On congruences of certain modular generalized MS-algebras

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Abstract:
In this paper, we define and characterize the concept of $\underline{K_2}$-congruence pairs of modular generalized MS-algebras from the subclass $\underline{\mathbf{K}_2}$ (briefly $\underline{K_2}$-algebras) due to A. Badawy [1]. We observe that every congruence relation θ of a $\underline{K_2}$-algebra L with $L^{\vee} = [d)$ associated with a unique $\underline{K_2}$-congruence pair (θ_1, θ_2) where θ_1 is a congruence on the Kleene algebra $L^{\circ\circ}$ and θ_2 is a lattice congruence on the modular lattice $L^{\vee}$. Also, we investigate special $\underline{K_2}$-congruence pairs of a $\underline{K_2}$-algebra L using its Boolean elements and derive their properties. We study the concept of n-permutability of a $\underline{K_2}$-algebra L with $L^{\vee} = [d)$in terms of K_2-congruence pairs. 1448

Key words: MS-algebras; K_2-algebras; Kleene algebras; Stone algebras; Boolean algebras; GMS-algebras; $\underline{K_2}$-algebras; n-permutability.
Mathematics subject classification (2010): $06D05, 06D30$.
DOI Number: 10.14704/NQ.2022.20.16.880143 **NeuroQuantology2022;20(16):1448-1464**

Introduction

T.S. Blyth and J.C. Varlet [9] introduced the class **MS** of all MS-algebras which generalized the classes **M** of all de Morgan algebras and **S** of all Stone algebras. In [10] T.S. Blyth and J.C. Varlet characterized all subvarieties of the class **MS** in terms of identities. Also, T.S. Blyth and J.C. Varlet [12,13] constructed MS-algebras from the subclass $\mathbf{K}_2$ by means of quadruples. T. Katriňák and et al. [19], [20] introduced the construction of modular p-algebras and MS-algebras by means of tripls, respectively. Later,M. Haviar [18] presented a simple quadruple construction of K_2-algebras with $L^{\vee} = [d)$.

D. Ševčovič [22] investigated the class **GMS** of generalized MS-algebras (briefly GMS-algebras) which containing the classes modular **GMS** of all modular GMS-algebras, **GM** of all generalized de Morgan algebras, modular **GM** of all modular GM-algebras and $\underline{\mathbf{S}}$ of all modular S-algebras. A. Badawy [1] introduced and constructed a certain class of modular GMS-algebras so called $\underline{K_2}$-algebras from Kleene algebras and modular lattices by means of $\underline{K_2}$-quadruples. Also, A. Badawy [2] introduced and constructed principal generalized K_2-algebras (briefly principal GK_2-algebras) from generalized Kleene algebras and bounded lattices by means of triples.

Several authors introduced and characterized the concept of congruence pairs of different algebras as MS-algebras and p-algebras (see [3], [6], [8], [15] and [16]).

The aim of this paper is to introduce the notion of $\underline{K_2}$-congruence pairs of a $\underline{K_2}$-algebra L with $L^{\vee} = [d)$. We show that every congruence relation θ on a $\underline{K_2}$-algebra L with $L^{\vee} = [d)$ can be uniquely determined by a $\underline{K_2}$-congruence pair (θ_1, θ_2) where $\theta_1 \in Con(L^{\circ\circ})$ and $\theta_2 \in Con(L^{\vee})$. Moreover, we investigate special $\underline{K_2}$-congruence pairs of a $\underline{K_2}$-algebra L in terms of Boolean elements of L. Finally, we study n-permutability of $\underline{K_2}$-algebras in terms of $\underline{K_2}$-congruence pairs.

2 Preliminaries

In this section, we recall some basic definitions and results taken from [11], [14] and [17] which we need throughout this paper.

An MS-algebra is an algebra $(L;\vee,\wedge,^{\circ},0,1)$ of type $(2,2,1,0,0)$ where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies:

$x \leq x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ} \text{and} 1^{\circ} = 0.$

A de Morgan algebra M is an MS-algebra satisfying the identity $x = x^{\circ\circ}$. A de Morgan algebra together with the identity $x \wedge x^{\circ} \leq y \vee y^{\circ}$ is called a Kleene algebra K. A Boolean algebra B is an MS-algebra satisfying the identity $x \vee x^{\circ} = 1$. An MS-algebra satisfying the identity $x \wedge x^{\circ} = 0$ is called a Stone algebra.

A K_2-algebra is an MS-algebra satisfying the following two identities:

$x \wedge x^{\circ} = x^{\circ} \wedge x^{\circ\circ} \text{and} x \wedge x^{\circ} \leq y \vee y^{\circ}.$

A generalized MS-algebra (simply GMS-algebra) is an algebra $(L;\vee,\wedge,^{\circ}, 0,1)$ where $(L;\vee,\wedge,0,1)$ is a bounded lattice and the unary operation $^{\circ}$ satisfies the identities:

$x \leq x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ} \text{and} 1^{\circ} = 0.$

A generalized de Morgan algebra (or GM-algebra) is a GMS-algebra satisfies the identity $x = x^{\circ\circ}$. A modular GM-algebra L is a GM-algebra where $(L;\vee,\wedge,0,1)$ is a modular lattice.

A modular GMS-algebra L is an GMS-algebra, where $(L;\vee,\wedge,0,1)$ is a modular lattice. The class of all GMS-algebras containing the classes of GM-algebras, modular GM-algebras and the class $\underline{\mathbf{S}}$ of all modular S-algebras which characterized by the identity $x \wedge x^{\circ} = 0$.

Theorem 1 [22] *For any two elements a, b of GMS-algebra L, we have*
(1) $0^{\circ} = 1$,
(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$,
(3) $a^{\circ\circ\circ} = a^{\circ}$,
(4) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$,
(5) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$,
(6) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$.

The lattice $(F(L);\vee,\wedge)$ of all filters of L is distributive (modular) iff L is distributive (modular) lattice, where $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \{x \in L: x \geq f_1 \wedge f_2,$ for some $f_1 \in F_1$ and $f_2 \in F_2\}$, for all $F_1, F_2 \in F(L)$.

A. Badawy [1], introduced $\underline{K_2}$-algebras as a generalization of Kleene algebras and modular S-algebras (p-algebras satisfying the Stone identity $x^{\circ} \vee x^{\circ\circ} = 1$) as follows:

Definition 2 [1] *A modular GMS-algebra L is called a $\underline{K_2}$-algebra if $L^{\circ\circ}$ is a distributive lattice and L satisfies the identities*
$x \wedge x^{\circ} = x^{\circ} \wedge x^{\circ\circ}, x \wedge x^{\circ} \leq y \vee y^{\circ}$.

We will denote by $\underline{\mathbf{K}}_2$ for the class of all $\underline{K_2}$-algebras. It is clear that $\underline{\mathbf{K}}_2$ contains the classes $\mathbf{K}_2, \mathbf{S}, \mathbf{M}, \mathbf{K}, \mathbf{B}$ and $\underline{\mathbf{S}}$.

Theorem 3 [1,4] *Let L be a $\underline{K_2}$-algebra. Then we have*
(1) $x = x^{\circ\circ} \wedge (x \vee x^{\circ}), \forall x \in L$,
(2) $L^{\circ\circ} = \{x \in L: x = x^{\circ\circ}\}$ is a Kleene algebra,
(3) $L^{\vee} = \{x \vee x^{\circ}: x \in L\} = \{x \in L: x \geq x^{\circ}\}$ is a filter of L,
(4) $L^{\wedge} = \{x \wedge x^{\circ}: x \in L\} = \{x \in L: x \leq x^{\circ}\}$ is an ideal of L,
(5) $D(L) = \{x \in L: x^{\circ} = 0\}$ is a filter of L, and $D(L) \subseteq L^{\vee}$.

Definition 4 [1] *Let K be a Kleene algebra and D be a modular lattice with 1. A mapping $\varphi(a): K \to F(D)$ is called a polarization if φ is a (0,1) - homomorphism such that $\varphi(a) = D$ for every $a \in K^{\vee}$ and $\varphi(a)$ is a principal filter of D for every $a \in K^{\wedge}$.*

Let L be a $\underline{K_2}$-algebra, $L^{\circ\circ}$ be a Kleene algebra and $L^{\vee}$ is a modular lattice. Then $F(L^{\vee})$ is also a modular lattice. Define a map $\varphi_L: L^{\circ\circ} \to F(L^{\vee})$ such that

$$\varphi_L(a) = [a^{\circ}) \cap L^{\vee}, a \in L^{\circ\circ}.$$

Lemma 5 [1] *Let $L \in \underline{\mathbf{K}}_2$. Then $\varphi_L(L^{\circ\circ})$ is a polarization of $L^{\circ\circ}$ into $F(L^{\vee})$.*

Definition 6 [12] *An equivalence relation θ on a lattice L is called a lattice congruence if $(a, b) \in \theta$, $(c, d) \in \theta$ implies $(a \vee c, b \vee d) \in \theta, (a \wedge c, b \wedge d) \in \theta$.*

Definition 7 [15] *Let θ be a lattice congruence on a bounded lattice L. Then we have the following important*

subsets
(i) The Kernel of θ (briefly $Ker\ \theta$) is the set $\{x \in L : (x,0) \in \theta\}$, which is an ideal of L,
(ii) The Cokernel of θ (briefly $Coker\ \theta$) is the set $\{x \in L : (x,1) \in \theta\}$, which is a filter of L.

