Common Fixed Point Theorems Of Composite Involutions In Banach Spaces

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Abstract

Some results on fixed point of certain composite involutions in Banach spaces have been obtained. These are partially generalizations of certain previously known results due to Ciric[1], Goebel and Zlotkiewicz[2], Khan-Imdad[6] and Iseki{[4],[5]}.

Keywords And Pharases: Banach Spaces, Composite Involutions, Fixed Point, Coincidence Point.

DOI Number: 10.48047/nq.2016.14.2.958 **NeuroQuantology 2016; 14(2): 447-449**

1. INTRODUCTION AND PRELIMINARIES:

In (1971) Goebel and Zlotkiewicz [2] have obtained an interesting result for involution mappings which reads as follows:

Theorem A : Let K be a closed convex subset of a Banach Space X if $T:K\to K$ satisfies the conditions

(a) $T^2=I$ (the identity mapping) and

(aa) $\|Tx-Ty\|\le h\,\|x-y\|$

holds for all x, y and K, where $0\le h<2$. Then T has at least one fixed point.

The extension of theorem A were given by several authors, e.g. Ciric [1], Iseki [4], Part-Yie [7], Khan et. al. [6] and many others.

Iseki [4] replaced condition (aa) by

(*) $\|Tx-Ty\|\le a\|x-y\|+b\big[\|x-Tx\|+\|y-Ty\|\big]$

where $a,b\ge 0,\ a+4b<2$

whereas Khan et al. [5] replaced the condition (aa) by

(**) $$\|Tx-Ty\|\le a\|x-y\|+b\big[\|x-Tx\|+\|y-Ty\|\big]+c\big[\|x-Ty\|+\|y-Tx\|\big]$$

where $a,b,c\ge 0, a+4b+3c<2$

Imdad and Khan [6] have given the definition of composite involution stated as:

1.1 Definition:

A pair of self-mappings (F,G) on a set X is said to be **composite involution** if their composition is an involution.

One may not that the notion of composite involution coincides with usual involution if one of the component maps is identity involution if the pair of maps (F, G) is a composite involution then the component maps need not be involutions as is evident from the following examples.

1.2 Example

Consider $X=\{x,y,z,w\}$. Define $F,G:X\to X$ as

$$Fx=y,\ Fy=z,\ Fz=w,\ Fw=x,$$
$$Gy=x,\ Gz=y,\ Gw=z,\ Gx=w,$$

So that

$$FGx=x,\ \ FGy=y,\ \ FGz=z,\ \ FGw=w,$$
$$F^2x=z,\ \ F^2y=w,\ F^2z=x,\ \ F^2w=y,$$

and $G^2x=z,\ \ G^2y=w,\ G^2z=x,\ \ G^2w=y,$

Thus also

$$F^2\ne I, G^2\ne I \text{ but } (FG)^2=I=FG.$$

1.3 Example:

Define $F,G:R\to R$ as

$$Fx=\begin{cases}-3x & if\ \ x\ge 0\\ -x/2 & if\ x<0\end{cases},\quad Gx=\begin{cases}2x & if\ \ x\ge 0\\ x/3 & if\ \ x<0\end{cases}$$

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Relevant conflicts of interest/financial disclosures: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

So that

$$FGx = \begin{cases} -6x & if \quad x \geq 0 \\ -x/6 & if \ x < 0 \end{cases}$$

Note that $F^2 \neq I,\ G^2 \neq I\ but\ (FG)^2 = I$

It is straight forward to note that if two maps are involutions and commuting then their composition is also an involution.

2. MAIN RESULT:

Motivated from the contractive conditions given by ciric [1], we prove the following.

2.1 Theorem :

Let *F,G,S* and *T* be self mappings of *a* Banach space *X* satisfying

(i) The pair (*ST,FG*) commute,

(ii) The pairs (*S,T*) and (*F,G*) are composite involutions,

(iii)

$$\|STx - STy\| \leq \frac{h}{2}\max\Big(\|FGx - FGy\|, \frac{1}{2}\|FGx - STx\|, \frac{1}{2}\|FGy - STy\|, \frac{1}{2}\|FGx - STy\|, \frac{1}{2}\|FGy - STx\|\Big) \quad ....(1)$$

for every $x, y \in X$ where $0 \leq h < 4$, then FG and ST have a coincidence point x_0 i.e., $FGx_0 = STx_0$. Moreover if the pairs (S,T), (ST,G) (ST,F), (F,G), (FG,S) and (FG,T) commute at the foregoing fixed point x_0, then x_0 also remains the unique common fixed point of *S,T, F* and *G*.

