STUDIES ON OPERATOR BETWEEN SPACES OF GENERALIZED MEANS

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ABSTRACT

In this paper, we apply our previous results to Hausdorff measures of non-compactness of some matrix operators between normalized means spaces and to obtain some identities or approximations for operator criterion. Further, we deduce the necessary and sufficient conditions for such operators to be compact.

Keywords : *BK space, Generalized means, Classical sequence spaces etc.*

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Introduction :

Given the fact that the matrix between BK spaces is continuous, it seems quite natural to define a compact operator between BK spaces to find the necessary and sufficient terms for entries of infinite the matrix. This can to be achieved, in many cases by applying the Hausdorff measure of non-compactness.

Let *r, t, r′, t′* ∈ U and s, s′ ∈ *Uo*. For an arbitrary *BK* space *X;* we write $\bar{X}$ = *X*(*r, s, t*) for short. If *Y* is any of the classical sequence spaces, we put $\hat{Y}$ = *Y* (*r′, s′, t′*), that is, $\hat{Y}$ is the matrix domain of triangle $\bar{A}$ (*r′, s′, t′*) in *Y*. Further, for an infinite matrix *A* = (*ank*), we define the matrix $B=(b_{nk})$ by :

$$b_{nk}=\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,a_{mk;}\quad (n,k\in\square),$$

that is, $B=\bar{A}(r',s',t')A$.

we have

$$B_n=\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,A_m=\left(\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,a_{mk}\right)_{k=0}^{\infty};\quad (n\in\square).$$

Moreover, we write $\tilde{A}=(\tilde{a}_{nk})$ and $\tilde{B}=(\tilde{b}_{nk})$ for associated matrices of *A* and *B*, respectively. Then it can easily to be seen that $\tilde{B}=\bar{A}(r',s',t')\tilde{A}$

i.e.

$$\tilde{b}_{nk}=\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,\tilde{a}_{mk;}\quad (n,k\in\square),$$

Here,

$$\tilde{B}_n=\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,\tilde{A}_m=\left(\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\,t'_m\,\tilde{a}_{mk}\right)_{k=0}^{\infty};\quad (n\in\square).$$

Now by using the above notation that $A\in(\bar{X},\hat{Y})$ if and only if $B\in(\bar{X},Y)$. Further, if $A\in(\bar{X},\hat{Y})$ then $\|L_A\|=\|L_B\|$ and the evaluation of LA_X can be reduced to that of LB_X.

Theorem 1.1 : *Let X be a BK space with AK or X* = *ℓ∞. If* A *is in any of the classes* $(\bar{X},\hat{l}\infty)$, $(\bar{X},\hat{c})$ *or* $(\bar{X},\hat{c}_0)$.

So that,

$$\|L_A\| = \|A\|_{(\bar{X},\hat{l}_\infty)} = \sup_n \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right\|_X^* < \infty.$$

Theorem 1.2 : *Let X be a BK space with AK or* $X = l_\infty$ *If* $A \in \left(\bar{X},\ \hat{l}_1\right).$

Then

$$\|A\|_{(\bar{X},\hat{l}_1)} \le \|L_A\| \le 4.\|A\|_{(\bar{X},\hat{l}_1)},$$

where

$$\|A\|_{(\bar{X},\hat{l}_1)} = \sup_{N\in F} \left\| \sum_{n\in N} \left(\frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right) \right\|_X^* < \infty.$$

Theorem 1.3 : *Let* $1 \le p < \infty.$ *If* $A \in \left(\bar{l}_1,\ \hat{l}_p\right)$

Then

$$\|L_A\| = \|A\|_{(\bar{l}_1,\hat{l}_p)} = \sup_k \left(\sum_{n=0}^{\infty} \left| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{a}_{mk} \right|^p \right)^{1/p} < \infty.$$

Furthermore, we define the sequence $\tilde{\beta} = \left(\tilde{\beta}_k\right)_{k=0}^{\infty}$

$$\tilde{\beta}_k = \lim_{n\to\infty} \left(\frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{a}_{mk} \right); \qquad (k \in \square)$$

provided these limits exist for all $k \in \square$ which case is whenever $A \in \left(\hat{X},\ \hat{c}\right)$, where *X* is a *BK* space with *AK* or *X*=ℓ∞.