Definition 8 *A lattice congruence θ on a $\underline{K_2}$-algebra L is called a congruence if $(a,b) \in \theta$ implies $(a^\circ, b^\circ) \in \theta$.*

For any $\underline{K_2}$-algebra L, we denote the lattice of all congruences of L by $Con(L)$. If $\theta \in Con(L)$, then $\theta_{L^{\circ\circ}} \in Con(L^{\circ\circ})$ and $\theta_{L^\vee} \in Con(L^\vee)$ are the restrictions of θ on $L^{\circ\circ}$ and $L^\vee$, respectively. Clearly, $(\theta_{L^{\circ\circ}}, \theta_{L^\vee}) \in Con(L^{\circ\circ}) \times Con(L^\vee)$. Also, we denote the identity congruence and the universal congruence on L by Δ_L and ∇_L respectively, where $\Delta_L = \{(a,a) : a \in L\}$ and $\nabla_L = \{(x,y) : x, y \in L\}$.

For more information about congruences on lattices and MS-algebras, we refer the reader to [4], [5], and[7].

3 $\underline{K_2}$-Congruence pairs of a $\underline{K_2}$- algebra

In this section, we introduce the concept of $\underline{K_2}$-congruence pairs of a $\underline{K_2}$-algebra L, where $L^\vee$ is a principal filter of L, that is, there exists $d \in L$ such that $L^\vee = [d)$. Many properties of $\underline{K_2}$ - congruence pairs will be investigated.

Lemma 9 *Let L be a $\underline{K_2}$-algebrawith $L^\vee = [d)$. Then we have the following:*
(i) $x = x^{\circ\circ} \wedge (x \vee d), \forall x \in L,$
$(ii) (a \wedge b) \vee \mathrm{d} = (a \vee d) \wedge (b \vee d),$
$\forall a, b \in L^{\circ\circ},$
$(iii) (x \wedge y) \vee \mathrm{d} = (x \vee d) \wedge (y \vee d),$
$\forall x, y \in \mathrm{L}.$

Proof. (i) For every $x \in L$ we have $x = x^{\circ\circ} \wedge (x \vee x^\circ)$. Then

$x^{\circ\circ} \wedge (x \vee d)$
$= x^{\circ\circ} \wedge ((x^{\circ\circ} \wedge (x \vee x^\circ)) \vee d)$
$= x^{\circ\circ} \wedge ((x^{\circ\circ} \vee d) \wedge (x \vee x^\circ)),$
by modularity of L as $x \vee x^\circ \geq d$
$= x^{\circ\circ} \wedge (x^{\circ\circ} \vee d) \wedge (x \vee x^\circ)$
$= x^{\circ\circ} \wedge (x \vee x^\circ)$, by absorption law
$= x.$

(ii) Let $a, b \in L^{\circ\circ}$. Since $L^\vee = [d)$, then $\varphi_L(a) = [a^\circ \vee d)$ and $\varphi_L(b) = [b^\circ \vee d)$. Thus we have
$(a \wedge b) \vee d$
$= [(a^{\circ\circ} \wedge b^{\circ\circ}) \vee d)$, as $a = a^{\circ\circ}, b = b^{\circ\circ}$
$= [(a^\circ \vee b^\circ)^\circ \vee d)$
$= \varphi_L(a^\circ \vee b^\circ)$
$= \varphi_L(a^\circ) \vee \varphi(b^\circ),$
as φ_L is ahomomorphism
$= [a^{\circ\circ} \vee d) \vee b^{\circ\circ} \vee d)$
$= [a \vee d) \vee b \vee d)$, as $a = a^{\circ\circ}, b = b^{\circ\circ}$
$= [(a \vee d) \wedge (b \vee d)).$
Then $(a \wedge b) \vee d = (a \vee d) \wedge (b \vee d),$ $\forall a, b \in L^{\circ\circ}.$

(iii) Let $x, y \in L$. Then $x = x^{\circ\circ} \wedge (x \vee x^\circ), y = y^{\circ\circ} \wedge (y \vee y^\circ)$. Thus by modularity of L with $(x \vee x^\circ), (y \vee y^\circ) \geq d$ we have
$(x \wedge y) \vee d$
$= ((x^{\circ\circ} \wedge (x \vee x^\circ)) \wedge (y^{\circ\circ} \wedge (y \vee y^\circ))) \vee d$
$= \left((x^{\circ\circ} \wedge y^{\circ\circ}) \wedge (x \vee x^\circ) \wedge (y \vee y^\circ)\right) \vee d$
$= \left((x^{\circ\circ} \wedge y^{\circ\circ}) \vee d\right) \wedge \left((x \vee x^\circ)\right) \wedge (y \vee y^\circ)$, by modularity of L
$= ((x^{\circ\circ} \vee d) \wedge (y^{\circ\circ} \vee d)) \wedge (x \vee x^\circ) \wedge (y \vee y^\circ)$, by (ii)
$= (x^{\circ\circ} \vee d) \wedge (x \vee x^\circ) \wedge (y^{\circ\circ} \vee d) \wedge (y \vee y^\circ)$
$= \left((x^{\circ\circ} \wedge (x \vee x^\circ)) \vee d\right) \wedge$

$\left((y^{\circ\circ} \wedge (y \vee y^{\circ})) \vee d\right)$, by modularity of L
$= (x \vee d) \wedge (y \vee d)$.
Then $(x \wedge y) \vee d = (x \vee d) \wedge (y \vee d), \forall x, y \in L$.

Definition 10 *Let L be a $\underline{K_2}$- algebra with $L^{\vee} = [d)$. We say that a pair $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con(L^{\vee})$ is a $\underline{K_2}$-congruence pair if it satisfies the following conditions:*

(CP_1) $x \equiv y(\theta_2) \Rightarrow x^{\circ} \equiv y^{\circ}(\theta_1), x, y \in L^{\vee}$,

(CP_2) $a \equiv b(\theta_1) \Rightarrow a \vee d \equiv b \vee d(\theta_2), a, b \in L^{\circ\circ}$.

Lemma 11 *Let L be a $\underline{K_2}$- algebra with $L^{\vee} = [d)$. Let $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con(L^{\vee})$ satisfying (CP_2). Then we have*

(i) $a \equiv b(\theta_1)$ and $x \equiv y(\theta_2)$ imply $a \vee x \equiv b \vee y(\theta_2)$,

(ii) $a \equiv b(\theta_1)$ implies $a \vee a^{\circ} \equiv b \vee b^{\circ}(\theta_2)$.

Now, a characterization of congruences of a $\underline{K_2}$ -algebra L by means of $\underline{K_2}$-congruence pairs is given.

Theorem 12 *Let L be a $\underline{K_2}$- algebra with $L^{\vee} = [d)$. Then every congruence relation θ on L determines a $\underline{K_2}$ -congruence pair $(\theta_{L^{\circ\circ}}, \theta_{L^{\vee}})$. Conversely, every $\underline{K_2}$- congruence pair (θ_1, θ_2) uniquely determines a congruence relation θ on L, satisfying $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_{L^{\vee}} = \theta_2$ by the following rule:*

$$x \equiv y(\theta) \Leftrightarrow x^{\circ} \equiv y^{\circ}(\theta_1) \text{ and } x \vee d \equiv y \vee d(\theta_2).$$

Proof. Let θ be a congruence relation on L. Let $x \equiv y(\theta_{L^{\vee}})$. Then $x \equiv y(\theta)$ and hence $x^{\circ} \equiv y^{\circ}(\theta)$. This implies that $x^{\circ} \equiv y^{\circ}(\theta_{L^{\circ\circ}})$. Then (CP_1) holds. For (CP_2), let $x \equiv y(\theta_{L^{\circ\circ}})$. Then $x \equiv y(\theta)$ and $x \vee d \equiv y \vee d(\theta)$. Hence $x \vee d \equiv y \vee d(\theta_{L^{\vee}})$. Therefore $(\theta_{L^{\circ\circ}}, \theta_{L^{\vee}})$ is a $\underline{K_2}$-congruence pair.
Conversely, let θ be a relation defined on L by

$$x \equiv y(\theta) \Leftrightarrow x^{\circ} \equiv y^{\circ}(\theta_1) \text{ and } x \vee d \equiv y \vee d(\theta_2).$$

It is clear that θ is an equivalence relation. In order to show that θ is a lattice congruence on L, let $x \equiv y(\theta)$ and $x_1 \equiv y_1(\theta), x, y, x_1, y_1 \in L$. Then $x^{\circ} \equiv y^{\circ}(\theta_1), x \vee d \equiv y \vee d(\theta_2)$, and $x_1^{\circ} \equiv y_1^{\circ}(\theta_1), x_1 \vee d \equiv y_1 \vee d(\theta_2)$.

We show that $(x \wedge x_1, y \wedge y_1) \in \theta$. Since $(x \wedge x_1)^{\circ} = x^{\circ} \vee x_1^{\circ} \equiv y^{\circ} \vee y_1^{\circ} = (y \wedge y_1)^{\circ}(\theta_1)$, and $(x \wedge x_1) \vee d$
$= (x^{\circ\circ} \wedge (x \vee d) \wedge x_1^{\circ\circ} \wedge (x_1 \vee d)) \vee d$
$= \left(x^{\circ\circ} \wedge x_1^{\circ\circ} \wedge (x \vee d) \wedge (x_1 \vee d)\right) \vee d$
$= \left((x^{\circ\circ} \wedge x_1^{\circ\circ}) \vee d\right) \wedge (x \vee d) \wedge (x_1 \vee d)$, by modularity of L
$= (x^{\circ\circ} \vee d) \wedge (x_1^{\circ\circ} \vee d) \wedge (x \vee d) \wedge (x_1 \vee d)$, by distributivity of d
$\equiv (y^{\circ\circ} \vee d) \wedge (y_1^{\circ\circ} \vee d) \wedge (y \vee d) \wedge (y_1 \vee d)(\theta_2)$
$= ((y^{\circ\circ} \wedge y_1^{\circ\circ}) \vee d) \wedge (y \vee d) \wedge (y_1 \vee d))$
$= \left((y^{\circ\circ} \wedge y_1^{\circ\circ}) \wedge (y \vee d) \wedge (y_1 \vee d)\right) \vee d$, by modularity of L
$= ((y^{\circ\circ} \wedge (y \vee d)) \wedge (y_1^{\circ\circ} \wedge (y_1 \vee d))) \vee d$
$= (y \wedge y_1) \vee d$,
then $x \wedge x_1 \equiv y \wedge y_1(\theta)$.

