



# The Representation of Spacetime in the Medial Entorhinal Cortex Derives from an Underlying Model of Computation over the Complex Field

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## ABSTRACT

The cortex forms a model of computation over the complex field. Mutually inhibitory logical primitives normalize amplitudes under the 2-norm. A sufficient class of linear unitary operators exists to support a universal model of computation. Recent results show that the medial entorhinal cortex constructs a representation of spacetime from this underlying model. The lattice-like computational history is automatically generally covariant and background independent under transformations providing for derivation of an Einstein-Regge spacetime.

**Key Words:** Entorhinal Cortex, Spacetime, Computational Model

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## Introduction

This paper has three parts:

Part One describes the neurophysiological substrate to the cortical model of computation. Our cornerstone for this description is Koch (1999). The goal here is to give a general overview of the physical mechanisms much like an introductory computer logic course to understand the mathematical mappings.

Part Two then describes how the physical architecture of Part One forms computational primitives for a model of computation. We will pay careful attention to what is required for this model of computation as stated in broadly accepted postulates. The requirements for this model are quite minimal and clearly present in the cortex.

Part Three then examines the recent discovery of a spacetime representation in the medial entorhinal cortex (MEC). We present evidence that the cortical model we describe in Part Two is exactly the kind of model that is found to generate the MEC geometry. We then tie

this in to the work of Lloyd (2006) who showed this geometry obeys the Einstein-Regge equation constructed from the underlying information processing of the cortical model.

## Part 1: The Neural Architecture

The cortex unfolded reveals a computational sheet roughly a square meter wide and 2.5mm thick (Merker, 2007). Different sections may be encoded for specialized functions over various input-output domains (Douglas and Martin, 1991; Braitenbury and Schuz, 1998; Shipp, 2007). Although the terminology and details are still under debate there is compelling evidence that the neurons of the cortical sheet cooperate to some extent in functional units, groups or columns (Mountcastle, 1997).

An argument could be made that structure does not equal function (Horton and Adams, 2005; Rakic, 2008). Strictly speaking that is correct. However, we are to argue that structure enables function, starting from physical

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structure, investigating its performance, and asking what is the most general computational formalism it can enable. We will adopt the column or group of cells as our logical primitive, while leaving the door open to single neuron functional units, because the parallel group architecture (Koch, 1999) allows us to bring fault tolerance in under our model's umbrella. Let's then accept this ubiquitous column of cells structure as our candidate logical primitive and see where it leads.<sup>1</sup>

The cortical macrocolumn is a group of cells that form a functional set perpendicular to the cortical surface. Each macrocolumn is itself composed of an estimated 50 to 100 minicolumns with each minicolumn containing 80-120 neurons, except in the visual cortex where there can be twice that number (Buxhoeveden and Casanova, 2002b). This same basic organization is evident throughout the cortex with an estimated  $2 \times 10^8$  minicolumns or  $2 \times 10^6$  macrocolumns. Functionally, the neurons in a minicolumn encode a common feature or aspect of some information source with neighboring minicolumns encoding related aspects. These neurons cooperate to explore all possible consistent interpretations of different source input (Tsunoda, *et al.*, 2001; Douglas and Martin, 2004). Within this set of cooperating minicolumns is the complete (mutually exclusive and exhaustive) representation of some specific source.

What can be said about the computational properties of a group of cells such as the minicolumn? Given a population of cells with identical threshold distributions, if all receive the same input we can define an output firing rate for the unit (Knight, 1972). The threshold for firing, generating axonal spikes, is not fixed but is some probability distribution so that every time a spike is generated the threshold is set to a new value drawn from the probability distribution (Gabbiani and Koch, 1998). It has been shown that for such an ensemble of cells the relationship between the input and the output firing rate is a linear one (Gestri, 1971). Koch (1999) expounds on the remarkable maintenance of linearity for neuronal functions, saying, "despite all the nonlinear mechanisms known to exist in the nervous system, evolution conspires to construct neurons that act in a perfectly linear manner". So, for this

<sup>1</sup> The parallel recurrent group architecture (Koch, 1999) incorporates fault tolerance even for faulty gates by the threshold theorem (Nielsen and Chuang, 2000).

candidate logical primitive, a firing rate ensemble of cells, we have linearity.<sup>2</sup>

#### *Cortical neurons are linear operators*

The minicolumn's firing rate neurons can be modeled by a charging RC element and passing the resulting membrane potential through a stationary nonlinearity, implemented by an ideal op-amp (Arbib, 1995). Interacting firing cells in a network are coupled through synaptic connections that have a scalar weight,  $W_{ij}$ , for presynaptic neuron  $i$  and postsynaptic neuron  $j$  (Bernander, 1993; Bernander, *et al.*, 1994; Zador and Dobrunz, 1997). For such a network we can list these variable inputs in connectivity matrices (Koch, 1999).

Looking at the minicolumn's volume we see that it is taken up mostly by the dendritic arbors of the cells, as is the cortex itself (Segev, 2006). Dendrites make up 98% of the cell's surface area. The dendrites are *where* the cortex computes (Mel, *et al.*, 1998; London and Segev, 2004; Branco and Hausser, 2010). Koch (1999) emphasizes that they are nontraditional computational devices and calls the common perception of them as threshold sum units "moronic". The actual operation is based upon their morphology and the synaptic input (Koch, 1999; Douglas and Martin, 2004). The input synaptic weights to the neuron's dendrite introduce voltage attenuation between the synaptic site and the soma. In general, this is a complex number (Rall and Rinzel, 1973). For a general complex number input both the amplitude and phase parts are available for encoding information (Fries, *et al.*, 2007). We can think of the dendrites as spanning the neuron's oscillatory cycle with branching allowing for an operation of positive or negative amplitudes to be applied up and down the phases of the wave's period. Thus, we have a candidate logical primitive with input described by matrices of complex numbers.

#### *Cortical scalars are complex numbers.*

The minicolumn cells' membrane potential can be the result of many different logical or

<sup>2</sup> We mean here inclusive of rate and temporal coding. Koch (1999) prefers the term correlation population code. We emphasize how important linearity of the neuron is for this model. All the following discussion of the superposition of complex numbers in the model of computation depends on this fact. For extensive reflection on this "deep principle" in neural function see Koch (1999).



mathematical operations encoded in dendritic morphology. A substantial body of evidence supports not only addition and subtraction but multiplication, division, and correlation-like operations (Koch and Poggio, 1992). Also available are polynomial functions, so-called second order synapses, such that connectivity matrices specify the function by specific spatial dendritic relationships (Bruck, 1990; Bruck and Smolensky, 1992; Mel, 1992; Mel, 1993; Mel, 1994). Boolean functions are available such as the exclusive-or, XOR, which may be generalized as a controlled-not, CNOT (Fromherz and Gaede, 1993).

The neurons are also able to implement linear phase shift or delay operations on input. The physical mechanisms available include the morphoelectrotonic transform of the cell and moving the synaptic input site within the dendritic morphology. (Agmon-Snir and Segev, 1993; Zador *et al.*, 1995). A linear operation that involves the multiplication of two outputs, one temporally delayed is possible (Reichardt, 1961). Relative phases will make a *crucial* difference in what follows.

Let's review. We've taken the cortex sheet out and observed the ubiquitous macrocolumn structure. Each macrocolumn is comprised of minicolumn clusters of neurons. These minicolumn units represent a single feature or aspect of some source by taking complex number input matrices and performing a linear operation that is represented in the membrane potential or amplitude related to the output firing rate. This physical architecture is set up to implement mathematical functions, especially multiplication and phase shift operations, on matrices of complex numbers. Any computational model we propose can rest on these physically realized operations.

