Numerical Analysis in Entanglement Swapping Protocols

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Abstract

Entanglement has recently been one of the most essential elements in the development of various quantum technologies. In fact, a swapping protocol was introduced to create a long-distance entanglement from multiple shorter ones. Extending the previous work, this paper provides a more detailed numerical analysis to help create long-distance entanglement out of the two non-maximal three-level states. Specifically, it shows that while the protocol does not always yield optimal results, namely, the weaker link, there is a substantial number of states that yield an optimal result. Moreover, we discuss the numerical approach in showing the existence of states that provide a result close to the optimal outcome, which may be useful in realizing the long-distance entanglement used in quantum technology.

Key Words: Quantum, Entanglement, Numerical Method, Information Network.

DOI Number: 10.14704/nq.2022.20.2.NQ22083 **NeuroQuantology 2022; 20(2):153-157**

Introduction

Signaled by Max Planck at the turn of the 20th century, the development of quantum theory has startled many and changed the way people think about nature. Indeed, the classical deterministic world view with strong causality has been destroyed by the probabilistic nature of quantum theory at the fundamental level. Moreover, the traditional practice of science, which was often considered to reveal truth about our universe, has changed to the pursuit of a mere interaction between an observing party, or a measuring apparatus, and the object being observed, such as an electron, at least in the context of the Copenhagen interpretation.

In fact, one of the most surprising features in quantum theory that is unseen in its classical counterpart is entanglement. Since Bell introduced his now renowned inequalities (Bell, 1964), entanglement has been shown to be useful not only in showing a unique characteristic of quantum theory but also in practical applications, including secret communication (Ekert, 1991) and practical computation (Ladd *et al*, 2010). Indeed, entanglement has been at the heart of recent developments in quantum technology.

Manipulating Entanglement

Let us first review the entanglement swapping protocol. The scheme was introduced in (Zukowski *et al*, 1993), which helped connect two entanglements by creating a longer one between distant parties. Initially, three parties, namely Alice, Bob, and Charlie, share two maximally entangled Bell pairs (see Fig. 1), as follows:

$$|\phi\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \quad (1)$$

$$|\psi\rangle_{34} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{34} \quad (2)$$

Next, Charlie performs a measurement on qubits 2 and 3 in Bell basis, i.e.,

$$\left\{\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\right\} \quad (3)$$

This then yields one of the four Bell longer-distance states, specifically 1 and 4, between Alice and Bob, with equal probabilities.

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Relevant conflicts of interest/financial disclosures: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.
Received: 29 December 2021 **Accepted:** 24 January 2022

$$|\Phi^{(\pm)}\rangle_{14} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{14} \quad (4)$$

$$|\Psi^{(\pm)}\rangle_{14} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)_{14} \quad (5)$$

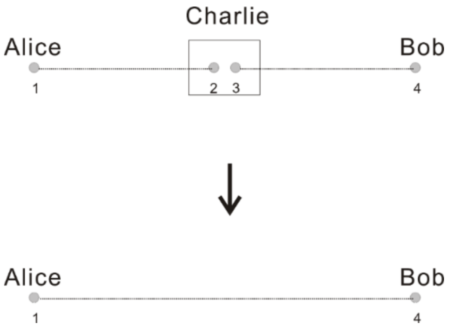

Figure 1. Alice and Charlie (set 1) and Charlie and Bob (set 2), initially share entangled states 1 and 2, and 3 and 4, respectively. Charlie performs a Bell measurement on the state 2 and 3, which then creates a long-distant entanglement between Alice and Bob

This protocol can be rather useful, since an entanglement generally tends to be sensitive to noise and easy to decohere, which makes it difficult to keep over a long distance. Therefore, short-distance parties can share entanglement and create a long-distance one by using the swapping protocol introduced above.

While entanglement swapping is useful, one of the difficulties in the actual implementation is that even short-distance states are unstable and may not usually be maximally entangled, as seen in (1) and (2). Indeed, various authors have considered the case when the initial states are not maximal. For instance, authors in (Bose *et al*, 1998) considered the case of two equal non-maximal states, and two different non-maximal qubits were studied in (Shi *et al*, 2000).