Now, we show that θ preserves join. We have $(x \vee x_1)^{\circ} = x^{\circ} \wedge x_1^{\circ} \equiv y^{\circ} \wedge y_1^{\circ}(\theta_1) = (y \vee y_1)^{\circ}$, and
$(x \vee x_1) \vee d = (x \vee d) \vee (x_1 \vee d)$
$\equiv (y \vee d) \vee (y_1 \vee d)(\theta_2)$
$= (y \vee y_1) \vee d$.
Therefore θ is a lattice congruence.

We prove that θ preserves the unary operation $^{\circ}$, let $x \equiv y(\theta)$. Then

eISSN1303-5150

$x^{\circ\circ} \equiv y^{\circ\circ}(\theta_1)$ and $x^{\circ} \vee d \equiv y^{\circ} \vee d(\theta_2)$, by Lemma 11$(i)$.
Thus $x^{\circ} \equiv y^{\circ}(\theta)$ and hence θ is a congruence on L.

Finally, we show that $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_{L^{\vee}} = \theta_2$. Let $x, y \in L^{\circ\circ}$. If $x \equiv y(\theta_1)$, then

$$x^{\circ} \equiv y^{\circ}(\theta_1) \text{ and } x \vee d \equiv y \vee d(\theta_2), \text{ by Lemma } 11(i).$$

This implies that $x \equiv y(\theta)$ and hence $x \equiv y(\theta_{L^{\circ\circ}})$. Then $\theta_1 \subseteq \theta_{L^{\circ\circ}}$.

For the converse, let $x \equiv y(\theta_{L^{\circ\circ}})$. Then $x \equiv y(\theta)$ This implies that$x^{\circ} \equiv y^{\circ}(\theta_1)$ and $x = x^{\circ\circ} \equiv y^{\circ\circ}(\theta_1) = y$. Then$\theta_{L^{\circ\circ}} \subseteq \theta_1$.Therefore $\theta_{L^{\circ\circ}} = \theta_1$.
To show that $\theta_{L^{\vee}} = \theta_2$.Let $x, y \in L^{\vee}$. If $x \equiv y(\theta_2)$, then $x^{\circ} \equiv y^{\circ}(\theta_1)$ and $x \vee d \equiv y \vee d(\theta_2)$, by (CP_1).
This implies that $x \equiv y(\theta)$ and hence $x \equiv y(\theta_{L^{\vee}})$. Then $\theta_2 \subseteq \theta_{L^{\vee}}$.

Conversely, if $x \equiv y(\theta_{L^{\vee}})$, then $x \equiv y(\theta)$. Thus we have
$x^{\circ} \equiv y^{\circ}(\theta_1)$and $x \vee d \equiv y \vee d(\theta_2)$.
Since $x = x \vee d \equiv y \vee d(\theta_2) = y$, as $x, y \geq d$ then $\theta_{L^{\vee}} \subseteq \theta_2$. Therefore $\theta_{L^{\vee}} = \theta_2$.

For the uniqueness, suppose that $\theta, \psi \in Con(L)$, such that $\theta_{L^{\circ\circ}} = \psi_{L^{\circ\circ}}, \theta_{L^{\vee}} = \psi_{L^{\vee}}$. Let $x \equiv y(\theta)$. Then $x^{\circ} \equiv y^{\circ}(\theta_{L^{\circ\circ}})$ and $x \vee d \equiv y \vee d(\theta_{L^{\vee}})$.
This implies that $x^{\circ} \equiv y^{\circ}(\psi_{L^{\circ\circ}})$ and $x \vee d \equiv y \vee d(\psi_{L^{\vee}})$.
Therefore $x \equiv y(\psi)$ and $\theta \subseteq \psi$. Similarly, we can show that $\psi \subseteq \theta$. Hence $\theta = \psi$.

The following Lemma gives equivalent rules of the congruence relation θ which defined in the above Theorem.

Lemma 13 *Let* L *be a* $\underline{K_2}$*-algebra with* $L^{\vee} = [d)$ *and* (θ_1, θ_2) *be a* $\underline{K_2}$*-congruence pair of* L. *Then we can determine* $\theta \in Con(L)$ *by the following equivalent rules:*
(i) $x \equiv y(\theta) \Leftrightarrow x^{\circ} \equiv y^{\circ}(\theta_1)$ and $x \vee d \equiv y \vee d(\theta_2)$,
(ii) $x \equiv y(\theta) \Leftrightarrow x^{\circ} \equiv y^{\circ}(\theta_1)$ and $x \vee u \equiv y \vee u(\theta_2), \forall u \in L^{\vee}$,
(iii) $x \equiv y(\theta) \Leftrightarrow x^{\circ} \equiv y^{\circ}(\theta_1)$ and $x \vee x^{\circ} \equiv y \vee y^{\circ}(\theta_2)$.

Proof.$(i) \Rightarrow (ii)$
Let (i) holds and $u \in L^{\vee}$. Since $u \geq d$, then $x \vee u = x \vee d \vee u$
$\equiv y \vee d \vee u(\theta_2) = y \vee u$.
Hence we get (ii). To prove $(ii) \Rightarrow (iii)$, Consider (ii) holds. From Lemma 11 (i), we have
$x^{\circ} \vee (x \vee u) \equiv y^{\circ} \vee (y \vee u)(\theta_2), \forall u \in L^{\vee}$.
Since $x \vee x^{\circ}, y \vee y^{\circ} \in L^{\vee}$, then we obtain $x \vee x^{\circ} \equiv y^{\circ} \vee y \vee x \vee x^{\circ}(\theta_2)$,
And $x^{\circ} \vee x \vee y \vee y^{\circ} \equiv y \vee y^{\circ}(\theta_2)$.
Then $x \vee x^{\circ} \equiv y \vee y^{\circ}(\theta_2)$ and hence (iii) is proved. In order to show $(iii) \Rightarrow (i)$
As $x, y \in L$, we have
$x = x^{\circ\circ} \wedge (x \vee x^{\circ})\,, y = y^{\circ\circ} \wedge (y \vee y^{\circ})\,.$
Then
$x \vee d = \left(x^{\circ\circ} \wedge (x \vee x^{\circ})\right)$
$= (x^{\circ\circ} \vee d) \wedge (x \vee x^{\circ})$,
by modularity as $x \vee x^{\circ} \geq d$
$\equiv (y^{\circ\circ} \vee d) \wedge (y \vee y^{\circ})(\theta_2)$, by$(iii)$
$= \left(y^{\circ\circ} \wedge (y \vee y^{\circ})\right) \vee d$,
by modularity as $y \vee y^{\circ} \geq d$
$= y \vee d$.
Therefore $x \vee d \equiv y \vee d(\theta_2)$ and (i) is valied.

We will denote the set of all $\underline{K_2}$-congruence pairs of a $\underline{K_2}$-algebra L by $A(L)$.

Corollary 14 *Let* L *be a* $\underline{K_2}$*-algebra with* $L^{\vee} = [d)$. *Then the set* $A(L)$ *is a sublattice of* $Con(L^{\circ\circ}) \times Con(L^{\vee})$ *and* $Con(L)$ *isomorphic to* $A(L)$.

Proof. Let $(\theta_1,\theta_2),(\psi_1,\psi_2) \in A(L)$.In order to show that
$(\theta_1 \wedge \psi_1, \theta_2 \wedge \psi_2) \in A(L)$, let $a \equiv b(\theta_1 \wedge \psi_1)$ and $x \equiv y(\theta_2 \wedge \psi_2)$. Then by using Theorem (12)we have
$x^\circ \equiv y^\circ(\theta_1)$and $a \vee d \equiv b \vee d(\theta_2)$,
and
$x^\circ \equiv y^\circ(\psi_1)$and $a \vee d \equiv b \vee d(\psi_2)$.
Then we have
$x^\circ \equiv y^\circ(\theta_1 \wedge \psi_1)$and
$a \vee d \equiv b \vee d(\theta_2 \wedge \psi_2)$.
Therefore (CP_1) and (CP_2) are valied. Hence $(\theta_1 \wedge \psi_1, \theta_2 \wedge \psi_2) \in A(L)$.

Next, we show that $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$. For (CP_1), let
$x \equiv y(\theta_2 \vee \psi_2)$. Then there exist a sequence
$x = x_0, x_1, \ldots, x_n = y$ in$L^\vee$,
such that $x_{j-1} \equiv x_j(\theta_2)$or
$$x_{j-1} \equiv x_j(\psi_2), 1 \le j \le n.$$
Since $(\theta_1,\theta_2),(\psi_1,\psi_2) \in A(L)$, then using Definition 10, we have
$x_{j-1}^\circ \equiv x_j^\circ(\theta_1)$or $x_{j-1}^\circ \equiv x_j^\circ(\psi_1)$.
Thus there are the sequence
$x^\circ = x_0^\circ, x_1^\circ, \ldots, x_n^\circ = y^\circ$in$L^{\circ\circ}$,
Therefore $x^\circ \equiv y^\circ(\theta_1 \vee \psi_1)$.For$(CP_2)$, Let $a \equiv b(\theta_1 \vee \psi_1)$. Then there are a sequence
$a = a_0, a_1, \ldots, a_m = b$ in$L^{\circ\circ}$,
such that $a_{i-1} \equiv a_i(\theta_1)$or
$a_{i-1} \equiv a_i(\psi_1), 1 \le i \le m$.
Since $(\theta_1,\theta_2),(\psi_1,\psi_2) \in A(L)$, then we have $a_{i-1} \vee d \equiv a_i \vee d(\theta_2)$, or
$a_{i-1} \vee d \equiv a_i \vee d(\psi_2)$.
Hence we get the sequence
$$a \vee d = a_0 \vee d, a_1 \vee d, \ldots, a_m \vee d = b \vee d \text{ in} L^\vee.$$
Then $a \vee d \equiv b \vee d(\theta_2 \vee \psi_2)$.
Therefore $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$.
Hence $A(L)$ is a sublattice of $Con(L^{\circ\circ}) \times Con(L^\vee)$.
Using Theorem (12), we observe that a map $\theta \to (\theta/L^{\circ\circ}, \theta/L^\vee)$ is an isomorphism from $Con(L)$ into $A(L)$.