Proof : From (i) and (ii) it follows that $(STFG)^2 = I$. Now using (4.15), we have

$$\|STFGFx - STFGFy\| \leq \frac{h}{2}\max\Big(\|(FG)^2Fx - (FG)^2Fy\|, \frac{1}{2}\|(FG)^2Fx - (STFG)Fx\|, \frac{1}{2}\|(FG)^2Fy - (STFG)Fy\|, \frac{1}{2}\|(FG)^2Fx - (STFG)y\|, \frac{1}{2}\|(FG)^2Fy - (STFG)Fy\|\Big)$$

If we set $Fx = z$ and $Fy = w$, then we get

$$\|STFGz - STFGw\| \leq \frac{h}{2}\max\Big(\|z - w\|, \frac{1}{2}\|z - (STFG)z\|, \frac{1}{2}\|w - (STFG)w\|, \frac{1}{2}\|z - (STFG)w\|, \frac{1}{2}\|w - (STFG)z\|\Big).$$

Since the map *STFG* is an involution and $0 \leq h < 4$, therefore by Theorem 2.1 due to Khan and Imdad [6], *STFG* has at least one fixed point say x_0 in *X* i.e., $STFGx_0 = x_0$. Now using $(ST)^2 = I$, we get $FGx_0 = STx_0$ i.e. x_0 is a coincidence point of *ST* and *FG*. Now

$$\|STx_0 - x_0\| = \|STx_0 - ST(FGx_0)\|$$
$$\leq \frac{h}{2}\max\Big(\|FGx_0 - FG(STx_0)\|, \frac{1}{2}\|FGx_0 - STx_0\|, \frac{1}{2}\|FG(STx_0) - ST(STx_0)\|, \frac{1}{2}\|FGx_0 - ST(STx_0)\|, \frac{1}{2}\|FG(STx_0) - STx_0\|\Big)$$
$$\leq \frac{h}{2}\|STx_0 - x_0\|$$

yielding thereby $STx_0 - x_0 = 0$, or $STx_0 = x_0$ i.e. x_0 is a fixed point of *ST* and hence of *FG*.

To prove the uniqueness of common fixed point x_0. Let y_0 be another fixed point of *ST* and *FG*, Then

$$\|x_0 - y_0\| = \|STx_0 - STy_0\|$$
$$\leq \frac{h}{2}\max\Big(\|FGx_0 - FGy_0)\|, \frac{1}{2}\|FGx_0 - STx_0\|, \frac{1}{2}\|FGy_0 - STy_0)\|, \frac{1}{2}\|FGx_0 - STy_0\|, \frac{1}{2}\|FGy_0 - STx_0\|\Big)$$
$$\leq \frac{h}{2}\|x_0 - y_0\|$$

Giving thereby $x_0 - y_0 = 0$ or $x_0 = y_0$ i.e, x_0 is a unique common fixed point of *ST* and *FG*.

Now using the commutativity of the pairs (*F,G*), (*S,T*), (*FG,S*), (*FG,T*), (*ST,F*) and (*ST,G*) at x_0 one can write.

$Sx_0 = S(TSx_0) = ST(Sx_0)$, $Fx_0 = F(GFx_0) = FG(Fx_0)$,

$Tx_0 = T(TSx_0) = ST^2x_0 = ST(Tx_0)$, $Gx_0 = G(GFx_0) = FG(Gx_0)$,

$Sx_0 = S(FGx_0) = FG(Sx_0)$, $Fx_0 = F(STx_0) = ST(Fx_0)$,

$Tx_0 = T(FGx_0) = FG(Tx_0)$, $Gx_0 = G(STx_0) = ST(Gx_0)$,

which show that Fx_0, Gx_0, Sx_0 and Tx_0 is a common fixed point of the pair (*ST,FG*) which due to uniqueness of the common fixed point of the pair (*ST,FG*) get us.

$$x_0 = Sx_0 = Tx_0 = Fx_0 = Gx_0$$

This completes the proof.

If we take *FG* = *I* and *S* = *I* in Theorem 2.1, we get the following result of Khan and Imdad [6].

2.2 Corollary:

Let *T* be self mappings of a Banach space *X* satisfying

(i) $T^2 = I$

(ii) $\|Tx - Ty\| \leq \frac{h}{2}\max\Big(\|x - y\|, \frac{1}{2}\|x - Tx\|, \frac{1}{2}\|y - Ty\|, \frac{1}{2}\|x - Ty\|, \frac{1}{2}\|y - Tx\|\Big)$

for every $x, y \in X$ where $0 \leq h < 4$, then *T* has at least one fixed point.

2.3 Remark:

Theorem 2.1, remains true if we replace condition (1) as follows

eISSN1303-5150 www.neuroquantology.com

$\|STx - STy\| \leq h\|FGx - FGy\|$ for every $x, y \in X$, where $0 \leq h < 2$

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eISSN1303-5150