Interestingly, it turns out that even when the states are not maximal in entanglement, the Bell measurement yields the optimal outcome. That is, when the states are equal, the average entanglement is equal to the initial non-maximal state. Moreover, when the two non-maximal states are not the same, the average entanglement between Alice and Bob turns out to be the same as the weaker link between the two initial entanglements.

Figure 2. If it is possible to obtain entanglements larger than one of the initial states, then Alice and Bob could employ a scheme shown here to increase entanglement between distant parties by having Bob perform the Bell measurement on states 2 and 3, thereby creating a larger entanglement between 1 and 4 compared to the initial entanglement between 1 and 2.

Why is obtaining the average entanglement of the weaker initial state the best one can do? One of the essential properties of entanglement is known to be that the amount of entanglement does not increase due to local operations and classical communications (LOCC) with Alice and Bob at each end (Popescu *et al*, 1997; Bennett *et al*, 1996). To increase the entanglement, Alice and Bob have to bring the qubits together and perform appropriate operations on them, but not when the qubits are apart from each other.

Therefore, if obtaining a weaker entanglement is not optimal, then Bob can increase entanglement by performing the Bell measurement to bring additional entangled pairs and increase the entanglement between Alice and Bob (see Fig. 2) without bringing qubits together. This is not possible, and the swapping protocol is the optimal way of creating entanglement between 1 and 4 when the states are not maximal, as in Fig. 1.

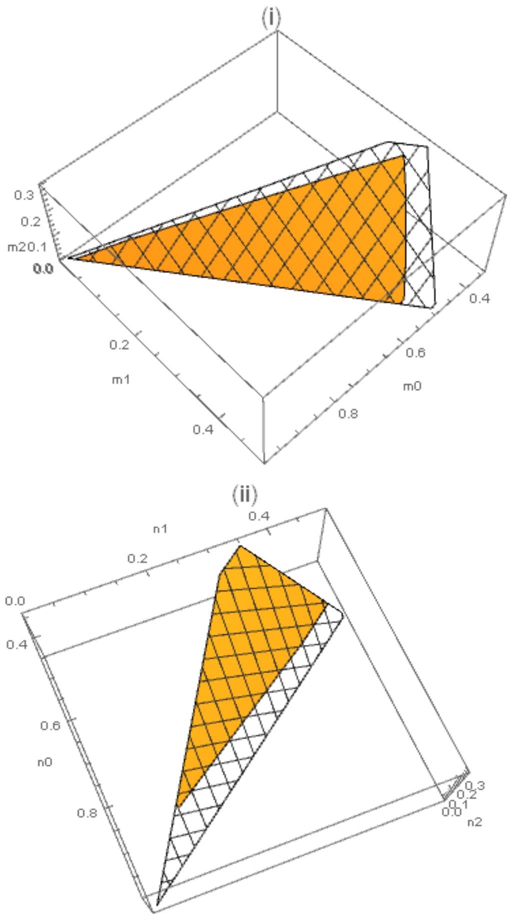

Figure 3. In the case of two 3-level states, unlike 2-level states, the Bell measurement does not always yield a weaker link. In (i), it shows the coefficients for state $|\phi\rangle_{12}$ in (6) that satisfy the conditions in (8-10) with a smaller triangle, thereby providing an optimality compared to the general states (larger triangle). A similar result is shown in (ii) for coefficients n_0, n_1 and n_2 of $|\psi\rangle_{34}$ in (7).

While entanglement swapping definitely provides a useful way of creating long-distance entanglement between Alice and Bob, this approach does not yield the optimal result when the non-maximal states are no longer 2-level, or qubits. For instance, for two 3-level states, i.e.,

$$|\phi\rangle_{12} = \sum_{i=0}^{2} \sqrt{m_i}\, |ii\rangle_{12} \quad (6)$$

$$|\psi\rangle_{34} = \sum_{j=0}^{2} \sqrt{n_j}\, |jj\rangle_{34} \quad (7)$$

Bell measurement on 2 and 3 does not always yield a weaker entanglement between $|\phi\rangle_{12}$ and $|\psi\rangle_{34}$. (For simplicity, $|\phi\rangle_{12}$ will be assumed to be the weaker link between the two.) In fact, various numerical analyses have been made for swapping of two 3-level states in (Song, 2019). In this paper, we wish to provide a similar but more general numerical analysis of the states in (6) and (7).