## Part 2: The Computational Model

The membrane potential is the result of varying amplitudes in dendritic branches (Douglas and Martin, 2004). This is the physical device upon which the operations listed above occur. These varying amplitudes include positive and negative changes to the resting potential due to two types of synaptic input: excitatory and inhibitory (Miles, 2000; Thompson, *et al.*, 2002; Contreras, 2004). The minicolumn unit of these excitatory pyramidal cells and inhibitory interneurons forms a naturally oscillating waveform that encodes the state of some system (Ward, 2003; Fries, *et al.*,

2007; Fries, 2009; Suppes, *et al.*, 2012). We can formalize an oscillatory system in terms of complex numbers, amplitude-phase, and the exponential-sinusoidal relationship. Thus, our candidate model logical primitive encodes a wave function. It is the phase part that gives it this "wavelike" character (Penrose, 2004).<sup>3</sup>

So each minicolumn encodes a particular state, feature, or aspect of a source stimulus. By "encodes" we mean through the function, such as those listed above, "wired" or "programmed" by the combination of dendritic morphology, synaptic weights, and their modulation. By "source" here we simply mean some "thing", some "system" that results in the attenuation appearing in the post-synaptic dendrite. We are considering the cortex in a "disconnected and program-agnostic" way. Any information processing system can be analyzed at several levels. This can seem almost overwhelmingly so for the cortex (Koch, 1999). We argue that the discrete minicolumn spaces, considered as a logical primitive, are the best candidates for a computationally meaningful level of representation *to the user*.<sup>4</sup>

Let's now try to understand what kind of model of computation may be formed by these minicolumns. We have that the macrocolumn contains within it the full set of values for any receptive field of input it operates over. Its minicolumns cooperate to explore all possible interpretations of receptive field input and select consistent interpretations (Douglas and Martin, 2004; Douglas and Martin, 2007; Tsunoda, *et al.*, 2001). What mathematical formalism seems appropriate? A probabilistic model is suggested because all outcomes are accounted for and encoded within the receptive field by the minicolumns. We have a complete measurement, meaning no further refinement of accounting for the matrix of complex number evidence (Peres, 1993). The completeness condition is satisfied by the physical architecture in that from the internal perspective of the model all evidence is accounted for. Subsequent evidence may change the probability distribution but that can't be known until it arrives. The minicolumns form an informationally complete set over their receptive

<sup>3</sup> The deep structure is a normed division algebra. These algebras, which generate the Lie Groups, specify the multidimensional symmetry foundations of physical theory (Wigner, 1960; Witten, 1998; Baez, 2002; Penrose, 2004; Stewart, 2008).

<sup>4</sup> We do not imply some kind of dualism. Universal models of computation can be self-referential.



fields. This suggests we can proceed to explore a probabilistic mathematical model on this cortical hardware as an information system for predictive encoding (Adams, et al., 2013; Cisek, 2007). However, we will see below that these probabilities do not represent ignorance in the way probabilities do in the classical theory. The complex amplitudes are a record of *intrinsically random* events (Bub, 2016).

The key in the physical realization to a probabilistic model will be the mutually inhibitory nature of coupled sets of minicolumns (Ben-Yishai, et al., 1995; Brunel, 2000). This inhibition allows for a divisive normalization among the coupled minicolumn's cells that is proportional to their excitatory amplitudes (Bloomfield, 1974; Torre and Poggio, 1978). Together with completeness this allows the network of cells to maintain a normalization condition, meaning that all discrete minicolumn amplitudes sum to 1, or a 100% representation of all evidence. But, if these are complex numbers then how can we formalize the normalization condition? Fortunately, we have a nice result from Gleason (1957) which tells us in general for probability spaces, as we are in the process of defining, that the only measure that works is that the sum of the squares of the magnitude of our complex numbers must equal to 1. That is, for  $n$  amplitudes that represent some system in our  $n$  minicolumns (encoded in the membrane potentials) the normalization condition maintains

$$\sum_i^n |a_i|^2 = 1$$

where  $a \in \mathbb{C}$  and  $|a| = a \cdot a^*$ . This should be understood as a statement about the oscillatory nature of the logical primitive for this model. If we use a complex basis as the measure then the waveform naturally imposes the normalization condition. The oscillatory waveform imposed by the interneurons connecting the mutually inhibitory columns maintains the 2-norm between the membrane potentials. The Pythagorean identity (unit circle or sphere) is a simple picture of this but Gleason (1957) gives the general measure for any such space.<sup>5</sup>

*Cortical normalization forms under the 2-norm.*

<sup>5</sup> For further reflection on this "2-norm" and the role of complex numbers see Aaronson (2004) and especially Penrose (2004).

What can we say formally about this "space" our model is working in? We have a set of mutually inhibitory minicolumns such that the set together represents the state of the system. We can think of these as an  $n$ -tuple list of complex number membrane potentials. Such a list is a complex number vector<sup>6</sup> in the vector space  $\mathbb{C}^n$ . This "state vector" of the system is naturally normalized to a unit vector by mutually inhibitory interneurons. We will represent such a vector by the symbols

$$|\Psi\rangle$$

Given our mathematical operators above we could also compute the following function for two vectors  $(y_1, \dots, y_n)$  and  $(z_1, \dots, z_n)$  expressed in the Dirac matrix (linear algebra) formalism (Nielsen and Chuang, 2000)

$$\langle Y|Z\rangle = \sum_i y_i^* z_i = [y_1^* \dots y_n^*] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = a \text{ where } a \in \mathbb{C}$$

We observe that the order of arguments for this function makes a difference,  $\langle Y|Z\rangle = \langle Z|Y\rangle^*$ . It is also linear in the second argument, by linearity in the neurons and operators as described above. And  $\langle Y|Y\rangle = 0$  if and only if  $|Y\rangle = 0$ . We say that such a function that takes as input two vectors from a vector space, produces a complex number, and satisfies the above requirements is an "inner product". We can think of this inner product as a measure of similarity between two vectors in our model (Busemeyer and Bruza, 2012). Orthogonal vectors are defined by the condition that  $\langle Y|Z\rangle = \langle Z|Y\rangle = 0$ . A norm or length for a vector  $\|Z\| = \sqrt{\langle Z|Z\rangle}$ . The inner product of a vector with itself equals the squared length  $\|Z\|^2 = \langle Z|Z\rangle = Z^\dagger \cdot Z$ , where  $Z^\dagger$  is the adjoint operation. This finite dimensional complex number vector space with inner product is called a Hilbert space (Horn and Johnson, 1985). Mel (1992, 1993) shows how this function may be implemented in neuronal dendrites with multiplication as a similarity detector.

This inner product is a Hebbian learning rule for synaptic potentiation (Hebb, 1949). In the neuroscience literature it is usually expressed as  $\Delta W_{ij} = V_i V_j$  where  $\Delta W_{ij}$  is the amplitude of the synaptic coupling between presynaptic neuron  $i$

<sup>6</sup> We usually picture vectors as arrows but they are very general entities (Maudlin, 2011).



and postsynaptic cell  $j$ ;  $V_i$  and  $V_j$  represent some “activity” of these neurons. There are variations on the rule but a key observation is that the presynaptic input precedes activity in the postsynaptic cell for the function. Thus, it is an asymmetric rule consistent with the inner product’s non-commutative nature. This “fire-together-wire-together” Hebbian learning is a measure then of the coherent coincidence between two vectors in the model. Two things can happen with this product: (1) the synapse transfers an amplitude and does not change its weight or (2) the synapse transfers an amplitude and changes its weight in the form of depression or enhancement. The first case of firing (or not) with no change in synaptic weight we identify below as unitary (linear) evolution. The second case of synaptic change we identify as a discontinuous jump in the state vector. Our understanding of synaptic plasticity is not complete but it is clear that it can occur on many different time scales as a form of memory in the computational model (Koch, 1999).

Let us summarize all we have learned so far about the model of computation for this device in the model’s first postulate (Nielsen and Chuang, 2000).

**Postulate 1:** The macrocolumn and minicolumn mutually inhibitory network over receptive fields in the cortex forms a complex-number valued realization of a vector space with inner product called a “state space”. Any system is described by an oscillatory “state vector” which is a normalized unit vector encoded in the membrane potentials of a distinct set of minicolumns.

Notice what this does *not* tell us: the dimension (number of minicolumns) of the state space or the state vector (the amplitudes of the membranes)! The cortex can only establish these through the network and memory mechanisms that set synaptic connections and weights. This “wiring up” process we could call programming or learning and it could result from a genetic (pre-installed) code or the environmental experiences of the user. We are describing the arena in which such programming may occur.

What is the simplest description possible of a system within this framework? Well, we could consider a single minicolumn state space. The minicolumn’s 80-120 neurons are bundled together to form a single primitive logical unit.

These cells could be jittered or noisy parallel representations such that the variable is a more fault-tolerant temporal average firing rate (Koch, 1999). The variable is the amplitude (potential) of the minicolumn’s membranes, encoded in the firing rate. To what degree subsets of a minicolumn’s neurons down to even a single neuron may satisfy this model is left open (Barlow, 1972; Quiroga, 2012). Because there is no mutual inhibition through the normalization condition with other minicolumns we may interpret the single stand-alone minicolumn’s behavior as a classical probabilistic bit. This means that its value is some real number, a random variable, in the interval of the unit’s operational frequency. Notice also due to the absence of excitatory and inhibitory coupling to other minicolumns that there can be no relative phase differences. Thus, the real part of the random variable captures all information in this single minicolumn state space. Global phase factors can be dropped. Of course, information can be encoded by varying the frequency, making use of interspike intervals, but that is still a “classical”, real number variable (Koch, 1999). This single “classical” bit may then oscillate as part of a code with other column vectors.

