Analytical Study Of Domination And Split Domination Number In Star Graphs, Bi Star Graphs, Petersen Graphs, And Middle Graphs

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Abstract

The paper is related on the analytical study of domination and split domination number in Star Graphs, Bi Star Graphs, Petersen Graphs, and Middle Graphs. Kulli and Janakirm [4] was introduced the split domination in graphs. In this paper, first we have present split domination of arithmetic graphs in the form of review study with some illustrations. Then, we have calculated split domination number of star graphs, Bi Star graphs, Petersen graphs, and middle graphs. We have developed several important and interesting results regarding the Star Graphs, Bi Star Graphs, Petersen Graphs, and Middle Graphs.

Keyword: Graph, Domination, Split Domination, Star Graphs, Bi Star Graphs, Petersen Graphs, Middle Graphs

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1. Introduction

Graph theory helps us to obtain the solution of many real-life problems in engineering and technology, physical science, social science and in biological science. In last century, theory of graphs has an unexpected growth in research field due to its extensive applications to inter-disciplinary domains. The notation of dominating sets in graph theory is one of the important concepts which are very simple but it has lots of important applications. Dominating sets also play a significant role in the improvement of exact algorithms and parameterized complexity.

The concept of domination in graphs is accepted from the complete analysis of the Topics on domination in graph that the theory of domination is one of the important concepts which are very simple but it has lots of important applications [6].In 1977 Cockayne and Hedetniemi gives some absorbing results related to dominating sets in graphs. According to them the domination number of any graph$G$ is denoted by $\gamma(G)$ and from this paper; Cockayne and Hedetniemi give rise to attentiveness study of domination in graphs. Till now more than 2000 research papers have been published related to domination in graphs and research paper is still growing. After that lots of graph theorists have finished extremely absorbing work in domination in graphs. In 1997 Kulli & Janakiram generalized the concept of domination in graphs as split domination in graphs. According to them the split domination number of any graph$G$ is denoted by $\gamma_s(G)$. They obtained the absorbing results related to split domination and find the split domination number $\gamma_s(G)$of cycle graph, wheel graph and complete bipartite graph. Vijayasaradhi & Vangipuram [10] discussed about the domination parameters in arithmetic Graphs and they gives a method for constructing arithmetic graph for a particular domination parameter. The applications of this type of construction are also important in real life problems so, that it is very important to

construct a graph corresponding to a known parameter. Vasumathi and Vangipuram [9], gives the method for construction of a graph corresponding to a known domination number. Similarly, Suryanarayana and Sreenivansan [6], gives the method for construction of a graph corresponding to a known split domination number of arithmetic graphs.

Inspired by the theory of domination and split domination we will look over some properties of split domination number of Star graph, Bi Star graph, Petersen Graph, and middle graph.

1.1 Domination and Split Domination:

(i) Dominating Set:
A set of vertices $D \in V$ is called a dominating set of graph$G$ if each vertex in the complement of D in V is in the neighborhood of some vertex in set D.

(ii) Dominating Number:
The dominating number $\gamma(G)$of any G is the least number of members in a dominating set of graph G.

(iii) Split Dominating Set:
Split dominating set of graph $G$is a dominating set D of graph Gif obtained sub graph $V - D$ of graph G is not connected.

(iv) Split Domination Number:
The possible least number of members in a split dominating set is called split dominating number $\gamma_S(G)$ of graph G.

2. Some Results on Split Domination of Standard Graphs:

Kulli and Janakiram have obtained the absorbing results related to split domination and find the split domination number $\gamma_S(G)$ of cycle graph, wheel graph and complete bipartite graph. Here we have few results which are given by Kulli and Janakiram [4].

(i) $\gamma_S(G) \leq \alpha_0(G)$, where $\alpha_0(G)$ is the cardinality of minimum vertex cover set.

Proof: Here $\alpha_0(G)$ is a vertex covering number and which is the cardinality of minimum vertex cover set. In a graph G, a maximum set of independent vertices is called maximum independent set hence in any graph G is $S_1$is the maximum independent vertex set then $V - S_1$ is a vertex covering set. So, each vertex in set S_1 has any neighboring vertex in $V - S_1$ hence, $V - S_1$ is not connected set. Thus, $V - S_1$ is split dominating set of graph G and $V - S_1$ is also a vertex cover. So, $\gamma_S(G) \leq \alpha_0(G)$.