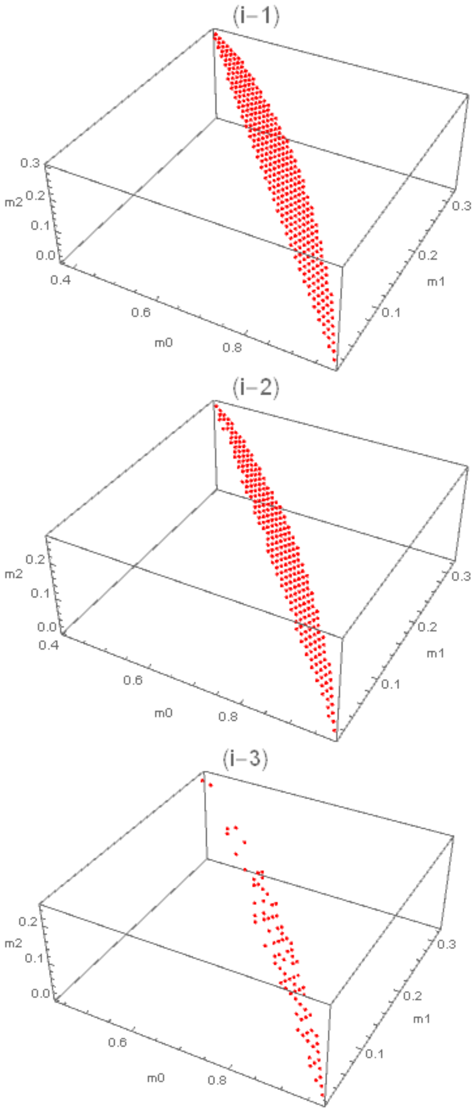

Figure 4. Distribution of coefficients in $|\phi\rangle_{12}$ that satisfy near-optimal conditions (11-13) are shown. The density of distribution is indicated where (i-1) shows the difference between the near optimal and optimal to be less than 0.01, while (i-2) and (i-3) show the difference less than 0.001 and 0.0001, respectively.