For modular S-algebras, we have the following properties.

Lemma 15 *Let L be a $\underline{K_2}$-algebra from the class $\underline{\mathbf{S}}$. Then*
(1)$L^{\circ\circ}$ is a Boolean algebra,
(2)$L^\vee = D(L)$.

Proof. (1) Since $L^{\circ\circ}$ is a kleene algebra, then $a = a^{\circ\circ}, \forall a \in L^{\circ\circ}$. Let $x \in L^{\circ\circ}$. Then $x \wedge x^\circ = 0, \forall x \in L$, as $L \in \underline{\mathbf{S}}$and $x \vee x^\circ = x^{\circ\circ} \vee x^\circ =$
$$(x \wedge x^\circ)^\circ = 0^\circ = 1.$$
Therefore $L^{\circ\circ}$ is a Boolean algebra.
(2) We have $D(L) \subseteq L^\vee$. On the other hand, let $x \in L^\vee$. Then we have
$$\begin{aligned} x^\circ &= (x \vee x^\circ)^\circ, \text{as } x \ge x^\circ \\ &= x^\circ \wedge x^{\circ\circ} \\ &= x \wedge x^\circ, \text{as L is a } \underline{K_2} - \text{algebra} \\ &= 0, \text{as } L \in \underline{S}. \end{aligned}$$
Then $x \in D(L)$ and hence $L^\vee = D(L)$.

Now we restrict the definition of congruence pairs of a modular p-algebra (see [8]) to modular S-algebra as follows:

Definition 16 *Let L be a $\underline{K_2}$- algebra from the class $\underline{\mathbf{S}}$. A pair $(\theta_1,\theta_2) \in Con(L^{\circ\circ}) \times Con(D(L))$ is called a congruence pair if it satisfies the condition:*
$$a \equiv 1(\theta_1) \text{ and } a \le d \in D(L) \Rightarrow d \equiv 1(\theta_2).$$

The following Lemma proves that the concept of $\underline{K_2}$-congruence pairs of a modular S-algebra is equivalent to the concept of congruence pair.

Lemma 17 *Let L be a $\underline{K_2}$- algebra* *from the class* $\underline{\mathbf{S}}$ *and let* $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con(D(L))$. *Then* $(\theta_1, \theta_2) \in A(L)$, *if and only if it is a congruence pair.*

Proof. Let (θ_1, θ_2) be a congruence pair. Let $x \equiv y(\theta_2)$. Since $x^\circ = y^\circ = 0$, then $x^\circ \equiv y^\circ(\theta_1)$ and (CP_1) holds. For (CP_2), let $a \equiv b(\theta_1)$ and let

$$\epsilon = (a \vee b^\circ) \wedge (b \vee a^\circ) \in L^{\circ\circ}.$$

Then we have,

$a \wedge \epsilon = a \wedge (a \vee b^\circ) \wedge (b \vee a^\circ)$
$= a \wedge (b \vee a^\circ)$, by absorption law
$= (a \wedge b) \vee (a \wedge a^\circ)$,
by distributivity of $L^{\circ\circ}$
$= (a \wedge b) \vee 0 = a \wedge b$,
as $L^{\circ\circ}$ is a Boolean algebra,
and
$b \wedge \epsilon = b \wedge (a \vee b^\circ) \wedge (b \vee a^\circ)$
$= b \wedge (a \vee b^\circ)$, by absorption law
$= (b \wedge a) \vee (b \wedge b^\circ)$,
by distributivity of $L^{\circ\circ}$
$= (b \wedge a) \vee 0 = a \wedge b$,
as $L^{\circ\circ}$ is a Boolean algebra.

Then $a \wedge \epsilon = a \wedge b = b \wedge \epsilon$ and $\epsilon \equiv 1(\theta_1)$. Since $\epsilon \leq \epsilon \vee d \in D(L)$, then $\epsilon \vee d \equiv 1(\theta_2)$and

$$(a \vee d) \wedge 1 \equiv (a \vee d) \wedge (\epsilon \vee d)(\theta_2).$$

Thus $(a \vee d) \equiv (a \wedge \epsilon) \vee d(\theta_2) = (a \wedge b) \vee d$. Similarly, $(b \vee d) \equiv (b \wedge \epsilon) \vee d(\theta_2) = (a \wedge b) \vee d$. Therefore $(a \vee d) \equiv (b \vee d)(\theta_2)$. Then $(\theta_1, \theta_2) \in A(L)$.

Conversely, let $(\theta_1, \theta_2) \in A(L), a \in B(L), a \leq c \in D(L)$ and $a \equiv 1(\theta_1)$. Then $a \vee c \equiv 1 \vee c(\theta_2)$ implies
$c \equiv 1(\theta_2)$.
Hence (θ_1, θ_2) is a congruence pair.

4 Congruence pairs of a $\underline{K}_2$-algebra via its Boolean elements

In this section, we define a Boolean element of a $\underline{K}_2$ -algebra L and investigate special $\underline{K}_2$-congruence pairs of L using its Boolean elements.

Definition 18 *Let* L *be a* $\underline{K}_2$*-algebra . An element* $a \in L$ *is said to be a Boolean element of* L *if* $a \vee a^\circ = 1$. *We denote the set of all Boolean elements by* $B(L)$, *that is,* $B(L) = \{a \in L: a \vee a^\circ = 1\}$.

Lemma 19 *Let* L *be a* $\underline{K}_2$*-algebra . Then we have the following:*
(1) $B(L)$ is the greatest Boolean subalgebra of $L^{\circ\circ}$,
(2) If $L \in \underline{\mathbf{S}}$then $B(L) = L^{\circ\circ}$.

Proof. (1) Let $a \in B(L)$. Then we have $a^{\circ\circ} = 1 \wedge a^{\circ\circ} = (a \vee a^\circ) \wedge a^{\circ\circ}$
$= a \vee (a^\circ \wedge a^{\circ\circ})$,
by modularity of L with $a^{\circ\circ} \geq a$
$= a \vee (a \vee a^\circ)^\circ = a \vee 1^\circ$
$= a \vee 0 = a$.
Then $a \in L^{\circ\circ}$ and hence $B(L) \subseteq L^{\circ\circ}$. Clearly, $0,1 \in B(L)$. Let $a, b \in B(L)$. Then $a \vee a^\circ = 1$ and $b \vee b^\circ = 1$.
By distributivity of $B(L)$, we get
$(a \wedge b) \vee (a \wedge b)^\circ$
$= (a \wedge b) \vee (a^\circ \vee b^\circ)$
$= (a \vee a^\circ \vee b^\circ) \wedge (b \vee a^\circ \vee b^\circ)$
$= (1 \vee b^\circ) \wedge (a^\circ \vee 1) = 1$.
Then $a \wedge b \in B(L)$. Similarly, one can show that $a \vee b \in B(L)$. Therefore $B(L)$ is a bounded sublattice of $L^{\circ\circ}$. Let $a \in B(L)$. Then $a^\circ \vee a^{\circ\circ} = a^\circ \vee a = 1$ and hence $a^\circ \in B(L)$. Also, we have $a \wedge a^\circ = a^{\circ\circ} \wedge a^\circ = (a^\circ \vee a)^\circ = 1^\circ = 0$. Thus $(B(L); \vee, \wedge, ^\circ, 0, 1)$ is a Boolean subalgebra of $L^{\circ\circ}$.
Now, we show that $B(L)$ is the greatest Boolean subalgebra of $L^{\circ\circ}$. Let H be any Boolean subalgebra of $L^{\circ\circ}$. Then for all $a \in H$, we have $a \vee a^\circ = 1$ and $a \wedge a^\circ = 0$. This implies that

$a \in B(L)$ and hence $H \subseteq B(L)$.
(2) Since $L \in \underline{\mathbf{S}}$, then $x \wedge x^\circ = 0$, $\forall x \in L$. Let $a \in L^{\circ\circ}$. Then we have

$$a \wedge a^\circ = 0 \Rightarrow (a \wedge a^\circ)^\circ = 0^\circ$$
$$\Rightarrow a^\circ \vee a^{\circ\circ} = 1$$
$$\Rightarrow a^\circ \vee a = 1, \text{as } a = a^{\circ\circ}.$$

Therefore $a \in B(L)$ and hence $L^{\circ\circ} \subseteq B(L)$. Since $B(L) \subseteq L^{\circ\circ}$, from (1) then $B(L) = L^{\circ\circ}$.

Let $L \in \underline{K_2}$, for $a \in L^{\circ\circ}$, we define a relation $\theta[(a]]$ on $L^{\circ\circ}$ by

$$(x, y) \in \theta[(a]] \Leftrightarrow x \wedge a^\circ = y \wedge a^\circ.$$

Theorem 20 *Let L be a $\underline{K_2}$-algebra. Then $\theta[(a]]$ is a congruence relation on $L^{\circ\circ}$ with $ker\theta[(a]] = (a]$ for some $a \in L^{\circ\circ}$ if and only if a is a Boolean element of L.*

Proof. Let $\theta[(a]]$ be a congruence relation on $L^{\circ\circ}$ with $ker\theta[(a]] = (a]$, for some $a \in L^{\circ\circ}$. Since $a \in Ker\theta[(a]]$, then $(a, 0) \in \theta[(a]]$. This implies that

$a \wedge a^\circ = 0 \wedge a^\circ = 0$ and hence

$$a^\circ \vee a = a^\circ \vee a^{\circ\circ} = (a \wedge a^\circ)^\circ = 0^\circ = 1.$$

Then a is a Boolean element of L. Conversely, let $a \in B(L)$. Then $a \vee a^\circ = 1$ and $a \wedge a^\circ = 0$. Clearly $\theta[(a]]$ is an equivalence relation on $L^{\circ\circ}$. Let $x, y, x_1, y_1 \in L^{\circ\circ}$. Suppose that (x, y) and $(x_1, y_1) \in \theta[(a]]$. Then

$x \wedge a^\circ = y \wedge a^\circ$ and $x_1 \wedge a^\circ = y_1 \wedge a^\circ$.