(ii) $\gamma(G) \leq \gamma_S(G)$

Proof:$\gamma(G)$ represents domination number of graph G and domination number is the least cardinality of dominating set. $\gamma_S(G)$ is split domination number of graph G and split dominating set is also a dominating set with condition that $V - D$ is disconnected. Hence, $\gamma(G) \leq \gamma_S(G)$.

(iii) $k(G) \leq \gamma_S(G)$

Proof: here $k(G)$ represents the connectivity of graph G and connectivity of graph G is the least number of vertices such that if we remove these vertices from the graph G then obtained graph is disconnected. So, if we remove $k(G)$ vertices from any graph then obtained graph is disconnected graph. Hence, $k(G) \leq \gamma_S(G)$.

(iv) $\gamma_S(G) \leq \frac{p.\Delta(G)}{\Delta(G)+1}$, such that $\Delta(G)$represents the maximum degree of graph G.

Proof: Assume that the set D is a split dominating set of graph G. A split dominating set of G is of minimum cardinality if corresponding to each vertex v of split dominating set$D$, we have a vertex $u \in V - D$ such that v is in the neighborhood of u. Hence, $V - D$ is dominating set of graph G.
So, $\gamma(G) \leq |V - D|$
$\gamma(G) \leq p - \gamma_S(G)$ And we know that $\gamma(G) \geq \frac{p}{\Delta(G)+1}$

$$\gamma_S(G) \leq p - \gamma(G)$$
$$\gamma_S(G) \leq p - \frac{p}{\Delta(G) + 1}$$
$$\gamma_S(G) \leq \frac{p.\Delta(G)}{\Delta(G) + 1}.$$

(v) $\gamma_S(G) \leq \delta(G)$; If Diam (G) = 2, such that $\delta(G)$ is the least degree of graph G.

Proof: let v be a vertex of minimum degree in graph G. now, we have given that Diam (G) = 2 so, there exist a vertex u not adjacent to v in graph G. it means u must

be adjacent to some vertex in $N(v)$. hence, $N(v)$ is a split dominating set of graph G. thus, $\gamma_S(G) \leq \delta(G)$.

(vi) $\gamma_S(C_p) = \left\lceil \frac{p}{3} \right\rceil$, where $\lceil x \rceil$ is the lowest positive integer less than x and C_p is a cycle graph having vertices p such that $p \geq 4$.

Proof: The domination number of cycle graph C_p is $\left\lceil \frac{p}{3} \right\rceil$. It means $\left\lceil \frac{p}{3} \right\rceil$ vertices in graph C_p are related with all the vertices of C_p and one vertex in graph C_p is related with three vertices including it. Hence, if we remove $\left\lceil \frac{p}{3} \right\rceil$ vertices from graph C_p then obtained graph $V - D$ must be disconnected graph. $V - D$ is the split dominating set of cycle graph. Hence, $\gamma_S(C_p) = \left\lceil \frac{p}{3} \right\rceil$.

(vii) $\gamma_S(W_n) = 3$, where W_n is a wheel graph with $n \geq 5$ vertices.

Proof: The domination number of wheel graph W_n is equal to one. Now, let $D = \{V_1, V_4, V_n\}$ is the dominating set of wheel graph W_n. If we remove a set of vertices which contains central vertex and two non-adjacent outer vertices from graph W_n then obtained graph $V - D$ is disconnected. So, $V - D$ is the split dominating set of wheel graph. Thus, $\gamma_S(W_n) = 3$.

3. Review study of Split Domination of Arithmetic Graphs with illustrations

In this section we present split domination number of arithmetic graphs in the form of review study with some illustrations.

The Arithmetic graph is denoted by V_m and it is a graph having its vertices as divisors of positive integer m except 1. The geometrical form of m is $m = {p_1}^{a_1}.{p_2}^{a_2} \ldots . {p_r}^{a_r}$, such that $p_i's$ are different primes and $a_i's \geq 1$ and two vertices a, b in arithmetic graph which are not of the same parity have an edge if greatest common factor(GCD) of vertices a, b is a prime number p_i for $1 \leq i \leq r$ and the parity of two vertices are same if they are the powers of the same prime number.