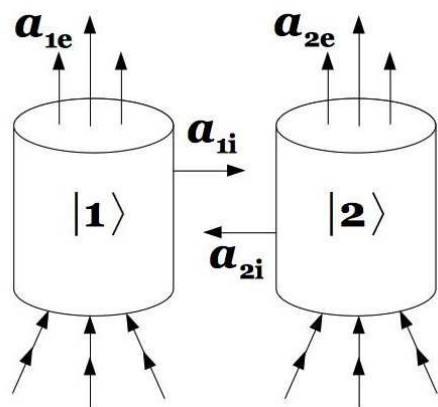
If a state space of one minicolumn functions as a classical probabilistic bit then what happens if two minicolumns are *coupled* to represent a state vector? This coupling is achieved by the two mutually inhibitory amplitudes under the 2-norm. This requires that the respective amplitudes interfere with each other in the sense that amplitude in one minicolumn must “push down” the other minicolumn’s amplitude through the action of their coupled inhibitory interneurons (Brunel, 2000). If the one-minicolumn state space is like one “bit” then this must be a “2-bit” state vector. But the two bits don’t behave like two classical bits. Classical bits do not interfere with each other’s amplitudes. Since these are “cortical bits” we will call this a “2-cobit”.<sup>7</sup> An arbitrary 2-cobit state vector then may be written as

$$|\Psi\rangle = a_1|1\rangle + a_2|2\rangle$$

<sup>7</sup> Busemeyer and Bruza (2012) reference Blutner and Hochnadel (2010) who coin the term “cubit”. Also, Moore and Mertens (2011) use the term “cobit” for “coherent bit” from Harrow (2004).



Where  $|1\rangle$  and  $|2\rangle$  are simply arbitrary labels for the two possible states of the system represented by the two minicolumns with respective membrane potentials  $a_1$  and  $a_2$ .



**Figure 1.** Mutually inhibitory minicolumns form a 2-cobit logical primitive with excitatory and inhibitory amplitudes. Note inhibitory connections are to basal dendrites. Recurrent connections from axonal projections back to basal dendrites are not shown.

We can then express a general  $n$ -dimensional state vector, with  $n$  minicolumns, as an  $n$ -cobit

$$|\Psi\rangle = a_1|1\rangle + a_2|2\rangle + \dots + a_n|n\rangle$$

Crucially, this is a digital representation as each cobit, due to the macrocolumn architecture, resides in separate degrees of freedom. The set  $\{a_1, a_2, \dots, a_n\}$  of complex numbers are the membrane potentials, reflected in the firing rates, of each of the  $n$  minicolumns which specify the state vector  $(a_1, a_2, \dots, a_n)$ . Together these represent the system in a superposition of the states for which the total amplitude is not in any one definite state (Gudder, 1988). Again, we are advancing under the notion that this is a probabilistic model and the amplitudes represent the probability of each state through the normalization rule. This is imposed by the natural oscillations of the inhibitory interneurons due to the relative membrane potentials of the pyramidal cells in the minicolumns. The probability amplitudes are then registered as complex numbers in the dendritic synaptic weights of other cells in the cortical circuit. Membrane potentials (amplitudes) being in a superposition of different states is computationally distinguishable from being in one or the other states.

For example, consider the state vector represented in

$$|\Psi\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$$

as a superposition in a 2-cobit. Here each minicolumn state has amplitude  $\frac{1}{\sqrt{2}}$  and probability  $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$  or 50%. In other words, the minicolumns have equal normalized amplitudes or membrane potentials. With equal potential they equally inhibit each other placing their firing equally at phase shifts of  $\frac{\pi}{4}$  from the peak (or minimum) amplitude. Notice also that the state vector

$$|\Psi\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$$

is the same in magnitude of the amplitudes but differs by a relative phase. In both cases each outcome has an equally likely probability of  $\frac{1}{2}$ . The relative phase difference is basis dependent. We can always construct an orthonormal basis set of vectors to represent this space (Strang, 1980). And we can always transform the basis such that the state is expressed only in positive amplitudes (Halmos, 1958). The phase difference tells the user the system is being “looked at” differently but still reflects a situation of two equally likely outcomes. Carandini and Heeger (1994) emphasized that to be a true superposition it must depend on the entire population of cells, summed over all possible aspects. Mel, et al., (1998) describe implementation of superposition in cells of the visual cortex. These phase-coupled states oscillate together as a coherent unit (Klausberger and Somogyi, 2008).

The user<sup>8</sup> must, of course, itself be a program represented in the cortex. Thus it must reside by this model in particular bases itself. We say more on this below. But for now, note that with this type of superposition the user is not *confused* about a state. It would be incorrect to say that the state is one, the other, both or neither (Albert, 1992). Neurophysiologically there is complex-valued amplitude in mutually inhibitory column’s membranes. The user is not *in phase*

<sup>8</sup> The literature refers to this as the “observer”. This entity has an *active* role in the computation. This “user” is then part of the cortex itself - part of a *physical* computer - a program running in *real hardware*. And we will point out that as such it can’t escape the laws of computation.



with any of the state's minicolumns when in superposition.

The above 1-cobit and 2-cobit examples are only the beginning. What is the upper limit? We know that extending the state space is necessary for describing an unknown state (Kraus, 1983; Royer, 1996). We also have that some neural operations may require multiple populations of cells with jittered thresholds (Suarez and Koch, 1989). Perhaps most significantly for the cortical architecture, ensemble average computing can result in the signal decreasing exponentially as the number of cobits increases (Nielsen and Chuang, 2000). These and other restrictions, such as overhead may mean the typical dimension of a cobit is much less than the number of minicolumns supposedly available in the macrocolumn and cortex as a whole. Thus, the extent to which the cortex utilizes non-classical bits in superposition in a set of coherently oscillating columns is an open question.

Furthermore, while multi-level spaces  $>2$  minicolumns do offer some computational advantage they do not produce a theoretical leap in computational complexity (Yao, 1993; Bernstein and Vazirani, 1997; Adleman, et al., 1997). Whether natural selection would find the extra resource overhead and diminishing return worth it in computational efficiency is unknown. We suspect that could be the case, however, for system domains, such as visual, where any edge in computational efficiency (even without a full jump to another complexity class) may make a survival difference. For examples of multi-cobit states in cognitive theory see Busemeyer and Bruza (2012). A high degree of network and synaptic plasticity does allow for rapid transformations of these spaces (Koch, 1999). We also leave open the question of to what extent neurons may group to form "ad hoc" minicolumns, or multi-unit arrays, independent of the minicolumn architecture. So long as such units work as mutually inhibitory logical primitives this model holds.

It may be worth pointing out the obvious that the computational unit of the state vector as described here is only a framework *associated with* some "system". Within the model we can only rather crudely say that the representation of the system in the cobit is all that the system can be said to be – except for a relative change of basis. At least it is the full measure of how a user of this cortical device may "speak of" the system. Before it has a representation within the cortical device

the "system" can not be said to have any dimension or a state vector. Once the user has a model  $n$ -cobit for the system developed then questions may be asked about the adequacy of its dimensional space or consistency with input – which may lead to revision of the model. Users are more likely to avoid philosophical inconsistencies by remembering that the "system" is a representation in the cortex. These are states of a cortical model associated with an external (or another internal cortical) system (Fuchs, 2010).

How does the state vector change? As described above the state vector is the complete complex "waveform" in all the state space's membrane potentials. These complex number amplitudes are the result of the positive or negative post-synaptic potentials encoded in the synaptic weights along neuronal dendrites. We are regarding the cortical column as a primitive computational unit that extends from its dendritic inputs to its axonal outputs and asking if this unit can be used to form a computational model. We are treating the synaptic weights like input to a computational device – it shows up somehow, somehow, from somewhere peripheral due to context. Of course, that input may come from other cortical units as well. Our focus is on what the cobit model can do with the input. There is somewhere on the order of  $10^4$  excitatory and  $10^3$  inhibitory inputs on a single cortical pyramidal cell's dendritic spines and shafts. How can we map this large set of complex numbers into a mathematical formalism? First, we will consider the function of the unit when synaptic weights do not change.