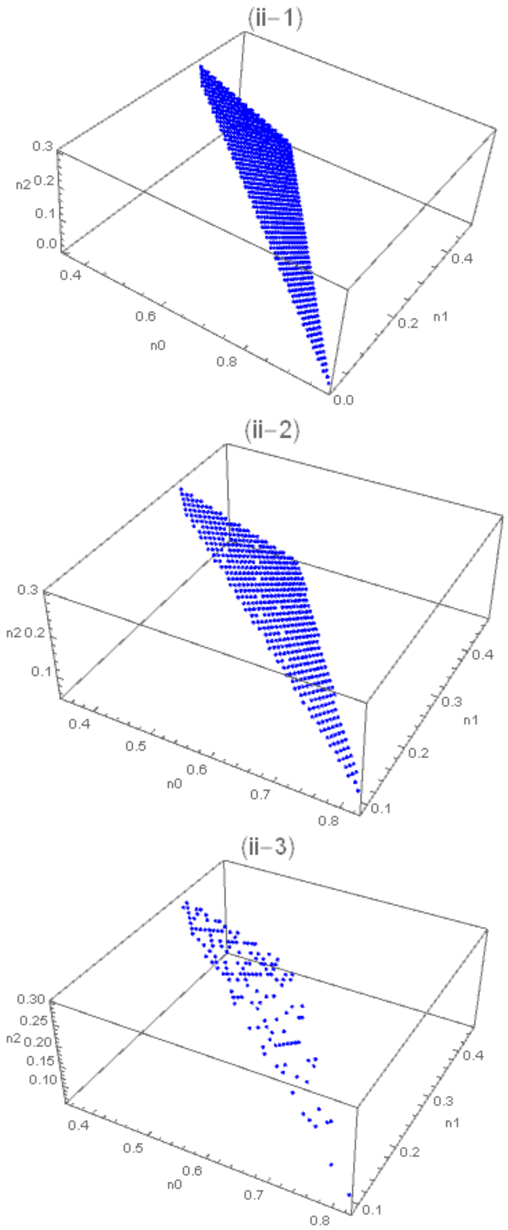

Figure 5. A similar result as in Fig. 4 is obtained for n_0, n_1, and n_2 in $|\psi\rangle_{34}$ in (7), which satisfy near optimality conditions. The difference corresponds to 0.01, 0.001, and 0.0001 in (ii-1), (ii-2), and (ii-3), respectively.

When the Bell measurement is made on 2 and 3, the resulting coefficients of entangled 3-level states between 1 and 4 are with m_0n_0, m_1n_1, m_2n_2, and m_0n_1, m_1n_2, m_2n_0 and m_0n_2, m_1n_0, m_2n_1 (see (Hardy, 1999; Hardy *et al*, 2000) for a review). In fact, it can be shown that when the following conditions are met:

$$m_0n_0 \geq m_1n_1 \geq m_2n_2 \quad (8)$$

$$m_0n_1 \geq m_1n_2 \geq m_2n_0 \quad (9)$$

$$m_0n_2 \geq m_1n_0 \geq m_2n_1 \quad (10)$$

The optimality is obtained, i.e., the average entanglement corresponds to that of $|\phi\rangle_{12}$ in (6).

To begin with, it is desirable to know roughly what range of initial states in (6) and (7) satisfies (8,9,10) such that the result is optimal, namely, the weaker link, $|\phi\rangle_{12}$. In Fig. 3, (a) shows the range of coefficients, m_0, m_1, m_2 which satisfy (8-10), which is substantial relative to the general coefficients in a larger triangle. Similar results were obtained for coefficients n_0 , n_1 , n_2 in (7). Therefore, although entanglement swapping does not always yield the weaker link given two 3-level initial states, a substantial number of non-maximal 3-level states, in fact, yield the optimal result, or the weaker link, as indicated in Fig. 3.

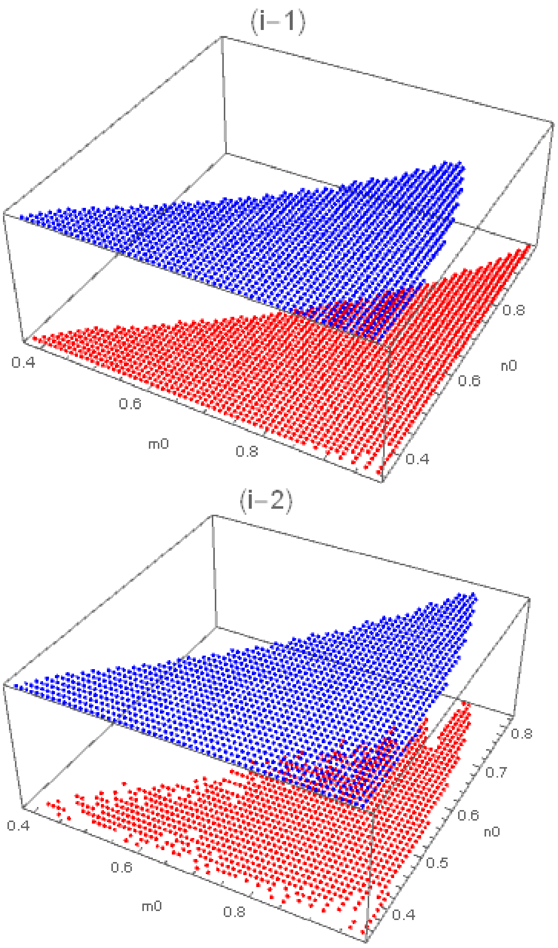

Figure 6. Optimal versus near optimal is shown for m_0 and n_0 in (6) and (7). (i-1) exhibits the comparison between the optimality (top) and near-optimality (bottom), where the difference in entanglement is less than 0.01, while (i-2) indicates the comparison with the difference to be less than 0.001; therefore, scarcer in its distribution at the bottom. The vertical difference between optimal and near optimal is provided only to make the comparison more visible.