For the motive of construction of a graph with some known parameters, Suryanarayana and Sreenivansan [6], gives the method for construction of a graph corresponding to a known split domination number of arithmetic graphs V_m with its vertices as divisors of m except 1 and they obtained the upper bound as $r + 1$ for the split domination number of arithmetic graphV_m, Where m is any positive integer and the geometrical form of m is $m = {p_1}^{a_1}.{p_2}^{a_2} \ldots . {p_r}^{a_r}$, such that$p_i's$ are different primes and $a_i's \geq 1$.

3.1 Theorem [6]:

If the geometrical form of m is $m = {p_1}^{a_1}.{p_2}^{a_2} \ldots . {p_r}^{a_r}$, such that$p_i's$ are different primes and $a_i's \geq 1$ corresponding to a arithmetic graph V_m then the upper bond of split domination number is $\gamma_S(V_m) \leq r + 1$.

Proof: Let the geometrical form of m is $m = {p_1}^{a_1}.{p_2}^{a_2} \ldots . {p_r}^{a_r}$, such that$p_i's$ are different primes and $a_i's \geq 1$ corresponding to an arithmetic graph V_m then the set $D = \{p_1, p_2, \ldots . p_r, p_1p_2 \ldots . p_r\}$ is a dominating set of arithmetic graphs V_m . The set $D = \{p_1, p_2, \ldots . p_r, p_1p_2 \ldots . p_r\}$ is also a split dominating set of arithmetic graphs V_m since $V - D$ is not connected. Now, if this split dominating set $D = \{p_1, p_2, \ldots . p_r, p_1p_2 \ldots . p_r\}$ is minimal split dominating set then the split domination number of arithmetic graphs V_m is $r + 1$ and if this split dominating set $D = \{p_1, p_2, \ldots . p_r, p_1p_2 \ldots . p_r\}$ is not minimal split dominating set then $r + 1$ is upper bound for the split domination number of arithmetic graphs V_m.

3.2 Construction of a Graph Corresponding to Known Split Domination Number

Now, we will construct an arithmetic graphs V_m corresponding to known split domination number with the help of upper bound of split domination number of arithmetic graphs V_m. The applications of this type of construction are also important in real life problems so, that it is very important to construct a graph corresponding to a known parameter. Let we have to construct an arithmetic graph V_m corresponding to know split domination number $\gamma_S(V_m) = n$ and the geometrical form of m is $m = {p_1}^{a_1}.{p_2}^{a_2} \ldots . {p_r}^{a_r}$, where $a'_is > 1$, for $i =$

1,2, … r and $p_1, p_2, \ldots . p_r$ are distinct primes then split domination number of graph V_m is $r+1$ and the split dominating set is $D = \{p_1, p_2, \ldots . p_{r-1}, p_1 p_2 \ldots . p_{r-1}\}$.

3.3 Illustrations

Now, here we have the illustration corresponding to split domination number $\gamma_S(V_m) = 3$ for construction of an arithmetic graphs V_m.

3.3.1 Here we have split domination number of arithmetic graph V_m as $\gamma_S(V_m) = 3$. Hence, $n-1=2$. Therefore assume two different primes numberp_1, p_2 and let the geometrical form of$m$ is $m = p_1^2 p_2^2$.Now, the members of vertex set of arithmetic graph V_m are the divisors of m except 1: $V = \{p_1, p_2, p_1^2, p_2^2, p_1 p_2, p_1 p_2^2, p_1^2 p_2, p_1^2 p_2^2\}$

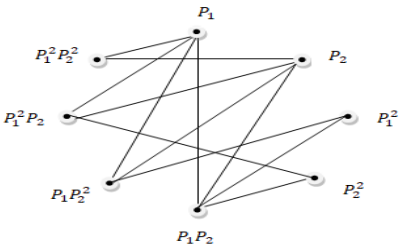

Figure 1: V_m Graph with m = $p_1^2 p_2^2$

Let $D = \{p_1, p_2, p_1 p_2\}$ is the dominating set and $V_m - D$ is not connected. So, D is split dominating set of an arithmetic graphs $V_m$and this is the minimal split dominating set. Hence, split domination number of an arithmetic graphs V_m with $m = p_1^2 p_2^2$is $\gamma_S(V_m) = 3$.