The mathematical operations we looked at above are linear operators between vector spaces and linear operators have matrix representations (Halmos, 1993). An operator is a prescription for how the cortex, using the set of mathematical and logical operations above, can make a new vector, new membrane potentials in the minicolumns, out of the old membrane potentials. For our computational model should there be restrictions on the operator? Certainly if we want to maintain the map to the system as it is defined then we will require an operator to preserve the inner product between vectors – don't break the Hilbert space! We would also like for the operator and its inverse to return us to the original state. The operators that satisfy these conditions are represented by unitary matrices. We can transform one set of coordinates into the other by using a unitary matrix that relates the two sets of



basis vectors. Unitary matrices are effectively then rotations that preserve the sum of squares, the 2-norm (Nielsen and Chuang, 2000). They are operators that do not change synaptic weight encodings. This is again the linearity condition (Penrose, 2004).

*Cortical functions include unitary operators.*

When a unitary matrix is encoded by synaptic weights over the minicolumn’s dendrites it can perform an operation on the state vector by encountering the “present state” of the vector. Massive recurrent connections feed back the present state of the unit (Douglas, et al., 1995; Douglas and Martin, 2007). This is the membrane potential of the postsynaptic neuron. Thus, the unitary matrix of input may operate on the present state to produce the next state’s membrane potential. The effect is to “rotate” the state vector within the space prescribed by the fixed synaptic weights. And this is our next postulate.

**Postulate 2:** The evolution of a system is described by a unitary transformation. That is, the state  $|\Psi\rangle$  of the system at time  $t_1$  is related to the state  $|\Psi'\rangle$  of the system at time  $t_2$  by a unitary operator  $U$  which depends only on times  $t_1$  and  $t_2$

$$|\Psi'\rangle = U|\Psi\rangle$$

where  $U$  is a matrix representation of the linear operator encoded in the dendritic tree morphology and synaptic coupling connection weights. Scalar products, recorded in synaptic weights, are preserved under unitary evolution.

As with the state space and state vector, the postulates do *not* tell the user a priori which unitary operator will describe successfully a system’s dynamics! As for the state space and state vector, the nature of a system’s unitary dynamics is what a cortical user means by, or what characterizes, the system. For 2-cobit systems it turns out that any unitary operator at all can be realized (Nielsen and Chuang, 2000).

For an example, consider the  $X$  matrix

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as the matrix representation of the linear operator that switches or flips the amplitudes on the two 2-cobit minicolumns. In matrix notation, for

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |2\rangle$$

This is the matrix formalism for the actual physical process in cortical minicolumns of a 2-cobit interchanging its state amplitudes. The cortex could do this by reversing the input assignments, put the input through inverter logic, or manipulate the external system – indeed this is what the operator formalism represents taking place. Per the operations above the necessary “dynamic VLSI” rules, so to speak, are available to assemble the operator hardware<sup>9</sup>. The matrix formalism is a higher-level way to represent hardware manipulation. Notice this  $X$  gate amounts to a 2-cobit **NOT** gate. Thus for a given state

$$|\Psi\rangle = a_1|1\rangle + a_2|2\rangle$$

the unitary operator  $X$ , the **NOT** gate, produces the state

$$X|\Psi\rangle = a_2|1\rangle + a_1|2\rangle.$$

Or, in the “computational basis” we have

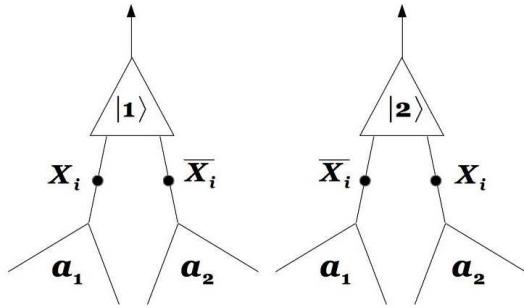
$$X \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}.$$

The **NOT** gate can be applied through controlled inhibition on two dendritic branches for each basis vector.

<sup>9</sup> Of course, whatever “VLSI layout” rules exist for the nervous system must ultimately be expressed in the DNA. All we are doing here is describing the computational model of a device “as found”. This would be like describing the model on a modern chip while ignoring whatever design and manufacturing processes may be required to produce it.



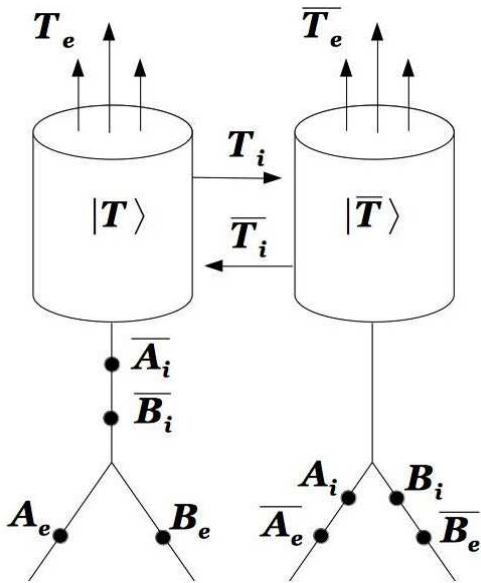




**Figure 2.** Realization of the NOT gate in the dendritic arbors of the minicolumn's pyramidal neurons.

In Figure 2 the  $\Delta$  represents the soma of the pyramidal cells. The  $a_1$  and  $a_2$  in the branches can be any function or logic (see Koch, 1999). Thus if  $X_i$  then left and right branches, respectively are inhibited and we have  $a_2|1\rangle + a_1|2\rangle$ . Remember that the cortical model is *not on a single clock phase*. Inhibition of the waveform can occur across phases – a complex number encoding under the 2-norm.

Furthermore, we can use the inverter X logic to create control-U gates such as the conjunctive control-U or Toffoli gate. Simple Toffoli control minicolumns are shown in Figure 3.



**Figure 3.** Programming for a Toffoli control gate by placement of excitatory and inhibitory synaptic amplitudes.

Thus, if (A and B) the T will switch the required amplitudes  $|ABX\rangle, |110\rangle \rightarrow |111\rangle$  (see Busemeyer and Bruza, 2012).

Importantly, note that the inhibitory effect is a *controllable* variable. Meaning, by clustering or separation (among other mechanisms) the

effect of inhibition, and thus of the amplitudes, may be variable and not simply all or none. Thus, amplitudes may continue to be superpositions of states. The cortex does not compute on a single phase clock but over a superposition of many phases, themselves embedded in multiple frequencies. Algorithms can thus produce structures on oscillatory interference. Furthermore, remember that these rotations in state space are possible with no changes to synaptic weights. Based on input the device can represent a superposition of possible states or toggle between them (Conte et al., 2009).

These linear operators operate on every component of the state *simultaneously* by axonal branching and through simultaneous excitatory and inhibitory connections to the dendritic trees of the minicolumns representing the state. The cortical model can thus combine parallelism with complex-number interference along any phase of the membrane potential oscillation. This parallel interference possibly allows this model to compute some things about a state by answering fewer questions than a classical model would need to as in evaluating a disjunction (Bub, 2016). An amplitude can be a linear combination of basis functions, each oscillating at a particular frequency, for the Fourier transform (Moore and Mertens, 2011), an operation tailor-made for efficient computation in the cortex (Koch, 1999). Other algorithms could use this property to efficiently detect periodicity or symmetry.

