Some Fixed Point Theorems In Banach Spaces

Neelmani Gupta[1]*

Abstract

In this paper,we have proved some fixed point theorems on coincidence points of certain composite involutions in Banach spaces employing Pachpatte [6], Delbosco [1] and Khan &Imdad [5] contractive conditions which seem to be a contribution to the existing results and which in turn generalize and unify several other results.

Key words and pharases: Banach Spaces, Composite Involutions, Fixed Point, Coincidence Point.

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1 INTRODUCTION:

By unifying several well known contractive conditions in fixed point theory, Delbosco [1] defined a **g – contraction** as follows

$d(Tx,Ty) \le g(d(x,y), d(x,Tx), d(y,Ty))$

Where g: $R_+^3 \to R_+$ is a continuous function having the properties:

(i) $g(1,1,1) = h < 1$ and

(ii) For $u,v \ge 0$ such that $u \le g(u,v,v)$ or $u \le g(v,u,v)$ or $u \le g(v,v,u)$ then $u \le hv$.

However, we shall assume function g to have somewhat different properties from that defined by Delbosco [1]

Let δ be the set all real valued continuous function of g : $R_+^3 \to R_+$ satisfies the condition

(i) $g(1,1,1) = h < 2$

(ii) if $u, v \ge 0$ are such that either $u \le g(v,2v,u)$ or $u \le g(v,u,2v)$ or $u \le g(u,2v,v)$, them $u \le hv$

Imdad and Khan [2] have given the definition of composite involution stated as:

1.1 Definition:

A pair of self-mappings (F,G) on a set X is said to be **composite involution** if their composition is an involution.

One may note that the notion of composite one of the component maps is identity involution if the pair of maps (F, G) is a composite involution then the component maps need not be involutions as is evident from the following examples.

1.2 Example

Consider $X = \{x, y, z, w\}$. Define $F, G : X \to X$ as

$Fx = y,\ Fy = z,\ Fz = w,\ Fw = x,$

$Gy = x,\ Gz = y,\ Gw = z,\ Gx = w,$

So that

$FGx = x,\ FGy = y,\ FGz = z,\ FGw = w,$

$F^2x = z,\ F^2y = w,\ F^2z = x,\ F^2w = y,$

and

$G^2x = z,\ G^2y = w,\ G^2z = x,\ G^2w = y,$

Thus also

$F^2 \ne I, G^2 \ne I\ but\ (FG)^2 = I = FG$.

2 MAIN RESULTS:

Let x be an arbitrary point in K and $A = \frac{1}{2}(T + I)$,

Define $y = Ax,\ z = Ty\ and\ \mu = 2y - z$, we shall make repeated use of the following equivalent values, where K stands for closed and convex subset of a Banach space X and $T : K \to K$. The following lemma is the key in

***Corresponding Author: -** Neelmani Gupta
Address:-*Department of Mathematics, Govt. Dungar College, Bikaner (Raj.)-334003,
Email: neelmanigupta04@gmail.com
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proving our main result.

2.1 Lemma:

$$\|y - Tx\| = \|x - y\| = 1/2\|x - Tx\|$$

and

$$\|x - Tx\| = 2\|Ax - x\|, \|y - Ty\| = 2\|A^2 x - Ax\|$$

Now we prove our main result.

2.2 Theorem:

Let *F, G, S* and *T* be self mappings of a Banach space *X* satisfying:-

(i) The pair (*ST, FG*) commute,

(ii) The pairs (*S, T*) and (*F, G*) are composite involutions,

(iii) $\|STx - STy\| \le g\left(\|FGx - FGy\|, \|FGx - STx\|, \|FGy - STy\|\right)$

for all $x, y \in X, g \in \delta$, then *FG* and *ST* have a coincidence point x_0, i.e. $FGx_0 = STx_0$, Moreover if the pairs (*S,T*), (*ST, F*), (*ST, G*), (*F, G*), (*FG, S*) and (*FG, T*) commute at the coincidence point x_0, then x_0 also remains the unique common fixed point of *S, T, F* and *G*.

Proof: From (i) and (ii) it follows that $(STFG)^2 = I$. Now using (1), we have

$$\|STFG\,Fx - STFG\,Fy\| \le g\left(\|(FG)^2 Fx - (FG)^2 Fy\|, \|(FG)^2 Fx - (STFG)Fx\|, \|(FG)^2 Fy - (STFG)Fy\|\right)$$

If we set $Fx = z$ and $Fy = w$, then we get

$$\|STFG\,z - STFG\,w\| \le g\left(\|(z - w\|, \|(z - STFG\,z\|, \|(w - STFG\,w\|\right)$$

Since the map *STFG* is an involution, therefore, we define $w = Az$, $\delta = (STFG)w$ and $\mu = 2w - \delta$ andnote the values given in Lemma 2.1.