3.3.2 Here we have split domination number of arithmetic graph V_m as $\gamma_S(V_m) = 3$. Hence, $n-1=2$. Therefore assume two different primes numberp_1, p_2 and let the geometrical form of m is $m = p_1^2 p_2^3$.

Now, the members of vertex set of arithmetic graph V_m are the divisors of m except $\{p_1, p_2, p_1^2, p_2^2, p_2^3, p_1 p_2, p_1 p_2^2, p_1 p_2^3, p_1^2 p_2, p_1^2 p_2^2, p_1^2 p_2^3\}$

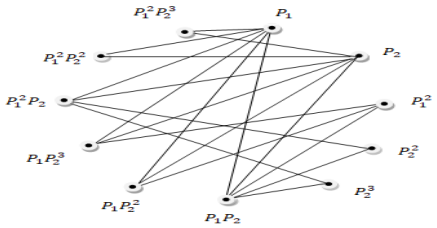

Figure 2: V_m Graph with $m = p_1^2 p_2^3$

Let $D = \{p_1, p_2, p_1 p_2\}$ is the dominating set and $V_m - D$ is not connected. So, D is split dominating set of an arithmetic graphs V_m and this is the minimal split dominating set. Hence, split domination number of an arithmetic graphs V_m with $m = p_1^2 p_2^3$ is $\gamma_S(V_m) = 3$.

3.3.3 Let $G = V_{30}$

Now, the vertices of arithmetic graph V_{30} are the divisors of 30(except 1).

So, $V = \{2, 3, 5, 6, 10, 15, 30\}$. Therefore, the graph of V_{30} is as follows:

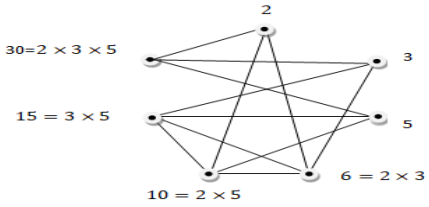

Figure 3: V_m Graph with $m = 30$

$D = \{2, 3, 5, 30\}$is the least split dominating set, since $< V_{30} - D >$ is not connected and $\gamma_S(V_{30}) = 4$.

3.3.4 Let $G = V_{60}$

Now, the vertex set of arithmetic graph V_{60} are the divisors of 60(except 1).

So, $V = \{2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$. Therefore, the graph of V_{60} is as follows:

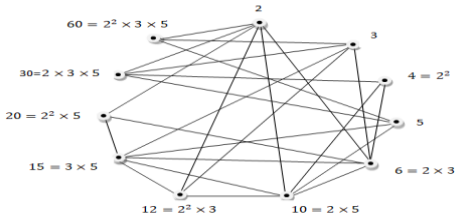

Figure 4: V_m Graph with $m = 60$

$D = \{2, 3, 5, 30\}$ is the least split dominating set, since $< V_{60} - D >$ is not connected and $\gamma_S(V_{60}) = 4$.

4. Split Domination Number of Star Graphs and Bi Star Graphs

In this section, we will investigate the split domination number of the star graph and the Bi Star graphs.

4.1 Star Graph:

Star graph (S_n) is a graph in form of a tree of order n with maximum diameter of graph is two. Let vertex set of star graph of order n is $V = \{a, a_1, a_2, a_3, \dots a_{n-1}, a_n\}$, where a is the central vertex of star graph S_n.

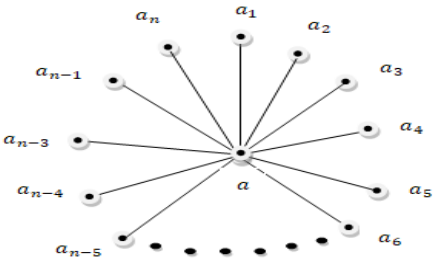

Figure 5: Star Graph of order n

4.2 Theorem:

The split domination number of star graph with $n + 1$ vertices is always1.

Proof: The dominating set of star graph of order n is $D = \{a\}$ and $V - D$ is disconnected graph. So, D is split dominating set of star graph of order n and this is the minimal split dominating set. Hence, split domination number of star graph of order n is equals to 1.

