GEOMETRIC CONSTRUCTIONS OF ITERATIVE FUNCTIONS TO SOLVE NONLINEAR EQUATIONS

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ABSTRACT

Numerical analysis has a significant challenge in attempting to solve nonlinear equations. Most of these versions are built by taking into account suitable quadrature formulae for the integral calculation. In this study, we provide a geometrical interpretation to numerous iterative approaches for solving a nonlinear scalar problem. We also discuss certain computational characteristics of these approaches and their generalization to generic Banach spaces.

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1. Introduction

Numerical analysis has a significant challenge in attempting to solve nonlinear equations. This paper explores the use of iterative methods for locating a single root.α, i.e., f(α)=0 and f′(α)≠0, of a nonlinear equation f(x)=0 that only makes use of f and f′ and not f's higher derivatives. We shall ignore the possibility of there being more than one root.

The most popular iterative method for calculating is most likely Newton's, which is defined as(1) xn+1=xn−f(xn)f′(xn). That it is the case is generally accepted

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

thus, the approach is quadratically convergent. In recent years, several third-order variations of Newton's technique have been developed and examined for solving nonlinear equations without the need to compute second derivatives (for examples, see and the references therein). Most of these versions are built by taking into account suitable quadrature formulae for the integral calculation. It has been shown that these techniques are just as fast as Newton's approach, if not faster.

Our focus in this paper is on developing iterative methods that converge at least cubically without resorting to the calculation of second derivatives. Our method can use any second-order iterative method to derive a third-order method, in contrast to the approaches used in the derivation of existing methods, which are based on complex geometric constructions, as will be described in detail below. To demonstrate the performance of the derived methods and to illustrate how our result is used to derive them, we provide several examples and the results of a numerical experiment.

2. Literature Review

Praveen Agarwal et.al (2022) In this paper, we develop a class of optimal eighth-order iterative algorithms in three steps for locating roots of nonlinear equations by means of the weight function method. In order to achieve the Kung-Traub optimality conjecture in terms of computing cost per iteration (i.e. 2n1), the recently published iterative techniques of eight order convergence need three function evaluations and one first derivative evaluation. As an added bonus, the theoretical convergence qualities of our schemes are extensively explored by employing the basic theorem that sets the convergence order. Several engineering applications are used to compare the effectiveness and efficiency of our optimum iteration algorithms to those of the state-of-the-art competition. Basins of attraction, dynamical planes, efficiency, log of residual, and numerical test cases show that the novel iterative techniques are more effective than the current approaches in the literature.

Mehdi Dehghan et.al (2022) This work presents seventh- and sixth-order approaches for solving systems of nonlinear equations. In this paper, we present an examination of the convergence of the suggested approaches. These procedures have a computing efficiency of 61/(3n+2n2) and 71/(4n+2n2), respectively. We evaluate the computational efficiency of various recently published algorithms, including Newton's, and some newer ones. Some numerical examples are provided to show how the approaches work in practise, and they are compared to previous findings.

Smmayya Iqbal et.al (2022) In this paper, we introduce a new family of optimal fourth- and sixth-order iterative methods for solving nonlinear equations, and we describe the convergence properties of this class of methods with respect to the original nonlinear equation. Nonlinear systems of equations of identical convergence order are included in the schemes' generalisation. Analysis of the stability features of the vectorial schemes reveals that they have symmetric, broad sets of converge estimates. Various examples from the actual world, including kinematic syntheses, boundary value issues, Fisher's and Hammerstein's integrals, etc., are used to demonstrate the versatility of our approaches in the multidimensional setting. We provide numerical comparisons to demonstrate the efficacy of our systems in comparison to the state-of-the-art effective approaches.

Shengfeng Li et.al (2019) Using Thiele's continued fraction as a starting point, we first present several one-step iterative methods for solving the root-finding problem. These include the classical Newton's method and the Halley's method. Second, we obtain a more efficient iterative approach, which is not only two-step iterative approach but also avoids calculating the higher derivatives of the function, by using approximants of the second and third derivatives. According to convergence analysis, the modified iterative method has a fourth order of convergence for a simple root of the equation. Finally, we provide some numerical experiments and comparisons to demonstrate the effectiveness and performance of the proposed approach.

