A STUDY ON FIXED POINT THEOREM IN CYCLIC CONTRACTION MAPPING IN METRIC SPACES

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ABSTRACT:
In this article we have proved some fixed point theorems for cyclic mapping with contractive and expansive conditions in Banach space. The presented results generalize the results proved in variou spaces and extend some results from the literature.

KEYWORDS: Fixed point theorem, Banach space, Cyclic Contration mapping, metric space
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INTRODUCTION

Many fields of mathematical analysis and its applications use fixed point theory as a particularly effective technique to solve existence problems. The study of the existence and uniqueness of solutions for a class of nonlinear integral equations has been conducted in physics and engineering using fixed point techniques in areas such as image retrieval, signal processing, and solution analysis. [17, 18] and the references therein provide some current research on fixed point theorems of integral type in G-metric spaces and stability of functional difference equations.

Kannan (see [6]) established a fixed point theorem in 1968 for operators that are not required to be continuous. Additionally, Chatterjea([4]) established a fixed point theory for discontinuous mapping—actually a dual of Kannan mapping—in 1972. In 1972, Zamfirescu ([20]) combined the three independent contraction conditions listed above to provide another fixed point result for operators that meet the criteria listed below. Rus ([17]) defined "contraction" in 2001. The author of [2] not only discovered a new contraction condition, but also developed a matching fixed point theorem. Berinde ([3]) discovered some results on contraction for an approximation of a fixed point in metric space in 2006. Some findings of approximating fixed points in metric space were made by Miandaragh et al. [7,8].

On the other hand, generalised metric spaces, or simply G-metric spaces, were first presented by Mustafa and Sims [15, 16] in 2006. Numerous scholars have discovered results for fixed point, linked fixed point, and coupled common fixed point in G-metric spaces (see [1,4, 17]).

The approximate best proximity pairs were first introduced in 2011 by Mohsenalhosseini et al [9], who also demonstrated the property of the approximate best proximity pairs. Additionally, Mohsenalhosseini et al. [10] proposed the approximation fixed point for the entirety of the norm space and map T and established the approximation fixed point condition for it in 2012. For contraction maps, Mohsenalhosseini [11] presented the Approximate best proximity pairs on metric space in 2014. Additionally, Mohsenalhosseini introduced the approximation fixed point in G-metric spaces for various.

A rough fixed point in G-metric spaces for different kinds of operators was also introduced by Mohsenalhosseini in [12]. The approximative fixed points of operators on G-metric spaces were recently introduced by Mohsenialhosseini [13] in 2017. This study introduces two new types of operators and contraction maps (not necessarily continuous) for cyclical contraction mapping on G-metric spaces, namely, diameter approximate fixed point and approximate fixed point with respect to the diameter. Additionally, we provide several examples that illustrate our key findings.

Consider a metric space P and its nonempty subset U. If there is a constant l $\in$[0, 1] such that g(Da, Db) ≤ k g(a, b) for every a,b$\in$X, then we know that a mapping D: X $\to$A is said to be a contraction.

Additionally, D is considered to be a nonexpansive if g(Da, Db) ≤ g(a, b), where a,b$\in$ X for all. It is evident that every mapping of a contraction is a nonexpansive, but not the other way around. Accordingly, the class of contration mappings. Is is intriguing to learn that there is a separate category to learn that there is a separate category of mappings called contractive mappigs called contractive mappings, which rightfully contains the class of contraction and belongs to the nonexpansive mappings category. For all a,b$\in$ X with g(Da, Db) ≤ g(a, b), a mapping D : X $\to$ A is said to be a contracive. If there are contractive mapping fixed points, it is simple to confirm that they are distinct.

Every contractive self mapping on a compact subset of a metric space has a unique fixed point, and Edelstein demonstrated this in showing that the sequence of subsequent iterations (Dna), starting at any a $\in$ X, converges to the unique fixed point of D. Later, Smithson used multivalued contractive mappings to Edelstein's fixed point theorem in the manner described below.

Let D: A $\to$ 2^A be a multivalued contractive mapping such that Da is closed for all a $\in$A, and let A be a compact metric space. As a result, D has a fixed point.

PRELIMINARIES

Theorem1:*LetUandVbenonemptysubsetsofmetricspaces(A,g)andacyclicmappingD*:*U*∪*V*→ *U*∪*Vsatisfiesg*(*Da,Db*)≤*l*[*D*(*a,Da*)+*g*(*b,Db*)]∀*a*∈*U,b*∈*V* *where 0≤ l ≤ ½. ThenDhasauniquefixedpointinU*∩*V.*

Theorem2:*LetUandVbeanonemptysubsetsofmetricspace*(*A,g*)*andacyclicmappingD*:*U*

$\cup V \to U \cup V$ *satisfies* $g(Da,Db) \leq l[g(a,Da)+g(b,Da)]\ \forall a \in U, b \in V$.