Theorem 1.4 : *Let X* to *be a BK space with AK or X = ℓ∞. Then we have*

(a) *If* $A \in \left(\hat{X},\ \hat{c}_0\right)$

Then

$$\|L_A\|_X = \limsup_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_{mk} \right\|_X^*$$

and

$$\lim_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right\|_X^* = 0.$$

(b) If $A \in \left(\hat{X}, \hat{c}\right)$

Then

$$\frac{1}{2}.\limsup_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m - \tilde{\beta} \right\|_X^* \le \|L_A\|_X \le \limsup_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m - \tilde{\beta} \right\|_X^*$$

LA is compact

if and only if

$$\lim_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m - \tilde{\beta} \right\|_X^* = 0.$$

(c) If $A \in \left(\bar{X}, \hat{l}_\infty\right),$

Then

$$0 \le \|L_A\|_X \le \limsup_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right\|_X^*$$

LA is compact

if

$$\lim_{n\to\infty} \left\| \frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right\|_X^* = 0.$$

Theorem 1.5 *If either* $A \in \left(\bar{l}_\infty, \hat{c}_0\right)$ *or* $A \in \left(\bar{l}_\infty,\ \hat{c}\right)$, *then the operator LA is compact.*

Theorem 1.6 *Let X be a BK space with AK. If* $A \in \left(\bar{X}, \hat{l}_1\right)$

Then

$$\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{X},\hat{l}_1)} \le \|L_A\|_X \le 4.\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{X},\hat{l}_1)}$$

LA is compact if and only if

$$\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{X},\hat{l}_1)} = 0,$$

where

$$\|A\|^{(r)}_{(\bar{X},\hat{l}_1)} = \sup_{N\in F_r} \left\| \sum_{n\in N} \left(\frac{1}{r'_n} \sum_{m=0}^{n} s'_{n-m} t'_m \tilde{A}_m \right) \right\|_X^*; \qquad (r \in \square).$$

Theorem 1.7 *Let* $1 \le p < \infty.$ *If* $A \in (\bar{l}_1,\ \hat{l}_p)$

Then

$$\|L_A\|_X = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}\left|\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\, t'_m\, \tilde{a}_{mk}\right|^p\right)^{1/p}\right)$$

and

$$\lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}\left|\frac{1}{r'_n}\sum_{m=0}^{n} s'_{n-m}\, t'_m\, \tilde{a}_{mk}\right|^p\right)^{1/p}\right)=0.$$

Now, it is important to note that Theorems 1.1-1.7 have many consequences with some particular sequences r, s, t, r', s' and t'.

For example, let us consider the special cases of Theorems 1.1, 1.4 and 1.5 when $r', s', t' = e$. Then, we have $\hat{l}\infty = bs$, $\hat{c} = cs$ and $\hat{c}_0 = cs_0$. Further, by taking $r, s, t = e$, that $\hat{a}_{nk} = a_{nk} - a_{n,k+1}$ for all n, $k \in \mathbb{N}$. Moreover, if we put $X = \ell\infty$ or $X = c_0$ in these results, then we have $\overline{l}\infty = \text{bs}$ and $\overline{c}_0 = cs_0$. Thus, we deduce the following corollaries:

Corollary 1.8 *Let X denote any of the spaces* bs *or* cs0. *If* A *is in any of the classes* (X, bs), (X, cs) *or* (X, cs_0),

then

$$\|L_A\| = \|L_A\|_{(X,bs)} = \sup_n\left(\sum_{k=0}\left|\sum_{m=0}^{n}\left(a_{mk} - a_{m,k+1}\right)\right|\right) < \infty$$

Corollary 1.9 *If either* $A \in (bs, \text{cs0})$ *or* $\text{A} \in (bs, cs)$, *then LA is compact.*

Corollary 1.10 *We have the following:*

(a) If $\text{A} \in (cs_0, cs_0)$, *then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=0}^{n}\left(a_{mk} - a_{m,k+1}\right)\right|\right)$$

and *LA is compact if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=0}^{n}\left(a_{mk} - a_{m,k+1}\right)\right|\right)=0.$$