Hongwei Lin et.al (2018) There is a family of iterative techniques for fitting curves and surfaces called geometric iterative methods (GIM), and two of these approaches stand out: the geometric interpolation/approximation method and the progressive-iterative approximation (PIA) method. In this study, we introduce the local characteristics and accelerating strategies of interpolatory and approximation geometric iteration algorithms, and we demonstrate their convergence. More importantly, GIM has found widespread usage in geometric design and related fields as a result of its ease of integrating geometric constraints in the iterative approach. We take a look at the various fields where geometric iterative methods have been put to good use, such as geometric design, data fitting, reverse engineering, and the generation of meshes and NURBS solids.

3. Methodology

Multiple iterative approaches for solving a nonlinear scalar equation are shown in this study, and their geometric meaning is discussed. The computational and generalization elements of these techniques are also discussed.

4. Data Analysis

The Real Case

Specifically, we solve a scalar nonlinear equation by analyzing the geometric construction of different iterative processes.

f(x)=0.

Newton's method has a well-known geometric interpretation: for any iterate xn, we can find the tangent line.

$$y(x) - f(x_n) = f'(x_n)(x - x_n),$$

to the f(xn, f(xn)) graph, and the following iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0.$$

There is a large body of literature that investigates this sequence and its properties, including the fact that converges to a root of . However, that is not the purpose of this article. In this paper, we only consider the geometric derivation of alternative methods if, in place of a straight line, we consider other tangent curves to the graph of f at (xn, f(xn)).

Newton's tangent line may be interpreted as the Taylor polynomial of f at x_n with degree Grst. Taking the parabola, a polynomial of degree 2, as an example is another popular construction.

$$y(x) - f(x_n) = f'(x_n)(x - x_n) + \frac{f'(x_n)}{2}(x - x_n)^2.$$

This series is found by starting at the point xn+1 where the graph of y meets the x-axis.

$$x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}}\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0,$$

Where

$$L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}.$$

It has been extensively researched and is known as Euler's method or irrational Halley's method. The convergence rate of Euler's method is cubic. The following hyperbola can be used in place of the parabola.

$$axy + y + bx + c = 0,$$

We also impose tangency constraints.

$$y - f(x_n) - f'(x_n)(x - x_n) - \frac{f''(x_n)}{2f'(x_n)}(x - x_n)(y - f(x_n)) = 0.$$

A third-order iterative process that corresponds to this is

$$x_{n+1} = x_n - \frac{2}{2 - L_f(x_n)}\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0,$$

Halley's famous method is this. "Halley's method must share with the secant method the distinction of being the most frequently rediscovered methods in the literature," Traub writes in. The abundance of articles written on the topic and the citations found within them lend credence to this claim. According to, Halley's method can be derived geometrically.

In terms of third-order methods, these two are among the most well-known.

The geometric interpretation of the others is less well known, if at all. To approximate f1(0), for instance, Chebyshev's method relies on a quadratic interpolation of the inverse function of f. Nonetheless, it can be derived geometrically, via a parabola, in the form

$$ay^2 + y + bx + c = 0,$$

that it is possible to write under certain tangency conditions

$$-\frac{f''(x_n)}{2f'(x_n)^2}(y - f(x_n))^2 + y - f(x_n) - f'(x_n)(x - x_n) = 0.$$

Thus, Chebyshev's method can be expressed as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n)\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0.$$

We now look at hyperbolas in the form

$$ay^2 + bxy + y + cx + d = 0,$$

Alternately,

$$x = -\frac{y + ay^2 + d}{by + c}.$$

Since the hyperbola passes through the point (xn; f(xn)), the point's coordinates can be written as an equivalent form.

$$x - x_n = -(y - f(x_n))\frac{1 + a_n(y - f(x_n))}{b_n(y - f(x_n)) + c_n}.$$

In order to determine the value of cn = f (xn) and the relationship between an and bn, we need only satisfy the remaining tangency conditions in .

$$a_n = -\frac{f''(x_n)}{2f'(x_n)^2} - \frac{b_n}{f'(x_n)}.$$

The point where these hyperbolas meet the x-axis at the subsequent iteration, xn+1, is indicated by the following expression.