Now consider

$$\begin{aligned}\|\delta - z\| &= \|(STFG)w - (STFG)^2 z\| \\ &\le g\left(\|w - (STFG)z\|, \|w - (STFG)w\|, \|(STFG)z - (STFG)^2 z\|\right) \\ &\le g\left(\|w - (STFG)z\|, \|w - (STFG)w\|, \|(STFG)z - z\|\right) \\ &\le g\left(\frac{1}{2}\|z - (STFG)z\|, \|w - (STFG)w\|, \|z - (STFG)z\|\right)\end{aligned}$$

by Lemma 2.1

Again

$$\begin{aligned}\|\mu - z\| &= \|2w - \delta - z\| = \|(STFG)z - (STFG)w\| \\ &\le g\left(\|z - w\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right) \\ &\le g\left(\frac{1}{2}\|z - (STFG)z\| \|z - (STFG)z\|, \|w - (STFG)w\|\right) \quad ...(4)\end{aligned}$$

But

$$\|\delta - \mu\| \le \|\delta - z\| + \|z - \mu\|$$

And so, using inequalities (3) and (4) we get

$$\|\delta - \mu\| \le 2g\left(\frac{1}{2}\|z - (STFG)z\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right)$$

Since $\|\delta - \mu\| = 2\|w - (STFG)w\|$, so that above inequality gives

$$\|w - (STFG)w\| \le g\left(\frac{1}{2}\|z - (STFG)z\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right)$$

So that

$$\|w - (STFG)w\| \le h/2\|z - (STFG)z\|$$

Thus from Lemma (2.1), we obtain

$$\|A^2 z - Az\| \le h/2\|Az - z\| \quad ....(1)$$

Thus, inductively, we obtain

$$\|A^{n+1}z - A^n z\| \le (h/2)^n \|Az - z\|$$

Since $h < 2$, it follows that $\|A^{n+1}z - A^n z\| \to 0$ as $n \to \infty$. Thus $\{A^n x\}$ is a Cauchy sequence and converges, to some point x_0 is *X*.

We obtains, therefore$Ax_0 = x_0$ and so$(STFG)\,x_0 = x_0$.

So (*STFG*) has at least one fixed point sayx_0 in *X* i.e., $(STFG)\,x_0 = x_0$. Now using $(ST)^2 = I$, ...(2) we get $FGx_0 = STx_0$i.e. is a coincidence point of *ST* and *FG*. Now

$$\begin{aligned}\|STx_0 - x_0\| &= \|STx_0 - ST(FGx_0)\| \\ &\le g\left(\|FGx_0 - FG(STx_0)\|, \|FGx_0 - STx_0\|, \|FG(STx_0) - ST(STx_0)\|\right) \\ &< g\left(\|STx_0 - x_0\|, 0, 0\right) \\ &< h\left(\|STx_0 - x_0\|\right)\end{aligned}$$

Yielding thereby $ST x_0$ - x_0 =0, or $ST x_0$ = x_0 i.e., x_0 is a fixed point of *ST* and hence of *FG*.

....(3

To prove the uniqueness of common fixed point x_0, let y_0 be another fixed point of *ST* and *FG*, then

$$\begin{aligned}\|x_0 - y_0\| &= \|STx_0 - STy_0\| \\ &\le g\left(\|FGx_0 - FGy_0\|, \|FGx_0 - STx_0\|, \|FGy_0 - STy_0\|\right) \\ &< g\left(\|x_0 - y_0\|, 0, 0\right) \\ &< h\left(\|x_0 - y_0\|\right)\end{aligned}$$

Giving thereby $x_0 - y_0 = 0$ i.e.x_0 is a unique

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common fixed point of ST and FG.

Now using the commutativity of the pairs (F,G), (S,T), (FG,S), (FG,T), (ST,F), (ST,F) and (ST,G) at x_0 one can write.
$Sx_0 = S\,(TSx_0) = ST\,(Sx_0)$, $Fx_0 = F\,(GFx_0) = FG\,(Fx_0)$,
$Tx_0 = T\,(TSx_0) = ST^2x_0 = ST\,(Tx_0)$, $Gx_0 = G\,(GFx_0) = FG\,(Gx_0)$,
$Sx_0 = S\,(FGx_0) = FG\,(Sx_0)$, $Fx_0 = F\,(STx_0) = ST\,(Fx_0)$,
$Tx_0 = T\,(FGx_0) = FG\,(Tx_0)$, $Gx_0 = G\,(STx_0) = ST\,(Gx_0)$,

which show that Fx_0, Gx_0, Sx_0 and Tx_0 is a common fixed point of the pair (ST,FG) which due to uniqueness of the common fixed point of the pair (ST,FG) get us.

$x_0 = Sx_0 = Tx_0 = Fx_0 = Gx_0$
This completes the proof.
After putting $FG = I$ and $S = I$, in Theorem 2.2, we get the following result.

2.3Corollary:
Let T be self mappings of a Banach space X satisfying
(i) $T^2 = I$,
(ii) $\|Tx - Ty\| \le g\left(\|x-y\|, \|x-Tx\|, \|y-Ty\|\right)$
for every $x, y \in X$ where $g \in \delta,$ then T has at least one fixed point

2.4 Remark:
The foregoing Theorem 2.2 can be conveniently used to corollaries the theorem of Iseki ([3,4]), if we choose $g(a,b,c) = (\alpha/2 + \beta)\max\{2a,b,c\}$ for all $a,b,c \ge 0$.

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