Now, we show that $\theta[(a]]$ preserves meet and join

$$(x \wedge x_1) \wedge a^\circ = (x \wedge a^\circ) \wedge (x_1 \wedge a^\circ)$$
$$= (y \wedge a^\circ) \wedge (y_1 \wedge a^\circ)$$
$$= (y \wedge y_1) \wedge a^\circ,$$

and

$$(x \vee x_1) \wedge a^\circ = (x \wedge a^\circ) \vee (x_1 \wedge a^\circ), \text{ by distributivity of } L^{\circ\circ}$$
$$= (y \wedge a^\circ) \vee (y_1 \wedge a^\circ)$$
$$= (y \vee y_1) \wedge a^\circ.$$

Then $\theta[(a]]$ is a lattice congruence.

For the unary operation $^\circ$, let

$(x, y) \in \theta[(a]]$
$\Rightarrow x \wedge a^\circ = y \wedge a^\circ.$
$\Rightarrow (x \wedge a^\circ)^\circ = (y \wedge a^\circ)^\circ$
$\Rightarrow (x^\circ \vee a^{\circ\circ}) = (y^\circ \vee a^{\circ\circ})$
$\Rightarrow (x^\circ \vee a) \wedge a^\circ = (y^\circ \vee a) \wedge a^\circ$, as $a = a^{\circ\circ}$
$\Rightarrow (x^\circ \wedge a^\circ) \vee (a \wedge a^\circ) = (y^\circ \wedge a^\circ) \vee (a \wedge a^\circ)$, by distributivity of $L^{\circ\circ}$
$\Rightarrow (x^\circ \wedge a^\circ) \vee 0 = (y^\circ \wedge a^\circ) \vee 0$, as $a \wedge a^\circ = 0$
$\Rightarrow (x^\circ, y^\circ) \in \theta[(a]]$.

Therefore $\theta[(a]]$ is a congruence relation on $L^{\circ\circ}$. Also, we see that

$Ker\theta[(a]] = \{x \in L^{\circ\circ} : (x, 0) \in \theta[(a]]\}$
$= \{x \in L^{\circ\circ} : x \wedge a^\circ = 0 \wedge a^\circ = 0\}$
$= \{x \in L^{\circ\circ} : x \leq a\} = (a]$.

Where $a = a \vee 0 = a \vee (x \wedge a^\circ) = (a \vee x) \wedge (a \vee a^\circ) = (a \vee x) \wedge 1 = a \vee x$ implies $x \leq a$. 1456

Lemma 21 *Let L be a $\underline{K_2}$-algebra with. Let $\theta[(a]]$ be a congruence on $L^{\circ\circ}$. Then we have the following:*

(1) $a \leq b \Leftrightarrow \theta[(a]] \subseteq \theta[(b]], \forall a, b \in B(L)$,
(2) $a = b \Leftrightarrow \theta[(a]] = \theta[(b]], \forall a, b \in B(L)$,
(3) $\theta[(0]] = \Delta_{L^{\circ\circ}}, \theta[(1]] = \nabla_{L^{\circ\circ}}$.

Proof. (1) Let $a \leq b$ and $(x, y) \in \theta[(a]]$. Then $x \wedge a^\circ = y \wedge a^\circ$. We have

$$x \wedge b^\circ = x \wedge a^\circ \wedge b^\circ, \text{as } a^\circ \geq b^\circ$$
$$= y \wedge a^\circ \wedge b^\circ$$
$$= y \wedge b^\circ.$$

This implies that $(x, y) \in \theta[(b]]$. Hence $\theta[(a]] \subseteq \theta[(b]]$. Conversely, suppose that $\theta[(a]] \subseteq \theta[(b]]$. Let $x \in Ker\theta[(a]]$. Then $(x, 0) \in \theta[(a]] \subseteq$

$\theta[(b]]$. Therefore $x \in Ker\theta[(b]]$ and hence $Ker\theta[(a]] \subseteq Ker\theta[(b]]$. Thus we have $(a] \subseteq (b]$. Then $a \leq b$.
(2) Let $a = b$ and $(x, y) \in \theta[(a]]$. Then $x \wedge a^\circ = y \wedge a^\circ$.This implies that $x \wedge b^\circ = y \wedge b^\circ$ and $(x, y) \in \theta[(b]]$. Hence $\theta[(a]] \subseteq \theta[(b]]$. Similarly, we get $\theta[(a]] \subseteq \theta[(b]]$. Then $\theta[(a]] = \theta[(b]]$. Conversely, let $\theta[(a]] = \theta[(b]]$. From (1), we get $a \leq b$ and $a \geq b$. Then $a = b$.
(3) For any $(x, y) \in \theta[(0]]$, we get $x \wedge 0^\circ = y \wedge 0^\circ$. This implies that $x = y$. Then $\theta[(0]] = \Delta_{L^{\circ\circ}}$. For all $x, y \in L^{\circ\circ}$, we have $x \wedge 1^\circ = 0 = y \wedge 1^\circ$. Then $(x, y) \in \theta[(1]]$ and hence $\theta[(1]] = \nabla_{L^{\circ\circ}}$.

Lemma 22 *Let L be a $\underline{K_2}$-algebra. Let $\theta[(a]], \theta[(b]]$ be congruences on $L^{\circ\circ}$. Then we have the following:*
(1) $\theta[(a]] \vee \theta[(b]] = \theta[(a \vee b]]$,
(2) $\theta[(a]] \wedge \theta[(b]] = \theta[(a \wedge b]]$,
(3) The set $Con\big(B(L)\big) = \{\theta[(a]]: a \in B(L)\}$ is a Boolean algebra and $Con\big(B(L)\big)$ isomorphic to $B(L)$.

Proof. (1) Since $a, b \leq a \vee b$, then $\theta[(a]], \theta[(b]] \subseteq \theta[(a \vee b]]$. Therefore $\theta[(a \vee b]]$ is an upper bound of $\theta[(a]]$ and $\theta[(b]]$. Suppose that $\theta[(c]]$ is an upper bound of $\theta[(a]]$ and $\theta[(b]]$. Then $\theta\big[(a]\big], \theta\big[(b]\big] \subseteq \theta\big[(c]\big]$

$$\Rightarrow a, b \leq c, \text{by Lemma 21 (1).}$$

Hence c is an upper bound of a and b. Since $a \vee b$ is the least upper bound of a and b, then $a \vee b \leq c$ $\Rightarrow \theta[(a \vee b]] \subseteq \theta[(c]]$.
Therefore $\theta[(a \vee b]]$ is the least upper bound of $\theta[(a]]$ and $\theta[(b]]$, that is $\theta[(a]] \vee \theta[(b]] = \theta[(a \vee b]]$.
(2)By duality of (1).
(3) Since $\theta[(0]] = \Delta_{L^{\circ\circ}}$ and $\theta[(1]] = \nabla_{L^{\circ\circ}}$ then $\Delta_{L^{\circ\circ}}, \nabla_{L^{\circ\circ}} \in Con(B(L))$. Let $\theta[(a]]$, $\theta[(b]] \in Con(B(L))$. Then

$$\theta\big[(a]\big] \wedge \theta\big[(b]\big] = \theta\big[(a \wedge b]\big] \in Con(B(L)),$$

and

$$\theta[(a]] \vee \theta\big[(b]\big] = \theta\big[(a \vee b]\big] \in Con(B(L)).$$

Since $B(L)$ is a Boolean algebra, then $\forall a \in B(L) \Rightarrow a^\circ \in B(L)$.Thus we have $\theta[(a^\circ]] \in Con(B(L)$.
Define $^\circ$ on $Con(B(L))$ by $(\theta[(a]])^\circ = \theta[(a^\circ]]$ and

$$\theta\big[(a]\big] \wedge \theta\big[(a^\circ]\big] = \theta\big[(a \wedge a^\circ]\big] = \theta[(0]] = \Delta_{L^{\circ\circ}},$$

$$\theta\big[(a]\big] \vee \theta\big[(a^\circ]\big] = \theta\big[(a \vee a^\circ]\big] = \theta[(1]] = \nabla_{L^{\circ\circ}}.$$

Therefore $\big(Con\big(B(L)\big); \vee, \wedge, ^\circ, \Delta_{L^{\circ\circ}}, \nabla_{L^{\circ\circ}}\big)$ is a Boolean algebra.

We can define a map from $B(L)$ to $Con(B(L))$ such that $a \to \theta[(a]]$.
It is obvious that such map is an isomorphism. Then the Boolean algebra $Con\big(B(L)\big)$ and $B(L)$ are isomorphic.

Definition 23 *A $\underline{K_2}$-algebra L is said to satisfy condition (1) if*

$$(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a), \quad \forall x, y \in L^\vee, \forall a \in B(L). (1)$$

Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$. For $a \in L^{\circ\circ}$, define a relation $\theta[\varphi_L(a)]$on $L^\vee$as follows:

$$(x, y) \in \theta[\varphi_L(a)] \Leftrightarrow x \wedge (a^\circ \vee d) = y \wedge (a^\circ \vee d).$$

Theorem 24 *Let L be a $\underline{K_2}$-algebra satisfying condition (1) and $L^\vee = [d)$. Then $\theta[\varphi_L(a)]$is a congruence relation on $L^\vee$ with $Coker\theta[\varphi_L(a)] = \varphi_L(a), \forall a \in L^{\circ\circ}$.*

Proof. Clearly $\theta[\varphi_L(a)]$ is an

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equivalence relation on $L^{\vee}$. Let $(x,y)$and $(x_1,y_1) \in \theta[\varphi_L(a)]$. Then $x \wedge (a^{\circ} \vee d) = y \wedge (a^{\circ} \vee d)$and $x_1 \wedge (a^{\circ} \vee d) = y_1 \wedge (a^{\circ} \vee d)$.