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 + b_n(f(x_n)/f'(x_n))}\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0,$$

where bn is an n-dependent parameter. As special cases, we have obtained the following third-order methods:

1. For bn=0 and bn>0, we get the well-known Chebyshev and Halley techniques, respectively.

$$b_n = -f''(x_n)/(2f'(x_n)),$$

respectively. 2.The super-Halley method is a third-order method that is not as well-known as the others.

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0.$$

Furthermore, it runs in our family, as b_n = $-f''(x_n)/f'(x_n).$ and even more so because

$$b_n = -\alpha f''(x_n)/f'(x_n),\ \alpha \in \mathbb{R},$$

through this, we obtain the class of third-order procedures discussed in

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0.$$

Finally, in the extreme case, $b_n = \pm\infty,$ We obtain Newton's method.

Our previous thoughts can be used as building blocks to create new iterative procedures.

$$a_n(y - f(x_n))^3 + b_n(y - f(x_n))^2 + (y - f(x_n)) + d_n(x - x_n) = 0,$$

$$a_n = C\frac{f''(x_n)^2}{(f'(x_n))^2},$$

$$b_n = -\frac{f''(x_n)}{2f'(x_n)^2},$$

when we force the tangency conditions upon the equation, we get $d_n = -f'(x_n).$

Finally, we have a new class of third-order procedures:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n) + CL_f(x_n)^2\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0.$$

Methods of this type have been the subject of research in

One Gnal observation in this section is that any of the procedures obtained here can be expressed as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n) + \mathrm{O}(L_f(x_n)^2)\right)\frac{f(x_n)}{f'(x_n)}, \quad n \geqslant 0,$$

in accordance with what Altman mentioned in

5. On The Generalization To Banach Spaces

Most of the procedures we have been studying up until this point can be formally extended to Banach spaces by writing inverse operators in place of quotients. It is possible, for instance, to express method (5) in terms of linear operators, as

$$x_{n+1} = x_n - (I + \tfrac{1}{2}(I + b_n F'(x_n)^{-1} F(x_n))^{-1} L_F(x_n)) F'(x_n)^{-1} F(x_n),$$

where X has an identity operator (I) and each $x \in X,\ L_F(x)$ has the form where for any X, it acts as a linear operator defined by

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x),$$

Taking it as read that $F'(x)^{-1}$ There is at least one. For this reason, we say that the above sequence deGned is well defined in Banach spaces if and only if

$b_n F'(x_n)^{-1} F(x_n)$ have the form of linear operators on X. When deciding on bn, a good set of parameters would include

$b_n = -\alpha F'(x_n)^{-1} F''(x_n),$ with r being a real number and bn being any bilinear operator from X2 to X.

The assessment of the second-order Fr(echet derivative is the key practical diNculty connected to the class of techniques we investigate. The GrstFr(echet derivative of an N-by-N nonlinear system is a N2 by N3 matrix, whereas the second Fr(echet derivative contains N2 by N3 entries. This requires a tremendous amount of work to assess each iteration. A number of writers have looked at alternatives to the second derivative as a means of getting around these difficulties. Methods that reevaluate the function and its Grst derivative several times provide an option. There is a comprehensive review of these techniques in . However, quite a few publications focus on third-order techniques in Banach spaces, namely Halley and Chebyshev methods. One example that may be given is. This is a proposal for a third-order recurrence, and it involves two steps. 3064

$$y_{n+1} = x_n - F'(x_n)^{-1} F(x_n),$$

$$x_{n+1} = y_{n+1} - F'(x_n)^{-1} F(y_{n+1}), \quad n \geqslant 0.$$

In comparison to third-order methods requiring the evaluation of the second derivative, this one is not only more cost-effective, but also more interesting from a practical standpoint. In this paper, we take into account these differences numerically. It is important to keep in mind that the two-stage procedure can be derived from the generalisation of if we consider bn the bilinear operator from X X into X such that

$$b_n F(y_{n+1}) F'(x_n)^{-1} F(x_n) = \tfrac{1}{2} L_F(x_n) F(x_n) - F(y_{n+1}).$$

