value of the Einstein gravitational constant \( \kappa \). It is usually derived by requiring that the weak and slow limit of the Einstein field equations must recover Newton’s theory of gravitation. In this way, the gravitational constant is determined by experience as a matter of fact. If the above argument is valid, the Einstein gravitational constant can also be determined in theory in terms of the minimum interval of space-time. Consider an energy eigenstate limited in a region with radius \( R \). The space-time outside the region can be described by the Schwarzschild metric by solving the Einstein field equations:

\[
\begin{align*}
\frac{ds^2}{L^2} & = (1 - \frac{r_s}{r})^{-2} dr^2 + dr^2 + d\theta^2 + (1 - \frac{r_s}{r}) \sin^2 \theta d\phi^2 \\
& + r^2 \sin^2 \theta d\phi^2 - (1 - \frac{r_s}{r}) \sin^2 \theta d\phi^2 dt^2
\end{align*}
\]

(8)

where \( r_s = \frac{\kappa E}{4\pi} \) is the Schwarzschild radius. By assuming the metric tensor inside the region \( R \) is the same as that on the boundary, the proper size of the region is

\[
L \approx 2 \int_0^r (1 - \frac{r_s}{R})^{-1/2} dr \approx 2 R + \frac{\kappa E}{4\pi}
\]

(9)

Therefore, the change of the proper size of the region due to the contained energy \( E \) is

\[
\Delta L \approx \frac{\kappa E}{4\pi}
\]

(10)

By comparing with Eq. (6) we find

\[
\kappa = \frac{2\pi L U T_U}{h}
\]

in Einstein’s field equations. When assuming \( L_U = 2L_p \) and \( T_U = 2T_p \) (as suggested by the black hole entropy formula and the holographic principle), this gives the right value of the Einstein gravitational constant. It can be seen that this formula itself seems to also suggest that gravity may originate from the discreteness of space-time (together with the quantum principle that requires \( h \neq 0 \)). In continuous space-time where \( T_U = 0 \) and \( L_U = 0 \), we have \( \kappa = 0 \), and thus Einstein’s gravity does not exist.

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5 Another route to deriving the Einstein field equations is through an action principle using a gravitational Lagrangian.
Possible implications
The above argument, if true, implies that the dynamical relationship between matter and space-time, which is described by GR, also holds true for microscopic particles, and thus gravity is at least as fundamental as the quantum, and as a geometric property of space-time it is also as fundamental as space-time itself. This result, if valid, may have some further implications for a complete theory of quantum gravity. As we know, there exists a fundamental conflict between the superposition principle of QM and the general covariance principle of GR (Penrose, 1996; Verlinde, 2002), Jacobson’s gravitational thermodynamics (Jacobson, 1995), and Verlinde’s latest idea of gravity as an entropic force (Verlinde, 2011; Gao, 2011). On the other hand, if gravity is not emergent but fundamental as the above argument suggests, then quantum and gravity may be combined in a way different from the string theory. Now that the general covariance principle of GR is universally valid, the superposition principle of QM probably needs to be compromised when considering the fundamental conflict between them (Penrose, 1996; Christian, 2001; Gao, 2006).

Conclusions
In this paper, we have argued that a certain kind of discreteness of space-time may imply the existence of gravity as a geometric property of space-time described by GR. In particular, the dynamical relationship between matter and space-time holds true not only for macroscopic objects, but also for microscopic particles. This argument may provide a possible basis for Einstein’s theory of gravity. Moreover, the Einstein gravitational constant in GR can be determined by the minimum intervals of discrete space-time. Lastly, we note that this new analysis may have some further implications for a complete theory of quantum gravity.

Acknowledgments
I am very grateful to Sabine Hossenfelder for helpful discussions. I am also grateful to the participants of Foundations of Physics Seminar at the University of Sydney for discussions.

6 Certainly, if space-time itself is emergent, then gravity must be also emergent. But even so, gravity is still fundamental in the emergent space-time.

7 This conflict between QM and GR can be regarded as a different form of the problem of time in quantum gravity. It is widely acknowledged that QM and GR contain drastically different concepts of time (and spacetime), and thus they are incompatible in nature. In QM, time is an external (absolute) element (e.g. the role of absolute time is played by the external Minkowski spacetime in quantum field theory). In contrast, spacetime is a dynamical object in GR. This then leads to the well-known problem of time in quantum gravity (Isham and Butterfield, 1999; Kiefer, 2004; Rovelli, 2004).
References