While the mutual inhibition of multiple joined minicolumns, as shown in the figures above, form non-classical logical primitives, the membrane potential of a single neuron results from classical diffusion in its dendrites, soma and axons. Let's look at the separate, classical matrix formalism for programming an operator in the dendrites as employed by Pearlmutter and Zador (1999). We have a membrane voltage matrix  $V$  representing the compartments of the dendritic "cable" for injected currents matrix  $J$ . The cable equation (Koch, 1999) evaluated at each point results in a system of coupled ordinary differential equations written compactly in matrix form as

$$\frac{dV}{dt} = BV - J_{inj}$$

where  $B = C^{-1} \left( \frac{d}{4R_i \Delta x^2} B' - G \right)$



for parameters of the membrane  $C$ ,  $R_i$  and  $G$ . Here  $B'$  is the second difference matrix. Eigenvalue analysis returns the solution as a vector sum of exponentials with constants determined by the projection of the initial conditions onto the eigenvectors

$$M^T c = V_0$$

This holds since the eigenvectors of  $B$  are orthogonal. If we denote by  $M$  the matrix that has the eigenvectors

$$V_i^{(z)} \propto \sin \frac{\pi z i}{n+1}$$

As its columns where  $V^{(z)}$  is the eigenvector corresponding to the  $z$ -th eigenvalue of any matrix  $B$ , the operator that governs the physical behavior of the dendritic cable equation, satisfying

$$BV^{(z)} = m_z V^{(z)}$$

The vector solution at time  $t$  can be written

$$V(t) = M^T V_0 M e^{m't}$$

where, if the eigenvalues of  $B$  are  $m_z$

$$m_z = \frac{4R_i \Delta x^2}{C_m d} \left( 2 - 2 \cos \frac{\pi z}{n+1} \right) + \frac{G_m}{C_m}$$

The eigenvectors can be understood as being arrived at by analogy to the discrete sine transform or the diffusion equation. The former as a linear operator matrix  $S$  with elements

$$S_{jz} = \sin \frac{\pi j z}{n+1}$$

The columns are precisely the eigenvectors of  $B$ . Thus,  $S$  diagonalizes  $B$

$$S^{-1} B S = \text{diag}(m'_1, m'_2, \dots, m'_n)$$

The eigen system acts as the link between dynamics and morphology and provides the means to compute response to stimuli from the physical description of the neuron. For arbitrary dendritic trees the graph can be converted into a Hines matrix (Hines, 1984). Thus, programming consists of specifying the operator matrix for each

minicolumn to position the state vector within the Hilbert space defined by the minicolumns. Note that this representation is for one “cable” or dendritic membrane. The superposition of a state represented in mutually inhibitory amplitudes “one level up” requires using the vector formalism described in the postulates above. The voltage diffusion *within* a dendrite is a classically described process. It is the mutual inhibition of membrane potential amplitudes *across* multiple coherent columns that maps to a complex-valued non-classical wave model. For a different approach to modeling the membrane voltage that is mathematically equivalent to this matrix formalism, see Cao and Abbott’s (1993) method based on the path integral (Feynman and Hibbs, 1965).

So again, let’s review. The primitive cortical bit of information is a complex number encoding. Mutually inhibitory column vectors form under the 2-norm. Unitary linear transformations can be realized in the dendritic morphology for state evolution. This combination of parallelism with interference of complex amplitudes, combining in or out of phase, describe a model of computation. *Only these three things are necessary for this model: complex amplitudes, the 2-norm and unitary operators* (Von Neumann, 1932; Bub, 2008; Pitowsky, 2008a; Chiribella, 2011; Moore and Mertens, 2011; Aaronson, 2013). There is clear evidence for all three in the cortex.

What about a case where an operator *changes* evolution of the vector in the state space? In the case where the operator reduced the space to a 1-cobit this results in no superposition of interfering amplitudes across outcomes but only a classical random variable for one outcome. One physical mechanism for this would be the disconnection or pruning of the synaptic connections that previously tied together mutually inhibitory columns. As described above, this would be indistinguishable from an architecture with no mutually inhibitory interference. Information is not present, or any longer potentially present, in a sum of many responses, but only as a real number magnitude in one. We will call such an operator, which for all practical purposes removes potential superposition and interference of amplitudes, a “measurement”.

Measurements discontinuously “jump” the state vector to a new state defined within the projected state space by the new synaptic weight.



In the language of linear algebra, a measurement changes the original state vector into a new state vector by projecting the original state into the subspace representing the observed event and dividing the projection by the length of the projection (Busemeyer and Bruza, 2012). Measurable properties of systems are called *observables* and are represented by linear operators on the vector space associated with those systems (Albert, 1992). A synaptic change represents the arrival of fundamentally new information. It represents the cortex (and possibly user) learning something about the world (or itself).

**Postulate 3:** Any operator on an  $n$ -cobit

$$|\Psi\rangle = a_1|1\rangle + a_2|2\rangle + \dots + a_n|n\rangle$$

that increases  $n$  is called an “expansion” of the state space while an operator that reduces  $n$  is called a “reduction”. The case of *reducing* the state space is called a “measurement”. For the collection of measurement operators  $\{M_n\}$  if the state vector before measurement is  $\Psi$  then the probability that result  $n$  occurs is

$$p(n) = \langle \Psi | M_n^\dagger M_n | \Psi \rangle = |a_n|^2$$

the membrane potential of the minicolumn. The state after a synaptic change is then renormalized

$$\frac{M_n|\Psi\rangle}{\sqrt{\langle \Psi | M_n^\dagger M_n | \Psi \rangle}}$$

The squared modulus rule<sup>10</sup> follows from normalization of total amplitude in the minicolumns. The notation  $\langle X|A|Y\rangle$  is the inner product on a linear operator notation (Nielsen and Chuang, 2000). In his description of this model, Von Neumann (1955) did not specify the mechanism for measurement. To be clear, we are here specifying measurement as a physical change in computational hardware.

Measurement operators are a different class of physical mechanism in the cortical hardware compared to the evolution operators. While the evolution operator does not change the

synaptic weights over the dendritic trees the measurement operator is a discontinuous physical change in the synaptic weight. The event represented in the measurement operator leads to a membrane potential, or firing rate, in only one minicolumn for the state of that part of the system. The other previously coherent minicolumns (dimensions) are now physically disconnected and can no longer interfere through the normalization condition. There is no longer a superposition from mutual inhibition. And a measurement operator may only be said to have been applied upon such a collapse of the state space. This ability of the cortex to dynamically project the state space by expansion or reduction distinguishes it from a classical computer, as Koch says, “like a machine that changes its instruction set as a function of its input” (Koch, 1999). Measurement does not reveal a pre-existing value. It is an intrinsically random change to the state vector evolution that the user can not pre-compute.

The  $X$  operator above (one of the so-called Pauli operators) is not only unitary,  $X^{-1} = X^*$ , but also self-adjoint,  $X = X^*$ , so that it also gives an example of a measurement operator that represents an observable. That gives us an example of what Dirac (1958) called the eigenvalue-eigenstate link. This says that a system has a property if and only if its state is represented by an eigenvector of the operator. The eigenvectors form a basis for the operator. The minicolumns are exactly such basis vector representations - the eigenstates of the property. This operator-eigenvector relation does not depend on the basis chosen. The same relation will obtain in any basis whatever (Albert, 1992).

A related but distinct computation is decoherence. Decoherence is best thought of as phase damping or “phase kicking” (Zurek, 1991; Nielsen and Chuang, 2000). When one minicolumn (basis dimension) controls all the mutually inhibitory interneurons then it will fire at the minimum of the inhibitory oscillation. All other coherently connected minicolumn amplitudes are “kicked down the waveform” to coincide with the maximum of the inhibitory cycle. Decoherence then has occurred by the physical mechanism of inhibition or shunting inhibition. The complex number-ness has been squeezed or filtered out so we have a classical model variable. But the  $n$ -cobit superposition is still “wired up” with the synaptic weights in place. They are only inhibited “as if” changed. In the

<sup>10</sup> This is called the Born rule. We encountered it above as Gleason’s measure.



matrix formulation the off-diagonal amplitudes are suppressed through inhibition. The “decohered” minicolumn dimensions are silenced by inhibition, except perhaps for background firing (Koch, 1999). And, although a distinct computation, decoherence can have the appearance of collapse of the wavefunction, the disconnection of minicolumns. Thus, distinct physical mechanisms among the cortical primitives explain why collapse implies decoherence but decoherence does not imply collapse (Pessoa, 1998; Conte, 2010).

Following a measurement the state vector wavefunction resumes evolving in its state space based on the linear, deterministic function encoded by the synaptic weights and dendritic morphology. Although measurement (decoherence) is the third postulate the first two postulates depend physiologically on this one. Measurement “shapes” the state space for state vectors and operators. Von Neumann (1955) referred to these two fundamental, yet incompatible, procedures as Process 1 (linear evolution) and Process 2 (nonlinear measurement or “collapse”). We emphasize that the wavefunction encoded by discrete amplitudes of multiple columns of neurons is *not* like a classical wave. A good appreciation of the difference can be had by comparing the classical voltage diffusion described by the Pearlmutter and Zador (1999) procedure above to the picture of a vector rotating around in the complex plane. The “waviness” is in the argument of this wavefunction as its phase. This produces a twisting round and round, corkscrew or wave packet picture (Penrose, 2004). The interference *within* a dendritic cable is classical wave diffusion but the interference *between* mutually inhibitory columns which encode a state vector is not classical. We must think of the entire set of columns as a single state vector, the logical primitive of the model (Figure 1). It is holistic in the sense that change in amplitude anywhere affects by mutual inhibition amplitude everywhere which is not true of classical waves. In the cortical wavefunction constructive interference and destructive interference are apparently instantaneous to the user. Until a measurement changes the state space the state vector deterministically rotates within the probability distribution encoded by the state space of dendritic morphology and synaptic weights.