Then, it is simple to verify that each iteration of the two-stage process can be read as the hyperbolic root.

$$x - x_n = -(y - F(x_n)) \frac{1 + a_n(y - F(x_n))}{b_n(y - F(x_n)) + c_n}.$$

Example 1

(Quadratic equations). For these equations the second derivative is constant. Therefore, the family of methods for

$b_n = -\alpha F'(x_n)^{-1} F''(x_n)$ is equivalent to, or can be expressed as

$$y_n = x_n - F'(x_n)^{-1} F(x_n),$$

$$z_n = x_n + \theta(y_n - x_n), \quad 0 \leqslant \theta \leqslant 1$$

$$x_{n+1} = y_n - F'(z_n)^{-1} F(y_n).$$

There is a correspondence between = 0 and the Chebyshev method, with the Halley method corresponding to = 1, and the super-Halley technique corresponding to = 1. Take note that the two-step procedure accords with the Chebyshev method for quadratic equations. More than that, having access to a variety of techniques enables us to pick and choose among them based on our needs and areas of focus.

For instance, the Chebyshev approach may be used to simplify the computations by requiring the least number of inverses feasible.
Calculus of inverses (second example): using the Chebyshev method, we can get a formula for finding the inverse of a regular matrix A. When applied to the equation F(x)=x1A, the method yields

$$x_{n+1} = (3I_n - (3I_n - x_nA)x_nA)x_n,$$

also, the number of matrix multiplications that must be calculated is relatively small. If we care about how quickly things converge, we can tweak the value of bn in such that convergence occurs at a higher order.

So, if $\text{if } b_n = b(x_n)$ with

$$b(x) = -\alpha(x)\frac{f''(x)}{f'(x)}, \quad \alpha(x) = 1 - \frac{L_{f'}(x)}{3},$$

We get a technique of the fourth order. Calculating this value typically results in very involved approaches. In some cases, however, we do learn more. For quadratic equations (Example 1), for example, (x) = 1. In that case, the super-Halley method requires four orders of magnitude to solve the equations in question. The following is another illustration that catches the eye.

Example 3(p-root calculus, Guti(errez and Hern(andez [11]) Solution of the equation f(t) = 0 for some time t yields the pth root of a positive integer a, where f(t) = t p a. This time around, 3065

$$L_{f'}(x) = (p-2)/(p-1)$$

$$\alpha(x) = \frac{2p-1}{3(p-1)}.$$ and then

The p-th root of operator A can be approximated using the same method. Typically, there will be several systems of partial differential equations involved in these types of issues. With this in mind, the answer to the following puzzle

$$x^{(\text{iv}}(t) + Ax(t) = 0,$$

$$x(0) = x_0$$

$$x(t) = \exp(-A^{1/4}t)x_0.$$

Is

Here we have to calculate the fourth-root of a matrix A

Finally, for the C-methods the order of convergence can be increased by taking

C = C(x) = [(1 − Lf (x))=3]=2.In the above cases C is constant: C = 1 2 for quadratic equations, C = 1 4 for the calculus of inverses and C = (2p − 1)=(6(p − 1)) for the calculus of pth roots.