Now, we show that $\theta[\varphi_L(a)]$ preserves meet and join.

$(x \wedge x_1) \wedge (a^{\circ} \vee d)$
$= (x \wedge x_1 \wedge a^{\circ}) \vee d$,
by modularity with $x \wedge x_1 \geq d$
$= ((x \wedge a^{\circ}) \wedge (x_1 \wedge a^{\circ})) \vee d$
$= \big((x \wedge a^{\circ}) \vee d) \wedge ((x_1 \wedge a^{\circ}) \vee d\big)$,
by distributivity of d
$= (x \wedge (a^{\circ} \vee d) \wedge (x_1 \wedge (a^{\circ} \vee d)$,
by modularity with $x, x_1 \geq d$
$= (y \wedge (a^{\circ} \vee d)) \wedge (y_1 \wedge (a^{\circ} \vee d))$
$= \big((y \wedge a^{\circ}) \vee d) \wedge ((y_1 \wedge a^{\circ}) \vee d\big)$,
by modularity with $y, y_1 \geq d$
$= \big((y \wedge a^{\circ}) \wedge (y_1 \wedge a^{\circ})\big) \vee d$,
$= (y \wedge y_1 \wedge a^{\circ}) \vee d$
$= (y \wedge y_1) \wedge (a^{\circ} \vee d)$,
by modularity with $y \wedge y_1 \geq d$.

Also

$(x \vee x_1) \wedge (a^{\circ} \vee d)$
$= ((x \vee x_1) \wedge a^{\circ}) \vee d$,
by modularity with $x \vee x_1 \geq d$
$= \big((x \wedge a^{\circ}) \vee (x_1 \wedge a^{\circ})\big) \vee d$,
bycondition (1)
$= \big((x \wedge a^{\circ}) \vee d) \vee ((x_1 \wedge a^{\circ}) \vee d\big)$,
by distributivity of d
$= \big(x \wedge (a^{\circ} \vee d)\big) \vee \big(x_1 \wedge (a^{\circ} \vee d)\big)$,
by modularity with $x, x_1 \geq d$
$= (y \wedge (a^{\circ} \vee d)) \vee (y_1 \wedge (a^{\circ} \vee d))$
$= \big((y \wedge a^{\circ}) \vee d) \vee ((y_1 \wedge a^{\circ}) \vee d\big)$,
by modularity with $y, y_1 \geq d$
$= \big((y \wedge a^{\circ}) \vee (y_1 \wedge a^{\circ})\big) \vee d$,
by distributivity of d
$= \big((y \vee y_1) \wedge a^{\circ}\big) \vee d$, by condition(1)
$= (y \vee y_1) \wedge (a^{\circ} \vee d)$,
by modularity with $y \vee y_1 \geq d$.

Then $\theta[\varphi_L(a)]$is a congruence relation on $L^{\vee}$. Also we get

$Coker\theta[\varphi_L(a)]$
$= \{x \in L^{\vee} : (x, 1) \in \theta[\varphi_L(a)]\}$
$= \{x \in L^{\vee} : x \wedge (a^{\circ} \vee d)$
$= 1 \wedge (a^{\circ} \vee d)\}$
$= \{x \in L^{\vee} : x \wedge (a^{\circ} \vee d) = a^{\circ} \vee d\}$
$= \{x \in L^{\vee} : x \geq (a^{\circ} \vee d)\}$
$= [a^{\circ} \vee d) = \varphi_L(a)$.

Lemma 25 *Let L be a $\underline{K_2}$-algebra with $L^{\vee} = [d)$. Then we have*
$(i) L^{\vee} \cap B(L) = \{1\}$ and $L^{\wedge} \cap B(L) = \{0\}$,
(ii) If $\varphi_L(x) = L^{\vee}, x \in B(L)$ then $x = 1$.

Proof. (i) Let $x \in L^{\vee} \cap B(L)$. Then $x \in L^{\vee}, B(L)$. Since $x \in L^{\vee}$, then $x \geq x^{\circ}$ implies $x = x \vee x^{\circ}$. But $x \vee x^{\circ} = 1$ as $x \in B(L)$, therefore $x = 1$ and hence $L^{\vee} \cap B(L) = \{1\}$. Similarly, we can show that
$L^{\wedge} \cap B(L) = \{0\}$.

(ii) Let $\varphi_L(x) = L^{\vee}, x \in B(L)$. Then
$(\varphi_L(x) = L^{\vee})$
$\Rightarrow [x^{\circ}) \cap L^{\vee} = L^{\vee}$
$\Rightarrow [x^{\circ}) \cap d) = [d)$, as $L^{\vee} = [d)$
$\Rightarrow [x^{\circ} \vee d) = [d)$
$\Rightarrow x^{\circ} \vee d = d$
$\Rightarrow x^{\circ} \leq d \in L^{\vee}$.
Since $x^{\circ} \leq d$, then $x^{\circ} \in L^{\wedge}$. Hence $x^{\circ} \in L^{\wedge} \cap B(L)$ implies $x^{\circ} = 0$. Therefore $x = 1$.

Lemma 26 *Let L be a $\underline{K_2}$-algebra, with $L^{\vee} = [d)$ satisfying condition* (1). *Then we have the following:*
(1) $a \leq b \Leftrightarrow \theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$,
(2) $a = b \Leftrightarrow \theta[\varphi_L(a)] = \theta[\varphi_L(b)]$,
(3) $\theta[\varphi_L(0)] = \Delta_{L^{\vee}}, \theta[\varphi_L(1)] = \nabla_{L^{\vee}}$.

Proof. (1) Let $a \leq b$. Then $a^{\circ} \geq b^{\circ}$. Let $(x,y) \in \theta[\varphi_L(a)]$. Then $x \wedge (a^{\circ} \vee d) = y \wedge (a^{\circ} \vee d)$. We have
$x \wedge (b^{\circ} \vee d) = x \wedge \big((a^{\circ} \wedge b^{\circ}) \vee d\big)$,
as$a^{\circ} \geq b^{\circ}$

$= x \wedge (a^\circ \vee d) \wedge (b^\circ \vee d)$,
by the distributivity of d
$= y \wedge (a^\circ \vee d) \wedge (b^\circ \vee d)$
$= y \wedge ((a^\circ \wedge b^\circ) \vee d)$
$= y \wedge (b^\circ \vee d)$, as $a^\circ \geq b^\circ$.
Therefore $(x,y) \in \theta[\varphi_L(b)]$. Hence $\theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$.
Conversely, suppose that $\theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$. Since $((a \wedge b) \vee d) \wedge (a^\circ \vee d) = (a \vee d) \wedge (a^\circ \vee d)$, then $((a \wedge b) \vee d, a \vee d) \in \theta[\varphi_L(a)]$. Then we have
$\left((a \wedge b) \vee d, a \vee d\right) \in \theta[\varphi_L(b)]$, as $\theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$. This implies that $\left((a \wedge b) \vee d\right) \wedge (b^\circ \vee d)$

$$= (a \vee d) \wedge (b^\circ \vee d)$$
$$(a \wedge b \wedge b^\circ) \vee d = (a \wedge b^\circ) \vee d$$
$$(a \wedge 0) \vee d = (a \wedge b^\circ) \vee d$$
$$d = (a \wedge b^\circ) \vee d$$
$$[d) = [(a \wedge b^\circ) \vee d)$$
$$L^\vee = \varphi_L((a \wedge b^\circ)^\circ)$$

Thus we get $(a \wedge b^\circ)^\circ = 1$, by Lemma 25, (ii). Then $a \wedge b^\circ = 0$.
Also, we have
$a^\circ = a^\circ \vee 0$
$= a^\circ \vee (a \wedge b^\circ)$
$= (a^\circ \vee a) \wedge (a^\circ \vee b^\circ)$
$= 1 \wedge (a^\circ \vee b^\circ)$
$= a^\circ \vee b^\circ$.
Therefore $a^\circ \geq b^\circ$ and $a \leq b$.
(2) Let $a = b$ and $(x,y) \in \theta[\varphi_L(a)]$. Then $x \wedge (a^\circ \vee d) = y \wedge (a^\circ \vee d)$. This implies that $x \wedge (b^\circ \vee d) = y \wedge (b^\circ \vee d)$ and $(x,y) \in \theta[\varphi_L(b)]$. Therefore $\theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$. Similarly, one can get $\theta[\varphi_L(a)] \subseteq \theta[\varphi_L(b)]$. Then $\theta[\varphi_L(a)] = \theta[\varphi_L(b)]$.
Conversely, let $\theta[\varphi_L(a)] = \theta[\varphi_L(b)]$. From (1), we get $a \leq b$ and $a \geq b$. Then $a = b$.
(3) For any $(x,y) \in \theta[\varphi_L(0)]$, we have $x \wedge (0^\circ \vee d) = y \wedge (0^\circ \vee d)$. Implies $x =$ y. Therefore $\theta[\varphi_L(0)] = \Delta_{L^\vee}$. For all $x, y \in L^\vee$, we have

$$x \wedge (1^\circ \vee d) = d = y \wedge (1^\circ \vee d).$$

Then $(x,y) \in \theta[\varphi_L(1)]$ and hence $\theta[\varphi_L(1)] = \nabla_{L^\vee}$.