The commutator of two matrix operators  $A$  and  $B$  is denoted  $[A, B] = AB - BA$ . If  $[A, B] = 0$  then  $A$  and  $B$  share at least one set of eigenvectors which form a basis. Operator matrices of incompatible observables can't share any complete basis. Thus, the commutators of incompatible observable matrices are nonzero. If knowing the value of  $A$  gives the user no information whatever about the outcome of measurement  $B$  then  $A$  and  $B$  are maximally incompatible (Bub, 2016). If the user being in an eigenstate of  $A$  simultaneously puts operator  $B$  in a totally undecidable position (an equal superposition as shown above) then the user and  $B$  are maximally incompatible or phase mismatched.

This model allows for correlations to appear that are structurally different from a classical model (Bub, 2016). Observables in different bases will, in general, not commute, meaning  $[A, B] \neq 0$ . If observables do commute,  $[A, B] = 0$ , then their properties form a Boolean simplex lattice (Pitowsky, 2008b). But, if the observables do not commute then it *is not possible* to join them together in one Boolean context. A Boolean logic can't get a grip on non-commuting bases (Kochen and Specker, 1967). Gleason (1957) described noncommuting observables as “intertwined”. So, what the cortex can have is a collection of distinct Boolean algebras (bases or perspectives) representing contexts intertwined in a particular way. Physiologically noncommuting minicolumns can not synchronously (inphase) oscillate together to represent a coherent state when one is being measured, meaning the user is in that basis. The user could, of course, switch contexts to the other basis. But then there is no information about the first. It would not be computationally possible to measure them simultaneously. However, critically important, amplitudes in *both* bases continue to exist in the membrane potentials and possibly produce interference regardless of the user's basis. If the environment for the cortex consists of correlations that will not fall within a single Boolean context then it stands to reason that this “information gathering and using” device (Gell-Mann and Hartel, 1990) would adapt to allow for alternative perspectives. This would prevent a user from using any particular Boolean logic (like causality) to explain extra-Boolean correlations. The model only relates correlations.

Very important interpretational note: just as for general states and operators, any



measurement operators and outcomes for a system are *not* encoded in the model a priori. The user must discover, find, or learn these (Abbott and Blum, 1996; Aaronson, 2007). However, it is certainly possible, as Gell-Mann and Hartel (1990) suggested, that a natural selection process could genetically encode cortical areas downstream from sensory or peripheral input domains preconfigured to adaptive bases. From the user's perspective there are two kinds of processes: (1) unitary evolution and (2) discontinuous jumps to a new state, whereupon evolution begins again. All of these depend upon the unique contextual, contingent nature of the interaction of the cortex with peripheral input, or neural input, and the nature of the biophysics – including the inherently probabilistic operation of the synaptic mechanisms and the condition of health. A measurement operator depends upon the particular basis representation of the system. The cortical encoding of a system is not, of course, the “thing” itself but, as stated in Postulate 1, it is *associated* with a system by the cortex. By such association to a state space the cortex defines that of which the user may speak (Wittgenstein, 1921). Furthermore, it seems reasonable that natural selection would adapt a computational device to approach the computational model of its environment if its purpose is to predict that environment (Wigner, 1960).

It is our opinion that the intrinsic randomness of the measurement (decoherence) process, which includes the cortex actively “choosing” a basis, is demanded by the nature of computation itself. A universal computer is a programmable device that can carry out any algorithm it is given (Moore and Mertens, 2011). Universal models of computation possess a “featurebug” of being self-referential. The cortex faces the problem of choosing a good basis for some measurement relative to the basis or bases with which it represents the “user”, a self-referential program running on its own hardware. Decoherence (measurement) should result in a preferred basis for the user, making certain facts determinate from that perspective (Barrett, 1999). But here we get into a self-referential “strange loop” because a program is calling *itself*. In Chapter 8 of Albert (1992) it is shown how this self-referencing nature of the model leads to an undecidable, paradoxical situation. This undecidability of a preferred basis is a “feature” because it allows for cognitive adaptation - the art of finding a good basis. But that comes with the

price, the “bug”, two logically contradictory processes, evolution and collapse, that cannot be reconciled. It had to be that way given that universal computing entails such paradoxes! The user may describe the situation as “randomness” or “free choice” in the basis (Conway and Kochen, 2006; Conway and Kochen, 2009; Bub, 2016). The computational consequence is the cortex (user) or natural selection is left with just trying solutions. If the user (cortex) learns something from its perspective it “collapses” to the distribution conditioned on that knowledge, just like for classical distributions (Moore and Mertens, 2011; Busemeyer and Bruza, 2012).

Everett (1973) appealed to Von Neumann's principle of psycho-physical parallelism to “consider the problem of observation itself.” He suggested we model observers as “mechanical devices ... equipped with memory registers ... These (memory) configurations can be thought of as punches in a paper tape, impressions on a magnetic reel, configurations of a relay switching circuit, or even configurations of brain cells ... It will appear to the observer ... that observation caused it to ‘jump’ into an eigenstate in a random fashion ... Relative to (the observer) the (observed) system states are the corresponding eigenfunctions ... Thus, with each observation (or interaction), the observer state ‘branches’ into a number of different states ... the ‘trajectory’ of the memory configuration is ... a branching tree ... In any memory device the branching does not continue indefinitely, but must stop at a point limited by the capacity of the memory” (Everett, 1957a, 1957b). It is perhaps unclear exactly how Everett thought of the “branches” (Barrett, 1999). The most conservative interpretation would be to stay within his computing model and think of these as branches in computational trees. He added that “all branches are actual” and the “total lack of effect of one branch on another (in measurement) implies no observer will be aware of any ‘splitting’ process” (Everett, 1957b). This suggests some sort of inhibition of branches in the model for the measurement (or decoherence) process. The amplitudes interfere with themselves, along the lines of a random walk propagating down a tree (Moore and Mertens, 2011), to mix for search or sampling tasks. Everett *never* used the phrase “many worlds”, although he let it pass without comment in DeWitt and Graham's anthology (Barrett, 1999). But, if we stick with Everett's computational model metaphor then our best

explanation for the paradox of Von Neumann's two processes may reside in the nature of computation itself: measurement entails self-reference.

Now, we know that the cortex probably never activates only one column for a complex cognitive state (Pellionisz and Llinás, 1985; Tsunoda, et al., 2001; Douglas and Martin, 2004; Douglas and Martin, 2007; Lundqvist, et al., 2010). Rather, cortical states are a coherent temporal binding of state spaces (Lundqvist, et al., 2006; Klausberger and Somogyi, 2008; Fries, 2009; Vitiello, 2009; Suppes, *et al.*, 2012). Following Nielsen and Chuang (2000), if we begin with the superposition principle that minicolumns form a linear combination of the basis vectors for the system state then we could form linear combinations of the minicolumn amplitudes of multi-cobit states. Suppose  $V$  and  $W$  are  $m$ -cobit and  $n$ -cobit dimensional spaces. Then we can form an  $mn$ -cobit dimensional vector space such that its elements are linear combinations of the amplitudes that describe the individual state vectors. Since this involves straightforward complex number matrix multiplication the combined system will have an inner product and inherit the other familiar notions of a Hilbert space we've described including unitary dynamics and measurement. This operation is known as a tensor product.

**Postulate 4:** The state of a composite system composed of distinct state vectors, each encoded in a set of minicolumns, is the tensor product of the state spaces of the component systems. For  $n$  systems we have the joint state

$$|\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \dots \otimes |\Psi_n\rangle$$

The joint state vector is encoded in a unique set of coherently oscillating minicolumns.