A numerical experiment

In this subsection, we aim to highlight a category of equations for which the investigated third-order methods are a viable replacement for the Newton and two-step methods. The Hammerstein equation, a specific instance of integral equations, will be the focus of our attention.

$$u(s) = \psi(s) + \int_0^1 H(s,t)f(t,u(t))\,\mathrm{d}t.$$

Boundary value problems for differential equations are related to these equations. Some of them benefit from an eJective (discretized) solution via third-order methods involving second derivatives. in its discrete form is

$$x^i = \psi(t_i) + \sum_{j=0}^{m} \gamma_j H(t_i,t_j)f(t_j,x^j), \quad i = 0,1,\ldots,m,$$

Where $0 \leqslant t_0 < t_1 < \cdots < t_m \leqslant 1$ the nodes of a quadrature formula's grid $\int_0^1 f(t)\,\mathrm{d}t \;\approx$

$\sum_{j=0}^{m} \gamma_j f(t_j)$, and $x^i = x(t_i)$. First, let's have a look at the Hammerstein equation from

$$x(s) = 1 - \frac{1}{4}\int_0^1 \frac{s}{t+s}\frac{1}{x(t)}\,\mathrm{d}t, \qquad s \in [0,1].$$

These nonlinear equations are found by integrating using the trapezoidal rule with the following step size, h = 1=m:

$$0 = x^i - 1 + \frac{1}{4m}\left(\frac{1}{2}\frac{t_i}{t_i+t_0}\frac{1}{x^0}\sum_{k=0}^{n}\frac{t_i}{t_i+t_k}\frac{1}{x^k} + \frac{1}{2}\frac{t_i}{t_i+t_m}\frac{1}{x^m}\right)$$

$$i = 0, 1, \ldots, m,$$

Where t_j = j=m.

Table 1

Exact solution of (10) with $m = 20$

9.658340375548916e − 001	8.383952084700058e − 001
9.418615742240362e − 001	8.331433414479326e − 001
9.231204383172876e − 001	8.283148314256238e − 001
9.077427356874352e − 001	8.238581489055439e − 001
8.947533453995619e − 001	8.197301553818098e − 001
8.835609377538635e − 001	8.158943977068417e − 001
8.737737155176020e − 001	8.123198157064024e − 001
8.651162715111133e − 001	8.089797476185554e − 001
8.573866000336692e − 001	8.058511541752204e − 001
8.504316843674818e − 001	8.029140058401950e − 001
8.441326442432634e − 001	

Table 2

$x_0 = 1.5$, l_∞-error, $m = 20$

Iter.	Newton	Chebyshev	Halley	2-step
1	0.0378	0.0180	0.0175	0.0042
2	2.33e − 04	1.26e − 06	1.03e − 06	4.05e − 09
3	9.98e − 09	0	0	0
4	0			

Table 3

$x_0 = 1.5$, l_∞-error, $m = 20$

Iter.	$C = \frac{1}{4}$	$C = \frac{1}{2}$	$C = 1$	$C = 2$	$C = 4$
1	0.0175	0.0171	0.0162	0.0144	0.0108
2	1.03e − 06	8.26e − 07	4.79e − 07	1.43e − 08	2.82e − 07
3	0	0	0	0	0

The second Fr(echet derivative in this situation is block-diagonal. The added cost is most noticeable with the two-stage process. In the quadrature trapezoidal formula, we set m to 20. Following the numerical computation using the Newton method, the exact solution is listed in Table 1. Tables 2 and 3 show a comparison of our results to those obtained using the two-stage procedure proposed in. In conclusion, this paper covers a broad category of third-order methods, namely, those used to compute the second Fr(echet derivative. We derived a geometric meaning for them, and we studied how they act in situations where calculating the second derivative doesn't take too much time.

Table 4

$x_0 = 1.5$, l_∞-error, $m = 20$

Iter.	$C = 6$	$C = 8$	$C = 10$	$C = 12$	$C = 14$
1	0.0072	0.0036	$2.98e-04$	0.0035	0.0071
2	$1.79e-07$	$3.79e-08$	$1.77e-11$	$4.54e-08$	$4.93e-07$
3	0	0	0	0	0

6. Conclusion

Our focus in this study is on building iterative algorithms that converge at least cubically without resorting to the calculation of second derivatives. There is a large body of literature that investigates this sequence and its properties, including the fact that converges to a root of. However, it is not the purpose of this article. In this paper, we focus solely on the geometrical origin of straight-line approaches. However, quite a few publications focus on third-order techniques in Banach spaces, namely Halley and Chebyshev methods. One example that may be given is. This is a proposal for a third-order recurrence, and it involves two steps.

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