Definition 3: *Let B be subset of a Banach space A. An operator D defined on B is said to belong to class* $T(x,y)$ *if* $\|Da-Db\| \leq x\|a-b\|+y[\|a-Da\|+\|b-Db\|]$ *for all a and b in l, where, if an operator D is in class* $T(l,0)$ *with* $0<l<1$*, then D is a contraction with* $0<l<1$.

Definition 4: *Let* Q_1 *and* Q_2 *be closed subsets of a Banach space D. An operator D defined on* Q_i *is said to belong to class* $T(x,y,z)$ *if* $\|Da-Db\| \leq x\|a-b\|+y[\|a-Db\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]$ *for all a and b in Q where* $0 \leq x,y,z \leq 1, x+2y+2z \leq 1$ *and* $y>0$.

Definition 5: *Let* Q_1 *and* Q_2 *be closed subsets of a Banach space A. An operator* $D: Q_1 \cup Q_2 \to Q_1 \cup K_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *is said to belong to class* $T(x,y)$ *if it satisfies* $\|Da-Db\| \leq x\|a-b\|+y[\|a-Da\|+\|b-Db\|]$ *for all* $a \in Q_1$ *and* $b \in Q_2$*, where* $0 \leq x,y \leq 1$. *It is clear that if D belongs to the class* $T(l,0)$ *with* $0<l<1$*, then D is a cyclic contraction.*

Definition 6: *Let* Q_1 *and* Q_2 *be closed subsets of a Banach space A. An operator* $D: Q_1 \cup Q_2 \to Q_1 \cup Q_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *is said to belong to class* $T(x,y,z)$ *if it satisfies* $\|Da-Db\| \leq x\|a-b\|+y[\|a-Da\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]$ *For all* $a \in Q_1$ *and* $b \in Q_2$ *where* $0 \leq x,y \leq 1$. *It is clear that if D belongs to the class* $T(l,0,0)$ *with* $0<l<1$. *Then D is cyclic contraction.* 1488

Definition 7: *A function* $\Phi:[0,\infty)^2 \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

(i) *Φ is non-decreasing and continuous*

(ii) $\Phi(u)=0$ *if and only if* $u=0$

Definition 8: *An ultra altering distance function is a continuous, nondecreasing mapping* $\psi:[0,\infty) \to [0,\infty)$ *such that* $\psi(u)>0, u>0$ *and* $\psi(0) \geq 0$. *We denote this set with* ψ_u.

MAINRESULTS

Proposition 9: *Let* Q_1 *and* Q_2 *be closed subsets of a Banach space A. an operator* $D: Q_1 \cup Q_2 \to Q_1 \cup Q_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *and satisfies*
$\|Da-Db\| \leq x\|a-b\|+y[\|a-Da\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]$
with $0 \leq x,y,z<1$ *and* $x+2y+2z<1$. *If* $R(D)=\{x \in Q_1 \cup Q_2, Dx=x\} /= \psi$*, then* $R(D)$ *consists of a single point.*

Proof. Assume the contrary, that is, let $x, w \in Q_1 \cup Q_2$ be two distinct fixed points of D, Then

$\|c-w\|=\|Dc-Dw\| \leq x\|c-w\|+ y[\|c-Db\|+ \|w-Dw\|] + z[\|c-Dw\|+ \|Dc-w\|]=(x + 2z)\|c-w\|$
Which implies $c=w$, since $x+2z<1$.

Theorem10: *Let* Q_1 *and* Q_2 *beclosedsubsetsofaBanachspaceA. SupposethattheoperatorD*:$Q_1 \cup Q_2 \to Q_1 \cup Q_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *andsatisfies*
$\psi(\|Da-Db\|) \leq R(\psi(x\|a-b\|+y[\|a-Da\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]),$
$\phi(a\|a-b\|+y[\|a-Da\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]))$
with $0 \leq x,y,z<1$ *and* $x + 2y + 3z \leq 1$. Φ *and* ψ *are altering distance and ultra altering distance functions respectively,* $R \in X$ *suchthat* $\psi(u+v) \leq \Phi(t) + \Phi(s)$. *Thenthesequence* $\{a_n\}$ *in* $Q_1 \cup Q_2$ *satisfies* $\lim_{n\to\infty} \|a_n - Da_n\| = 0$ *and thesequence* $\{a_n\}$ *convergestotheuniquefixedpointofD.*

Corollary11: *Let* Q_1 *and* Q_2 *beclosedsubsetsofaBanachspaceA.SupposethattheoperatorD*:$Q_1 \cup Q_2 \to Q_1 \cup Q_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *andsatisfies*

$\psi(\|Da-Db\|) \leq R(\psi(x\|a-b\|+y[\|a-Da\|+\|b-Db\|]), \phi(x\|a-b\|+y[\|a-Da\|+\|b-Db\|])$ *with* $0 \leq x,y<1$ *and* $x+2y=1$. Φ *and* ψ *andarealteringdistanceandultraalteringdistancefunctionsrespectively,* $R \in X$ *suchthat* $\Phi(u+v) \leq \Phi(u)+\Phi(v)$. *Thenthesequence* $\{a_n\}$ *in* $Q_1 \cup Q_2$ *satisfies* $\lim_{n\to\infty}(a_n - Da_n)=0$ *andthesequence* $\{a_n\}$ *convergestotheuniquefixedpointofD.*

Proof. Theproofofcorollaryfollowsimmediate,bytaking$c=0$intheabovetheorem.