Lemma 27 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$ satisfying condition (1). Then $\forall a, b \in L^{\circ\circ}$ we have the following:*
(1) $\forall a, b \in L^{\circ\circ}$
$\theta[\varphi_L(a)] \wedge \theta[\varphi_L(b)] = \theta[\varphi_L(a \wedge b)]$,
(2) $\forall a, b \in L^{\circ\circ}$
$\theta[\varphi_L(a)] \vee \theta[\varphi_L(b)] = \theta[\varphi_L(a \vee b)]$,
(3) The set $H = \{\theta[\varphi_L(a)] : a \in B(L)\}$ is a Boolean algebra on its own.

Proof. (1) Since $a \wedge b \leq a, b$, then $\theta[\varphi_L(a \wedge b)] \subseteq \theta[\varphi_L(a)], \theta[\varphi_L(b)]$. Therefore $\theta[\varphi_L(a \wedge b)]$ is a lower bound of $\theta[\varphi_L(a)]$ and $\theta[\varphi_L(b)]$. Assume that $\theta[\varphi_L(c)]$ is a lower bound of $\theta[\varphi_L(a)]$ and $\theta[\varphi_L(b)]$. Then
$\theta[\varphi_L(c)] \subseteq \theta[\varphi_L(a)], \theta[\varphi_L(b)]$
$\Rightarrow c \leq a, b$, by lemma (25).
Hence c is a lower bound of a and b. Since $a \wedge b$ is the greatest lower bound of a and b, then $c \leq a \wedge b$ this implies that $\theta[\varphi_L(c)] \subseteq \theta[\varphi_L(a \wedge b)]$. Therefore $\theta[\varphi_L(a \wedge b)]$ is the greatest lower bound of $\theta[\varphi_L(a)]$ and $\theta[\varphi_L(b)]$. Then
$\theta[\varphi_L(a)] \wedge \theta[\varphi_L(b)] = \theta[\varphi_L(a \wedge b)]$.
(2) By duality of (1).
(3) Since $\theta[\varphi_L(0)] = \Delta_{L^\vee}$ and $\theta[\varphi_L(1)] = \nabla_{L^\vee}$, then $\Delta_{L^\vee}, \nabla_{L^\vee} \in H$. Let $\theta[\varphi_L(a)]$, $\theta[\varphi_L(b)] \in H$. Then we have

$$\theta[\varphi_L(a)] \wedge \theta[\varphi_L(b)] = \theta[\varphi_L(a \wedge b)] \in H,$$

And

$$\theta[\varphi_L(a)] \vee \theta[\varphi_L(b)] = \theta[\varphi_L(a \vee b)] \in H.$$

Therefore H is a bounded lattice. Since $B(L)$ is a Boolean algebra, then

$a^\circ \in B(L)$ for every $a \in B(L)$. Thus $\theta[\varphi_L(a^\circ)] \in H$. Define $'$ on H as $\theta[\varphi_L(a)]' = \theta[\varphi_L(a^\circ)]$ then we have

$$\theta[\varphi_L(a)] \wedge \theta[\varphi_L(a)]' = \theta[\varphi_L(a)] \wedge \theta[\varphi_L(a^\circ)] = \theta[\varphi_L(a \wedge a^\circ)] = \theta[\varphi_L(0)] = \Delta_{L^\vee},$$

and

$$\theta[\varphi_L(a)] \vee \theta[\varphi_L(a)]' = \theta[\varphi_L(a)] \vee \theta[\varphi_L(a^\circ)] = \theta[\varphi_L(a \vee a^\circ)] = \theta[\varphi_L(1)] = \nabla_{L^\vee}.$$

Then $(H; \vee, \wedge, ', \Delta_{L^\vee}, \nabla_{L^\vee})$ is a Boolean algebra.

Theorem 28 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$ satisfying condition (1). Then for every $a \in B(L)$, a pair $(\theta[(a]], \theta[\varphi_L(a)])$ is a $\underline{K_2}$-congruence pair.*

Proof. Let $x, y \in L^\vee$. In order to prove (CP_1), let $(x,y) \in \theta[\varphi_L(a)]$. Then by distributivity of $L^{\circ\circ}$, we get

$(x,y) \in \theta[\varphi_L(a)]$
$\Rightarrow x \wedge (a^\circ \vee d) = y \wedge (a^\circ \vee d)$
$\Rightarrow (x \wedge (a^\circ \vee d))^\circ = (y \wedge (a^\circ \vee d))^\circ$
$\Rightarrow x^\circ \vee (a^{\circ\circ} \wedge d^\circ) = y^\circ \vee (a^{\circ\circ} \wedge d^\circ)$
$\Rightarrow (x^\circ \vee (a \wedge d^\circ)) \wedge a^\circ = (y^\circ \vee (a \wedge d^\circ)) \wedge a^\circ$, as $a = a^{\circ\circ}$
$\Rightarrow (x^\circ \wedge a^\circ) \vee (a \wedge d^\circ \wedge a^\circ) = (y^\circ \wedge a^\circ) \vee (a \wedge d^\circ \wedge a^\circ)$,
$\Rightarrow (x^\circ \wedge a^\circ) \vee (d^\circ \wedge 0) = (y^\circ \wedge a^\circ) \vee (d^\circ \wedge 0)$, as $a \wedge a^\circ = 0$
$\Rightarrow x^\circ \wedge a^\circ = y^\circ \wedge a^\circ$
$\Rightarrow (x^\circ, y^\circ) \in \theta[(a]]$.

For (CP_2), let $(b,c) \in \theta[(a]]$. Then

$(b,c) \in \theta[(a]]$
$\Rightarrow b \wedge a^\circ = c \wedge a^\circ$
$\Rightarrow (b \wedge a^\circ) \vee d = (c \wedge a^\circ) \vee$
$\Rightarrow (b \vee d) \wedge (a^\circ \vee d) = (c \vee d) \wedge (a^\circ \vee d)$, by distributivity of d
$\Rightarrow (b \vee d, c \vee d) \in \theta[\varphi_L(a)]$.

Therefore a pair $(\theta[(a]], \theta[\varphi_L(a)])$ is a $\underline{K_2}$-congruence pair of L.

We denote the set of all $\underline{K_2}$-congruence pairs of the form $(\theta[(a]], \theta[\varphi_L(a)]), a \in B(L)$ by $\grave{A}(L)$, that is $\grave{A}(L) = \{(\theta[(a]], \theta[\varphi_L(a)]) : a \in B(L)\}$.

Theorem 29 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$ satisfying condition(1). Then $\grave{A}(L)$ is a bounded sublattice of $A(L)$. Moreover, $\grave{A}(L)$ is a Boolean algebra on its own.*

Proof. Since $(\theta[(0]], \theta[\varphi_L(0)]) = (\Delta_{L^{\circ\circ}}, \Delta_{L^\vee})$ and $(\theta[(1]], \theta[\varphi_L(1)]) = (\nabla_{L^{\circ\circ}}, \nabla_{L^\vee})$, then $(\Delta_{L^{\circ\circ}}, \Delta_{L^\vee}), (\nabla_{L^{\circ\circ}}, \nabla_{L^\vee}) \in \grave{A}(L)$.

Let $(\theta[(a]], \theta[\varphi_L(a)]), (\theta[(b]], \theta[\varphi_L(b)]) \in \grave{A}(L)$. Then by using Lemma 22 and 27, we have

$$(\theta[(a]], \theta[\varphi_L(a)]) \vee (\theta[(b]], \theta[\varphi_L(b)]) = (\theta[(a]] \vee \theta[(b]], \theta[\varphi_L(a)] \vee \theta[\varphi_L(b)]) = (\theta[(a \vee b]], \theta[\varphi_L(a \vee b)]) \in \grave{A}(L),$$

and

$$(\theta[(a]], \theta[\varphi_L(a)]) \wedge (\theta[(b]], \theta[\varphi_L(b)]) = (\theta[(a]] \wedge \theta[(b]], \theta[\varphi_L(a)] \wedge \theta[\varphi_L(b)]) = (\theta[(a \wedge b]], \theta[\varphi_L(a \wedge b)]) \in \grave{A}(L).$$

Then $\grave{A}(L)$ is a bounded sublattice of $A(L)$. Define a unary operation $^\diamond$ on $\grave{A}(L)$ by $(\theta[(a]], \theta[\varphi_L(a)])^\diamond = (\theta[(a^\circ]], \theta[\varphi_L(a^\circ)]) \in \grave{A}(L)$.

In order to show that $\grave{A}(L)$is a Boolean algebra, we have

$$(\theta[(a]], \theta[\varphi_L(a)]) \vee (\theta[(a]], \theta[\varphi_L(a)])^\diamond$$

$= \left(\theta[(a]] \vee \theta[(a^\circ]], \theta[\varphi_L(a)] \vee \theta[\varphi_L(a^\circ)]\right)$
$= (\theta[(a \vee a^\circ]], \theta[\varphi_L(a \vee a^\circ)])$
$= (\theta[(1]], \theta[\varphi_L(1)]) = (\nabla_{L^{\circ\circ}}, \nabla_{L^\vee})$,
and
$\left(\theta[(a]], \theta[\varphi_L(a)]\right) \wedge \left(\theta[(a]], \theta[\varphi_L(a)]\right)^\diamond$
$= (\theta[(a]], \theta[\varphi_L(a)]) \wedge (\theta[(a^\circ]], \theta[\varphi_L(a^\circ)])$
$= \left(\theta[(a]] \wedge \theta[(a^\circ]], \theta[\varphi_L(a)] \wedge \theta[\varphi_L(a^\circ)]\right)$
$= (\theta[(a \wedge a^\circ]], \theta[\varphi_L(a \wedge a^\circ)])$
$= (\theta[(0]], \theta[\varphi_L(0)]) = (\Delta_{L^{\circ\circ}}, \Delta_{L^\vee})$.
Then
$\left(\grave{A}(L); \vee, \wedge, ^\diamond, (\Delta_{L^{\circ\circ}}, \Delta_{L^\vee}), (\nabla_{L^{\circ\circ}}, \nabla_{L^\vee})\right)$ is a Boolean algebra.