Note that for superposition and interference effects to remain that the amplitudes of the state vector of the joint state must be computed in discrete minicolumns. For example, suppose we form the tensor product of two 2-cobits. Each 2-cobit can be said to have two possible states,  $|1\rangle$  and  $|2\rangle$ . A two 2-cobit joint system would then have four possible states, listed side-by-side as 11, 12, 21, and 22. A pair of 2-cobits could then exist in a superposition of

these four states utilizing four minicolumn amplitudes, a 4-cobit

$$|\Psi\rangle = a_{11}|11\rangle + a_{12}|12\rangle + a_{21}|21\rangle + a_{22}|22\rangle$$

Probabilities and the normalization condition are preserved as before

$$\sum_x |a_x|^2 = 1$$

We can find probability and outcome states also as before. For example, the first 2-cobit gives an outcome of 1 with probability

$$p(1x) = |a_{11}|^2 + |a_{12}|^2$$

leaving the post-measurement state

$$|\Psi'\rangle = \frac{a_{11}|11\rangle + a_{12}|12\rangle}{\sqrt{|a_{11}|^2 + |a_{12}|^2}}$$

Observe as well that the joint state could undergo a synaptic or pruning operation. The four state joint state vector  $|\Psi\rangle$  above could undergo a state reduction operation from a four minicolumn to a two minicolumn superposition such as, for reduced state  $|\Psi_R\rangle$

$$|\Psi_R\rangle = \frac{|11\rangle + |22\rangle}{\sqrt{2}}$$

This type of joint state reduction results in what is called "entanglement". These are possible correlations that lie outside of any classical correlation, outside any Boolean simplex, on a complex manifold (Pitowsky, 2008b; Bub, 2016). There is no equivalent notion to this for 1-cobit states or classical random variables (Jozsa, 1997). In the cortical model it is realized as two mutually inhibitory minicolumns, a 2-cobit. It is something like a more efficient encoding of a state space when there is no amplitude evidence for other terms. A measurement on either of the two minicolumns thus encodes the "entangled" outcomes for the two encoded 2-cobits. And, as before, measurement "destroys" coherence and interference. Inhibitory decoherence or synaptic depotentiation can put it on hold over a wide range of time scales.

Because entanglement is a more efficient encoding it may offer some computational and



communication advantages (Nielsen and Chuang, 2000; Nelson and McEvoy, 2007; Kempe, *et al.*, 2008; Gisin, 2014). This comes at the cost of being able to say anything about the probabilities of the entangled cobits separately. Recall that the joint state vector is now a superposition of the amplitudes that represent the combined states in mutually inhibitory, interfering minicolumns. Until a measurement occurs the total amplitude can only be said to be “spread out” in the membrane voltages of the minicolumns. These states also oscillate coherently as those described above. The finite physical dimensions of the cortex certainly impose some upper bound on the possible dimensions in a joint tensored state.

Above we gave the example of how inhibitory and excitatory synaptic weights applied to a minicolumn’s dendrites can change values to effect the  $X$  or **NOT** operator. This unitary operator can be thought of as a computational “gate”, a manipulation that converts information from one form to another. Similarly to how we defined the **NOT** gate we can use the available set of mathematical and logical operators to construct other single 2-cobit gates such as

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S \equiv \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

which are known as the Hadamard, phase, and  $\frac{\pi}{8}$  gates, respectively (DiVincenzo, 1998).

The proto-typical multi-cobit gate is the controlled-not or **CNOT** gate, a generalization of **XOR**. This gate can be constructed from two 2-cobits, known as the control and target. The matrix representation for the 2-cobit case is

$$U_{CN} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be shown that these four gates above form a universal set of gates for 2-cobit computation (Boykin, *et al.*, 1999). Recall that the set of measurement operators reduce the state vector to a 1-cobit with no coherent oscillations or interference, that is no complex phase information binding, with other minicolumns for that system state. If we construct a circuit of many gates that includes such an operator we can

always move that measurement to the end of the circuit (Nielsen and Chuang, 2000). Otherwise the measurement operator destroys potentially useful coherent information encoded in the phases.

Importantly, a universal gate set allows for the formation of a programmable device that can carry out any algorithm. Unfortunately, although we have a universal set for approximating any unitary operator, actually constructing such gates is generically a hard problem. In fact, most unitary gates can only be implemented very inefficiently (Nielsen and Chuang, 2000). This may be evident in the difficulties experienced by the user in finding, recognizing, or learning an operator that will transform a state.<sup>11</sup> And any universal computer, by its self-referential nature, leads to undecidability and unprovability (Davis, 1965; Moore and Mertens, 2011).

Have we described a universal model of computation? We have assemblies of discrete sets of physical components, the cortical minicolumns, which can describe computational procedures. These operations, such as the  $X$  matrix represents, may be considered as equivalent to gates in a circuit model (DiVincenzo, 1998). A cortical circuit would then operate on some number,  $m$ , of cortical bits. A cortical bit, the cobit, is a set of mutually inhibitory minicolumns representing all possible states of the system. Where  $n$  is the largest dimension of  $m$  cobits, the state space is in general an  $n^m$ - dimensional Hilbert space. Product states of the form  $|x_1, \dots, x_m\rangle$  are known as computational basis states. The vector  $|X\rangle$  denotes a computational basis state, where  $X$  is the number whose  $n$ -nary representation is  $x_1, \dots, x_m$ . Any computational basis state can be prepared in at most  $m$  steps by setting the membrane potential via the synaptic weights. Such gates can be applied to any subset of  $n$ -cobits and a universal family of such gates can be implemented (DiVincenzo, 1995; Boykin, 1999).

Information processing is always subject to physical constraints. The number of minicolumns in the cortex is clearly finite. However, the bounds on the possible space dimensions, product states, and overhead costs are unknown. The cortical component’s time constants are slow compared to CMOS technologies. And the cortical architecture is set

<sup>11</sup> There are some interesting results, however, that a population of interacting, multi-prover, or refereed computational models of this type can achieve some additional gains in computational efficiency (Watrous 1999).



up for massively parallel algorithms typically executed in one or two time constants rather than serial processing (Koch 1999). The general  $\geq 2$ -cobit model is cooperative and dynamic across phases (bases) rather than (single phase) clocklike. Without a cooperative temporal framework the various synaptic potentials would decohere and lose their meaning (Klausberger and Somogyi, 2008).

There are results for the computational complexity of this model (Yao, 1993; Bernstein and Vazirani, 1997). And while some speedup results by increasing the cobit dimension beyond a 2-cobit this does not make a difference from the theoretical point of view for the complexity class of multi-cobit spaces (Nielsen and Chuang, 2000). It is presently unknown if extending the cobit dimension from 1 to 2 constitutes a difference in complexity classes – meaning 2-cobit or higher spaces could efficiently solve some problems that 1-cobit spaces could not. There is some evidence for that (Simon, 1997; Grover, 1997; Watrous, 1999). The reduction to the efficiency of entanglement may not be possible for exclusively 1-cobit models. To represent the state of an  $n$ -cobit system on a restricted 1-cobit only model would require  $4^n - 1$  encoded 1-cobit real numbers (Nielsen and Chuang, 2000). The potential improvement in efficiency for the  $\geq 2$ -cobit seems to only exist for certain classes of problems – generally, search and period finding. Phase encoding is the lever which enabled progress for many of these algorithms (Nielsen and Chuang, 2000). Efficiency gains for intermediate complexity instances in a class of problems does not necessarily grant efficiency for the so-called “complete” or hardest instances. The question then is if the extension to a more efficient model for problems over the dimensions available to the cortex provides enough of a difference for natural selection to work on. Given the potential importance to users of search and period finding in natural processes critical to survival we would suspect such a difference exists even if the speedup were only polynomial.

David DiVincenzo (2000) listed the following requirements for a physical realization of this universal model of computing:

1) Scalable physically to increase the number of bits

We have dynamic cortical network scaling (Pellionisz and Llinás, 1985; Koch, 1999; Douglas and Martin, 2004; Lundqvist *et al.*, 2010) and

evolution is believed to work on the cortex by adding minicolumns (Mountcastle, 1997; Buxhoeveden and Casanova, 2002a).

2) Bits can be initialized to arbitrary values  
Synaptic weights and dendritic tree morphology provide a means for setting the membrane potential by the user (For details of the physical mechanisms, see Koch, 1999.)

3) Gates faster than “decoherence” time  
Cortical circuits can maintain coherent oscillations of states over time scales relevant to the user. As we explain above, “decoherence” is a certain type of operator in this model. But for further evidence see (Tsunoda *et al.*, 2001; Ward, 2003; Behera *et al.*, 2005; Aerts 2007; Fries *et al.*, 2007; Klausberger and Somogyi, 2008; Aerts, 2009; Conte *et al.*, 2009; DeBarros and Suppes, 2009; Fries, 2009; Khrennikov, 2009; Sun and Dan, 2009; Cheon and Takahashi, 2010; Aerts *et al.*, 2011; Busemeyer and Bruza, 2012).