(b) If $\text{A} \in (cs_0, cs)$,

Then

$$\frac{1}{2}.\limsup_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=n}^{\infty}\left(a_{mk} - a_{m,k+1}\right)\right|\right) \le \|L_A\|_X \le \limsup_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=n}^{\infty}\left(a_{mk} - a_{m,k+1}\right)\right|\right)$$

and

$$\lim_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=n}^{\infty}\left(a_{mk} - a_{m,k+1}\right)\right|\right)=0.$$

(c) *If either* $A \in (cs_0, bs)$ *or* $A \in (bs, bs)$,

Then

$$0 \le \|L_A\|_X \le \limsup_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=0}^{n}\left(a_{mk} - a_{m,k+1}\right)\right|\right)$$

and

$$\lim_{n\to\infty}\left(\sum_{k=0}\left|\sum_{m=0}^{n}\left(a_{mk} - a_{m,k+1}\right)\right|\right)=0.$$

Similarly, if we put $r' = t' = e$ and $s' = (1, -1, 0, 0 \ldots)$ in Theorems 1.2, 1.3, 1.6 and 1.7, then we have $\hat{l}_1 = bv$ and $\hat{l}_p = bv^p$ for $1 < p < \infty$.

Moreover, let $r = t = e$ and $s = (1, -1, 0, 0 \ldots)$. Then we obtain from theorem that $\tilde{a}_{nk} = \sum_{j=k}^{\infty} a_{nj}$ for all $n, k \in \mathbb{N}$. Therefore, we get the conclusion of Corollary 1.8 from Theorems 1.2 and 1.6 in the special case $X = \ell p$ $(1 < p < \infty)$. Also, following corollaries are immediate by Theorems 1.3 and 1.7, respectively.

Corollary 1.11 *Let* $1 \le p < \infty$. *If* $A \in (bv, bvp)$.

$$\|L_A\| = \|L_A\|_{(bv,bv^p)} = \sup_k\left(\sum_{n=0}\left|\sum_{j=k}^{\infty}\left(a_{nj} - a_{n-1,j}\right)\right|^p\right)^{1/p} < \infty.$$

Corollary 1.12 *Let* $1 \le p < \infty$. *If* $A \in (bv, bv^p)$,

Then

$$\|L_A\|_X = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}\left|\sum_{j=k}^{\infty}\left(a_{nj} - a_{n-1,j}\right)\right|^p\right)^{1/p}\right)$$

LA is compact if and only If

$$\lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}\left|\sum_{j=k}^{\infty}\left(a_{nj} - a_{n-1,j}\right)\right|^p\right)\right)=0.$$

Finally, let us mention here that our results can be obtained from Theorems 1.1-1.7 the special case $r = \alpha$, $r' = \beta$, $s = s' = e^{(0)}$ and $t = t' = e$, where $\alpha = (\alpha k)$ and $\beta = (\beta k)$ are positive real sequences. For instance, we conclude this paper by the following corollaries:

Corollary 1.13 *If either* $A \in \left(s_\alpha, s_\beta^0\right)$ *or* $A \in \left(s_\alpha, s_\beta^{(c)}\right)$ *then LA is compact.*

Corollary 1.14 *Let* $1 \le p < \infty$.

If $L \in B\left(l_\alpha^1, l_\beta^1\right)$ *is given by matrix* $A \in B\left(l_\alpha^1, l_\beta^p\right)$

$$\Rightarrow \quad \|L_A\|_X = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}\left|\frac{\alpha_k\alpha_{nk}}{\beta_n}\right|^p\right)^{1/p}\right)$$

$$\Rightarrow \quad L \in C\left(l_\alpha^1, l_\beta^p\right) \text{ if and only if}$$

$$\lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}\left|\frac{\alpha_k\ \alpha_{nk}}{\beta_n}\right|^p\right)\right) = 0.$$

Same technique used in corollaries 1.8-1.14, one can derive some new results concerning matrix operators between the Nörlund, Cesáro or Euler sequence spaces.

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