5 Permutability of $\underline{K}_2$-Algebras

In this section, we introduce the notion of n-permutability of congruences of $\underline{K_2}$- algebras in terms of $\underline{K_2}$-congruence pairs.

Definition 30 *A $\underline{K_2}$-algebra L is said to have n-permutable congruences if every two congruences α, β on L are n-permutable, that is $\alpha \circ \beta \circ \alpha \circ \ldots = \beta \circ \alpha \circ \beta \circ \ldots, (n\ times)$.*

Lemma 31 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$. Consider α, β be two congruences on L. Then we have*
$(i)(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^{\circ\circ}} = \alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \ldots, (n \text{ times})$,
$(ii)(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^\vee} = \alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \ldots, (n \text{ times})$.

Proof. (i) Since $\alpha_{L^{\circ\circ}}$ and $\beta_{L^{\circ\circ}}$ are the restrictions of α and β to $L^{\circ\circ}$, respectively, then it is clear that
$\alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \ldots \subseteq (\alpha \circ \beta \circ \alpha \circ \ldots)_{L^{\circ\circ}}$.

Conversely, let $a, b \in L^{\circ\circ}$ such that $a \equiv b(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^{\circ\circ}}$. Then there exist $c_1, c_2, \ldots, c_{n-1} \in L$ such that
$a \equiv c_1(\alpha), c_1 \equiv c_2(\beta), \ldots, c_{n-1} \equiv b(\alpha)$ if n is odd, or
$a \equiv c_1(\alpha), c_1 \equiv c_2(\beta), \ldots, c_{n-1} \equiv b(\beta)$ if n is even.
Then we have
$a = a^{\circ\circ} \equiv c_1^{\circ\circ}(\alpha), c_1^{\circ\circ} \equiv c_2^{\circ\circ}(\beta), \ldots, c_{n-1}^{\circ\circ} \equiv b^{\circ\circ}(\alpha) = b$ if n is odd,
or
$a = a^{\circ\circ} \equiv c_1^{\circ\circ}(\alpha), c_1^{\circ\circ} \equiv c_2^{\circ\circ}(\beta), \ldots, c_{n-1}^{\circ\circ} \equiv b^{\circ\circ}(\beta) = b$ if n is even.
Since $c_1^{\circ\circ}, c_2^{\circ\circ}, \ldots, c^{\circ\circ}_{n-1} \in L^{\circ\circ}$, then
$a \equiv c_1^{\circ\circ}(\alpha_{L^{\circ\circ}}), c_1^{\circ\circ} \equiv c_2^{\circ\circ}(\beta_{L^{\circ\circ}}), \ldots, c_{n-1}^{\circ\circ} \equiv b(\alpha_{L^{\circ\circ}})$ if n is odd, or
$a \equiv c_1^{\circ\circ}(\alpha_{L^{\circ\circ}}), c_1^{\circ\circ} \equiv c_2^{\circ\circ}(\beta_{L^{\circ\circ}}), \ldots, c_{n-1}^{\circ\circ} \equiv b(\beta_{L^{\circ\circ}})$ if n is even.
Therefore $a \equiv b(\alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \ldots)$ and hence $(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^{\circ\circ}} \subseteq \alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \ldots$.
Thus we get $(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^{\circ\circ}} = \alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \ldots, (n \text{ times})$.
(ii) Clearly, $\alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \ldots \subseteq (\alpha \circ \beta \circ \alpha \circ \ldots)_{L^\vee}$. Conversely, let $x, y \in L^\vee$ such that $x \equiv y(\alpha \circ \beta \circ \alpha \circ \ldots)_{L^\vee}$. Then there exist $z_1, z_2, \ldots, z_{n-1} \in L$ such that
$x \equiv z_1(\alpha), z_1 \equiv z_2(\beta), \ldots, z_{n-1} \equiv y(\alpha)$ if n is odd, or
$x \equiv z_1(\alpha), z_1 \equiv z_2(\beta), \ldots, z_{n-1} \equiv y(\beta)$ if n is even.

Thus we have
$x = x \vee d \equiv z_1 \vee d(\alpha), z_1 \vee d \equiv z_2 \vee d(\beta), \ldots, z_{n-1} \vee d \equiv y \vee d(\alpha) = y$, if n is odd, or
$x = x \vee d \equiv z_1 \vee d(\alpha), z_1 \vee d \equiv z_2 \vee d(\beta), \ldots, z_{n-1} \vee d \equiv y \vee d(\beta) = y$, if n is even.
Since $z_1 \vee d, z_2 \vee d, \ldots, z_{n-1} \vee d \in L^\vee$, then
$x \equiv z_1 \vee d(\alpha_{L^\vee}), z_1 \vee d \equiv z_2 \vee d(\beta_{L^\vee}), \ldots, z_{n-1} \vee d \equiv y(\alpha_{L^\vee})$, if n is odd,
or

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$x \equiv z_1 \vee d(\alpha_{L^\vee}), z_1 \vee d \equiv z_2 \vee d(\beta_{L^\vee}),$
$\dots, z_{n-1} \vee d \equiv y(\beta_{L^\vee})$, if n is even.
Hence $x \equiv y(\alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \dots)$ and then $(\alpha \circ \beta \circ \alpha \circ \dots)_{L^\vee} \subseteq \alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \dots$. Therefore $(\alpha \circ \beta \circ \alpha \circ \dots)_{L^\vee} = \alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \dots$, ($n$ times).

Now, a characterization of n-permutable of congruences is given in the following Theorem.

Theorem 32 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$. Then L has n-permutable congruences iff $L^{\circ\circ}$ and $L^\vee$ have n-permutable congruences.*

Proof. Let L has n-permutable congruences. Let α, β be any two congruences on L. Then by lemma 31(i) and (ii), respectively, we obtain that

$$\alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ = (\alpha \circ \beta \circ \alpha \circ \dots)_{L^{\circ\circ}}$$
$$= (\beta \circ \alpha \circ \beta \circ \dots)_{L^{\circ\circ}}$$
$$= \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \dots,$$

and

$$\alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \dots = (\alpha \circ \beta \circ \alpha \circ \dots)_{L^\vee}$$
$$= (\beta \circ \alpha \circ \beta \circ \dots)_{L^\vee}$$
$$= \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \beta_{L^\vee} \dots.$$

Therefore $L^{\circ\circ}$ and $L^\vee$ have n-permutable congruences. Conversely, Let $L^{\circ\circ}$ and $L^\vee$ have n-permutable congruences. If $a \equiv b(\alpha \circ \beta \circ \alpha \circ \dots)$, then by Theorem 12, $(\alpha \circ \beta \circ \alpha \circ \dots)$ corresponds to the $\underline{K_2}$-congruence pair $((\alpha \circ \beta \circ \alpha \circ \dots)_{L^{\circ\circ}}, (\alpha \circ \beta \circ \alpha \circ \dots)_{L^\vee})$. Therefore

$$a^\circ \equiv b^\circ(\alpha \circ \beta \circ \alpha \circ \dots)_{L^{\circ\circ}} \text{ and}$$
$$a \vee d \equiv b \vee d(\alpha \circ \beta \circ \alpha \circ \dots)_{L^\vee}.$$

By Lemma 31(i) and (ii), respectively, we get

$$a^\circ \equiv b^\circ(\alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \dots) \text{and}$$
$$a \vee d \equiv b \vee d(\alpha_{L^\vee} \circ \beta_{L^\vee} \circ \alpha_{L^\vee} \circ \dots).$$

Since $\alpha_{L^{\circ\circ}}, \beta_{L^{\circ\circ}}$ and $\alpha_{L^\vee}, \beta_{L^\vee}$ are n-permutable on $L^{\circ\circ}$ and $L^\vee$, respectively, then

$$a^\circ \equiv b^\circ(\beta_{L^{\circ\circ}} \circ \alpha_{L^{\circ\circ}} \circ \beta_{L^{\circ\circ}} \circ \dots) \text{ and}$$
$$a \vee d \equiv b \vee d(\beta_{L^\vee} \circ \alpha_{L^\vee} \circ \beta_{L^\vee} \circ \dots).$$

Again by Lemma 31(i) and (ii), respectively, we have

$$a^\circ \equiv b^\circ(\beta \circ \alpha \circ \beta \circ \dots)_{L^{\circ\circ}} \text{ and}$$
$$a \vee d \equiv b \vee d(\beta \circ \alpha \circ \beta \circ \dots)_{L^\vee}.$$

Then by using Theorem 12, we obtain that $a \equiv b(\beta \circ \alpha \circ \beta \circ \dots.)$ and hence $\alpha \circ \beta \circ \alpha \circ \dots \subseteq \beta \circ \alpha \circ \beta \circ \dots$. Similarly, we can get $\beta \circ \alpha \circ \beta \circ \dots \subseteq \alpha \circ \beta \circ \alpha \circ \dots$. Therefore $\alpha \circ \beta \circ \alpha \circ \dots = \beta \circ \alpha \circ \beta \circ \dots$ and hence L has n-permutable congruences.

For 2-permutability of congruences we have the following:

Corollary 33 *Let L be a $\underline{K_2}$-algebra with $L^\vee = [d)$. Then L has 2-permutable congruences if and only if $L^{\circ\circ}$ and $L^\vee$ have 2-permutable congruences.*

If $L \in \underline{S}$, then $L^{\circ\circ}$ is a Boolean algebra and $L^\vee = D(L)$, by Lemma 15. It is known that each Boolean algebra has n-permutable congruences. So, we have the following:

Corollary 34 *Let $L \in \underline{S}$ with $L^\vee = [d)$. Then L has n-permutable congruences if and only if $L^\vee$ has n-permutable congruences.*

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