4.3 Bi Star Graph:

Bi star graph $B(n, n)$ is a graph which contains two star graph of order n and the central vertex of two star graphs is joined by an edge. Let vertex set of star graph of order n is $V = \{a, a_1, a_2, a_3, \dots a_{n-1}, a_n, b, b_1, b_2, b_3, \dots b_{n-1}, b_n\}$, where a and b are the central vertex of bi star graph $B(n, n)$.

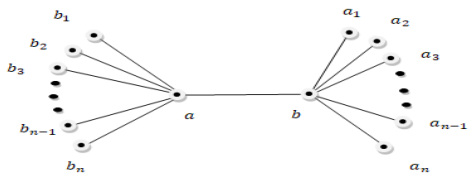

Figure 6: Bi Star Graph of order $2n$

4.4 Theorem:

The split domination number of bi star graph with $2n + 2$ vertices is always 2.

Proof: The dominating set of bi star graph of order n is $D = \{a, b\}$ and $V - D$ is disconnected graph. So, D is split dominating set of bi star graph of order n and this is the minimal split dominating set. Hence, split domination number of bi star graph of order n is equals to 2.

5. Split Domination Number of Petersen Graph

In this section, we will investigate the split domination of the Petersen graph.

5.1 Petersen Graph:

Peterson graph is a graph with 10 vertices and 15 non directional edges in the graph. Let the vertex set of Petersen graph is $V = \{1, 2, 3, \dots, 9, 10\}$.

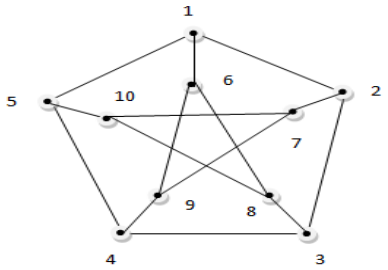

Figure 7: Petersen Graph

5.2 Theorem:

The split domination number of Petersen graph is 3.

Proof: The dominating set of Petersen graph is $D = \{1, 4, 10\}$ and this is the minimal dominating set. Hence, domination number of Petersen graph is 3 and $V - D$ is disconnected graph. So, D is split dominating set of Petersen graph and this is the minimal split dominating set. Hence, split domination number of Petersen graph is also 3. Therefore, domination number and split domination number of Petersen graph is same.

6. Split Domination Number of Middle Graphs

In this section we will investigate split domination of middle graphs.

6.1 Middle Graph:

The middle graph $M(G)$ of any graph G with vertex set $V = \{1, 2, \dots\}$ and edge set $E = \{a_1, a_2, \dots.\}$ is a graph with vertex set $V_{M(G)} = \{1, 2, \dots, a_1, a_2, \dots.\}$ and two vertices $a, b \in V_{M(G)}$ have an edge in middle graph $M(G)$ if one of the above conditions holds:

(i) If $a, b \in E(G)$ then (a, b) is an edge in middle graph $M(G)$ if (a, b) is an edge in G.

(ii) If $a \in V(G)$ and $b \in E(G)$ then (a, b) is an edge in middle graph $M(G)$ if a is an vertex on edge b.

Now, we will find the split domination number of middle graph of path graphs of order n, where $n \geq 3$.for finding this first we will find middle graph of path graphs for different values of n and then we will fine split domination number off each middle graph for the corresponding path graph.

Illustration1: let $n = 3$ then the corresponding path graph is p_3 with vertex set $V = \{1, 2, 3\}$ and edge set $E = \{a, b\}$ then the vertex set of middle graph $M(p_3)$ is $V_{M(p_3)} = \{1, 2, 3, a, b\}$. Therefore, the path graph p_3 and middle graph of $M(p_3)$ is as follows:

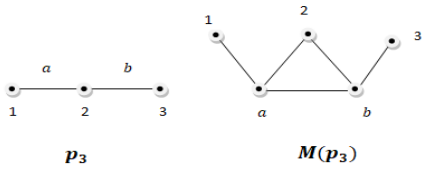

Figure 8: Path Graph $p_3$and middle graph $M(p_3)$

The dominating set of middle graph $M(p_3)$ is $D = \{a, b\}$ and $V - D$ is disconnected graph. So, D is split dominating set of middle graph $M(p_3)$ and this is the minimal split dominating set. Hence, split domination number of middle graph $M(p_3)$ is 2.