Theorem12: *Let* Q_1 *and* Q_2 *beclosedsubsetsofaBanachspaceA.SupposethattheoperatorD*:$Q_1 \cup Q_2 \to Q_1 \cup Q_2$ *with* $D(Q_1) \subset Q_2$ *and* $D(Q_2) \subset Q_1$ *andsatisfies*

$\Phi\|Da- Db\|) \leq R(\psi(x\|a- b\| + y[\|a- Da\| + \|b- Db\|] + z[\|a- Db\|+ \|Da- b\|]), \psi(x\|a-b\|+y[\|a-Da\|+\|b-Db\|]+z[\|a-Db\|+\|Da-b\|]))$
with $0 \leq x,y,z<1$ *and* $x+2y+3z<1$. *Then*

(i) *Dhasauniquefixedpointrin* $Q_1 \cap Q_2$.
(ii) $\|Da-r\|<\|a-r\|$ *forall* $x \in Q_1 \cup Q_2$ *whereristhefixedpointofD.*

Proof.

(i) Takeapoint$a_0 \in Q_1$.Define$a_{n+1}= Da_n$for$n= 0,1,2,...$thenwehave

$\Phi(\|a_{n+1}-Da_{n+1}\|)=\Phi(\|Da_n - Da_{n+1}\|)$
$\leq R(\Phi(x\|a_n-a_{n+1}\|+ y[\|a_n - Da_n\|+ \|a_{n+1} - Da_{n+1}\|]$
$+z[\|a_n-Da_{n+1}\|+\|Da_n-a_{n+1}\|]), \psi$
$(x\|a_n-a_{n+1}\|+y[\|a_n-Da_n\|+\|a_{n+1}-Da_{n+1}\|]+z[\|a_n-Da_{n+1}\|+\|Da_n-a_{n+1}\|]))$

Thisinequalityimplies

$\leq R(\Phi(x\|a_n-a_{n+1}\|+y[\|a_n-Da_n\|+\|a_{n+1}-Da_{n+1}\|]+z[\|a_n-Da_{n+1}\|+\|Da_n-a_{n+1}\|]),$
$\psi(x\|a_n-a_{n+1}\|+y[\|a_n-Da_n\|+\|a_{n+1}-Da_{n+1}\|]$
$+z[\|a_n-Da_{n+1}\|+\|Da_n-a_{n+1}\|]))$

Φ (‖a_{n+1} − Da_{n+1}‖) ≤ R (Φ ((x+y+z)‖a_n− Da_{n+1}‖ + (y+z)‖a_{n+1} − Da_{n+1}‖), ψ ((x=y+z)‖a_n−Da_n‖+(y+z)‖a_{n+1}−Da_{n+1}‖))

Hence,weobtain‖a_n −Da_n‖→0as$n$→∞,implyingthatlim$_{n\to\infty}a_n=r$,whereristhefixedpointof$D$.

Sincethe subsequence {a_{2n}} ∈$Q_1$and the subsequence {a_{2n+1}} ∈Q_2, then r ∈Q_1∩ Q_2.The uniqueness follows fromProposition.

(ii) Let$r$bethefixedpointof$D$and$a$∈Q_1∩Q_2. Thenusing(4)andthetriangleinequality,wehave

Φ (‖Da−r‖)≤ Φ (‖Da−Dr‖+Dr−r‖)≤ R (Φ (x‖a− r‖ + y[‖a− Da‖ + ‖r− Dr‖] + z[‖a− Dr‖ + ‖Da− r‖]), ψ(x‖a−r‖+x[‖a−Da‖+‖r−Dr‖] +z[‖a−Dr‖+‖Da−r‖]))
≤R(Φ (x‖a−r‖+y[‖a−r‖+ ‖r−Da‖]+ z[‖a−r‖+‖Da−r‖]), ψx‖a−r‖+y[‖y−r‖+‖r−Da‖]+z[‖a−r‖+‖Da−r‖]))
Theinequalityimplies

≤R(Φ ((x+y+z)‖a−r‖+(y+z)‖Da−r‖), ψ ((x+y+ z)‖a−r‖+(y+z)‖Da−r‖))‖Da−r‖≤ (x+y+z/1-y-z)‖a−r‖<‖a−r‖
As (x+y+z/1-y-z)<1,whichcompletestheproof.

Conclusion

For cyclic mapping in Banach space with contractive and expansive conditions, we have established a few fixed point theorems. The results that are provided expand certain findings from the literature and generalise those that have been demonstrated in diverse contexts.

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