4) Universal Gate Set  
Arithmetic, phase, and logical operations available (Koch, 1999) can be used to construct linear operators that form a universal gate set (Boykin *et al.*, 1999).

5) Bits can be read easily  
The set of measurement operators produce a minicolumn population amplitude as firing rate code available to user or other cortical areas through axonal and interneuron projectors (Rieke *et al.*, 1997; Koch, 1999; Molnar, *et al.*, 2008).

There are mathematically equivalent alternative statements of this model of computation (Feynman *et al.*, 1965; Conte, 2010; Arkani-Hamed and Trnka, 2014 for examples) and of the four postulates given above (Nielsen and Chuang, 2000). For example, the density operator (matrix) picture could be an especially helpful device for description of subsystems. In the neural architecture individual Hebbian synaptic weight changes would be the partial trace (Nielsen and Chuang, 2000) of observable quantities for subsystems of a composite system. But we leave open here if or when one of those may provide a better picture of cortical states and operations.

We have described the cortical model of universal computation. Mutually inhibitory, coherently oscillating columns of cells realize the logical primitive. The variable present in these columns is a vector of complex number valued

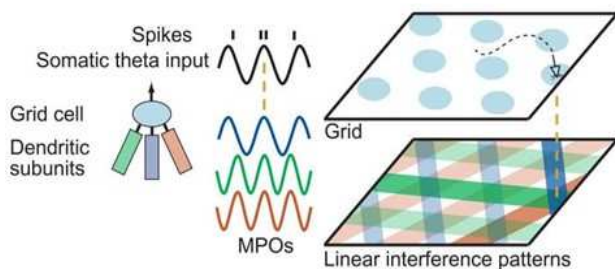




amplitudes. This complex number encoding results in phase interference that is fundamentally non-classical. Logical and mathematical operators can be programmed in synaptic weights over the various morphological arborizations of the cells. These operators, which can be represented as matrices of complex numbers, form a universal gate set that can support universal computation in the cortex.

### Part 3: The Cortical Spacetime Computation

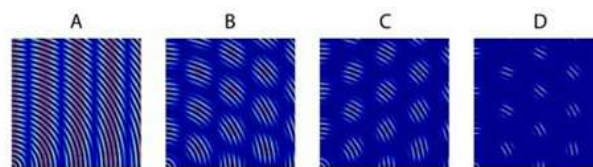
Phase encoding is confirmed in the cortex (Agmon-Snir and Segev, 1993; Suppes, *et al.*, 2012) and, specifically in the hippocampal and entorhinal areas (O'Keefe and Burgess, 2005). Inputs that vary across phase can result in synaptic potentiation or depotentiation. Rate and phase codes mean that the spike train can code for two variables at the same time. These complex-valued variables produce interference patterns or wavelets repeating at regular intervals based on the intrinsic oscillations of the cells (Figure 4).



**Figure 4.** Schematic of the interference model of grid cell firing. Left: Grid cell receiving input from dendritic subunits. Middle: Spikes, somatic theta input and dendritic membrane potential oscillations (MPOs). The locations of grid cell firing occur wherever the wave envelopes all peak together – in this case a rat traverses along the shown direction. (Taken from Burgess, Barry and O'Keefe, 2007.)

The oscillatory interference model produces multiple linear interference patterns in MEC grid cell firing (Burgess, Barry and O'Keefe, 2007). These grids are a lattice-like tessellation not in the physical placement of the grid cells but in their computational behavior (Moser and Moser, 2008). A representation of spacetime emerges from the computational interference of coherently oscillating cells. Any combination of the interference patterns produces the hexagonal grid (Figure 5). It is fundamentally the action of phase in the model that encodes and thus creates the representation of space, displacement or distance (Burgess, 2008), consequently the metric

(Barry, *et al* 2007), and the notion of time (Nunez and Cooperrider, 2013). The Hebbian inner product described above is asymmetric in time thus a stored path memory in the lattice will possess time asymmetry – a sequence encoded via phase (Montague and Sejnowski, 1994).



**Figure 5.** Directional interference tessellation patterns showing the product of two (B), three (C) or six patterns oriented at multiples of  $60^\circ$  to each other. (Taken from Burgess, Barry and O'Keefe, 2007).

The model may combine with attractor dynamics to generate standing waves. (Hasselmo and Brandon, 2012). The periodic lattice of grid fields minimizes the number of neurons required to encode space for a given resolution (Wei, Prentice and Balasubramanian, 2013). Phase noise can be controlled with recurrent inhibition ensuring fault tolerant coherence and stability (Bush and Burgess, 2014). Consistent with our model, grid cells in the same column (also referred to as modules or clusters) share functional connectivity (Dunn, Morreaunet and Roudi, 2014).

Lloyd (2006) describes how each computation of this type of graph corresponds to a discrete classical spacetime with a metric that obeys Einstein's equation. The geometry derives from the computation itself. Distance emerges from phases in a way analogous to signals passing between a set of GPS satellites. The metric is constructed by timing (phase angle) at local points (Giovanneti, Lloyd, Maccone, 2004.) Such a computation is automatically generally covariant and background independent: it is automatically invariant under general coordinate transformations. The geometry induced by this computation, with general covariance, implies Einstein's equation (Regge, 1961; Friedberg and Lee, 1984; Miller, 1997.)

The spacetime structure in the MEC emerges from cobits interacting and acquiring a phase. This phase binding creates the lattice pattern of vertices, a computational history, that corresponds to the phase. The computational history then corresponds to a discrete classical

spacetime.<sup>12</sup> The metric then determines the notions of space and duration for the user. These may, of course, give relativistic answers for lengths or simultaneity for different users depending on the state of motion (Wald, 1984).

The computational graph forms a discrete lattice picture of space. The derivation of the Einstein-Regge equation begins with requiring that actions be stationary under variations of the metric (Lloyd, 2006). Triangulation, which is optimal (Wei, 2013), constructs a simplicial lattice with edge lengths defined by the causal structure and local action of the computation.

From Lloyd (2006), begin with the gravitational action

$$I_G = \frac{1}{8\pi G} \sum_h \epsilon_h A_h$$

where  $\epsilon_h$  is the angle of the hinge  $h$  and  $A_h$  its area in the lattice. The variation in the action is

$$\delta I_G = \frac{1}{8\pi G} \sum_h \epsilon_h \delta A_h$$

The variation in  $\epsilon_h$  cancels out,  $l_{ph}$  is the  $p$ 'th edge of hinge  $h$  and  $\phi_{ph}$  the angle in the hinge opposite, giving by explicitly inserting the metric

$$\delta I_G = \frac{1}{16\pi G} \sum_h \sum_{p(h)} l_{ph} \frac{\delta l_{ph}}{\delta g_{ab}} \delta g_{ab} \cot \phi_{ph}$$

Taking the Lagrangian to be a function only of the computational "matter" together with the metric

$$L_l = -g_{ab} \hat{T}^{ab} / 2 - U_l$$

Where  $\hat{T}^{ab}$  is the energy-momentum tensor (Hawking and Ellis, 1973) and  $U_l$  is the energy density at the  $l$ 'th logic gate. Combining the last two equations gives

$$- \sum_{h \in N(l)} \sum_{p(h)} l_{ph} \frac{\delta l_{ph}}{\delta g_{ab}(l)} \cot \phi_{ph}$$

<sup>12</sup> A fascinating experiment would be to observe the MEC lattice under external manipulation of the user's visual environment, perhaps in a virtual reality setting, with non-uniform arrangements of the spacetime null cones. Can the MEC adapt to a non-uniform null cone arrangement of spacetime that would be radically unlike our local approximation of uniform null cones? What would the MEC lattice look like (distorted?) in such a virtual reality?

$$= 8\pi G \left( \hat{T}^{ab} - U_l g^{ab}(l) \right) \Delta V_l$$

This reduces to the Einstein-Regge Equation

$$- \sum_{h \in N(l)} \epsilon_h \frac{\delta A_h}{\delta g_{ab}} = 4\pi G T^{ab} \Delta V_l$$

The key point: The cortical representation of spacetime in a lattice is built on top of an underlying model of computation. Penrose (2004) describes how not only the Regge proposal but many others for deriving spacetimes are based upon the fundamental underlying role of complex numbers. It is the complex number encoding in the logical primitive that gives this model its distinct computational nature.

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