Illustration2: let $n = 4$ then the corresponding path graph is p_4 with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{a, b, c\}$ then the vertex set of middle graph $M(p_4)$ is $V_{M(p_4)} = \{1, 2, 3, 4, a, b, c\}$. Therefore, the path graph p_4 and middle graph of $M(p_4)$ is as follows:

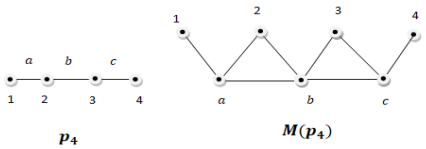

Figure 9: Path Graph $p_4$and middle graph $M(p_4)$

The dominating set of middle graph $M(p_4)$ is $D = \{a, b\}$ and $V - D$ is disconnected graph. So, D is split dominating set of middle graph $M(p_4)$ and this is the minimal split dominating set. Hence, split domination number of middle graph $M(p_4)$ is 2.

Illustration3: Now, we will discuss the general case. Let p_n be the path graph of order n with vertex set $V = \{1, 2, 3, 4, \dots, n\}$ and edge set $E = \{a_1, a_2, \dots, a_{n-1}\}$ then the vertex set of middle graph $M(p_n)$ is $V_{M(p_n)} = \{1, 2, 3, 4, \dots., n, a_1, a_2, \dots, a_{n-1}\}$. Therefore, the path graph p_n and middle graph of $M(p_n)$ is as follows:

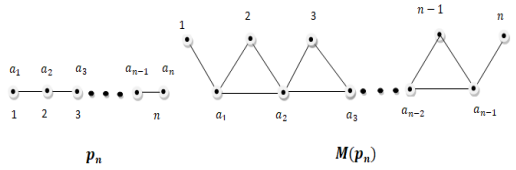

Figure 10: Path Graph p_n and middle graph $M(p_n)$

6.2 Theorem:

The split domination number of middle graph of path graph $M(p_n)$ is

$$\gamma_S(M(p_n)) = \begin{cases} \frac{n}{2} & when\ n\ is\ even \\ \frac{n+1}{2} & when\ n\ is\ odd \end{cases}.$$

Proof: Let p_n be the path graph of order n with vertex set $V = \{1, 2, 3, 4, \dots, n\}$ and edge set $E = \{a_1, a_2, \dots, a_{n-1}\}$ then the vertex set of middle graph $M(p_n)$ is $V_{M(p_n)} = \{1, 2, 3, 4, \dots., n, a_1, a_2, \dots, a_{n-1}\}$. Now for finding the split domination number of middle graph of path graph $M(p_n)$ we will take two cases:

Case I: When n is even-

When the order of path graph is even then the edges in the graph is odd. So, total vertices in middle graph $M(p_n)$ are odd and the graph $M(p_n)$ contains $n - 2$ cycles of three vertices

and $n-2$ is even. Hence, minimum $n/2$ vertices dominate the whole middle graph $M(p_n)$ and the corresponding set $V-D$ must be disconnected. So, this dominating set containing $n/2$ vertices is also a minimum split dominating set. Hence, split domination number of middle graph $M(p_n)$ is $n/2$, where n is even.

Case II: When n is odd-

When the order of path graph is odd then the edges in the graph is even. So, total vertices in middle graph $M(p_n)$ are odd and the graph $M(p_n)$ contains $n-2$ cycles of three vertices and $n-2$ is odd. Hence, minimum $(n+1)/2$ vertices dominate the whole middle graph $M(p_n)$ and the corresponding set $V-D$ must be disconnected. So, this dominating set containing $(n+1)/2$ vertices is also a minimum split dominating set. Hence, split domination number of middle graph $M(p_n)$ is $(n+1)/2$, where n is odd.

7. Concluding Remarks

We have investigated the split domination number of star graphs, Bi star graphs, Peterson graphs, and middle graphs. The implement of domination number and split domination number allow us to develop a simple method of constructing a graph with a known cardinality of the dominating set and spilt dominating set. It is also astonishing to examine that how such a graph with a given domination number and split domination number can be extended to add more vertices and edges in a well-organized manner without affecting the domination number and split domination number.

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