While it is good that a substantial number of 3-level states exist that yield the optimal link, it is still desirable to seek some states in (6) and (7) that yield the entanglement between Alice and Bob that is close to the optimal protocol. Indeed, in (Song, 2019), it was found that one can introduce another condition as follows:

$$m_0n_0 \geq m_1n_1 \geq m_2n_2 \quad (11)$$
$$m_0n_1 \geq m_2n_0 > m_1n_2 \quad (12)$$
$$m_0n_2 \geq m_1n_0 \geq m_2n_1 \quad (13)$$

That is, when the second condition in (9) is replaced by (12), while the other two remain the same, the output after swapping will not be the weaker link, i.e., $|\phi\rangle_{12}$, but very close to it. Let us first examine what we mean by *very close* to the optimality.

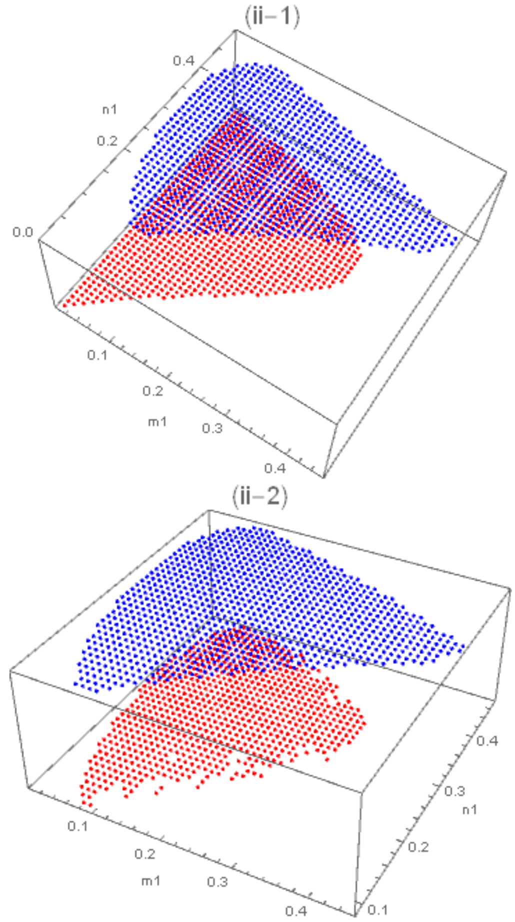

Figure 7. A similar result is obtained, as in Fig. 6, with coefficients m_1 and n_1. (ii-1) and (ii-2) indicate the difference of average entanglement to be less than 0.01 and 0.001, respectively.

Indeed, the distribution of the coefficients m_0, m_1, and m_2 (Fig. 4) as well as n_0, n_1, and n_2 (Fig. 5), that satisfy the conditions (11-13) are shown. The coefficients that satisfy the average entanglement to be less than 0.01 compared to the entanglement of $|\phi\rangle_{12}$ are shown in (i-1). For a better outcome, i.e., closer to the weaker link, the distribution of coefficients becomes scarcer, and (ii-2) and (ii-3) indicate the difference to be 0.001 and 0.0001, respectively. Therefore, it can be seen that there are fewer states that yield the entanglement closer to the optimality in Fig. 5 for coefficients n_0, n_1, and n_2 for $|\psi\rangle_{34}$.

Next, we wish to consider how wide the states are that satisfy the conditions in (11-13) and, thus, are near optimal compared to the states that satisfy (8-10), i.e., optimal cases. Fig. 6, 7, and 8 provide comparisons between near-optimal versus optimality. For instance, the coefficients m_0 and n_0 from $|\phi\rangle_{12}$ and $|\psi\rangle_{34}$ are compared between near optimal (bottom) and optimal (top), with the difference being less than 0.01 in (i-1) and 0.001 in (i-2). Comparable results are shown in Fig. 7 and 8 for coefficients m_1 versus n_1, and m_2 and n_2, respectively.

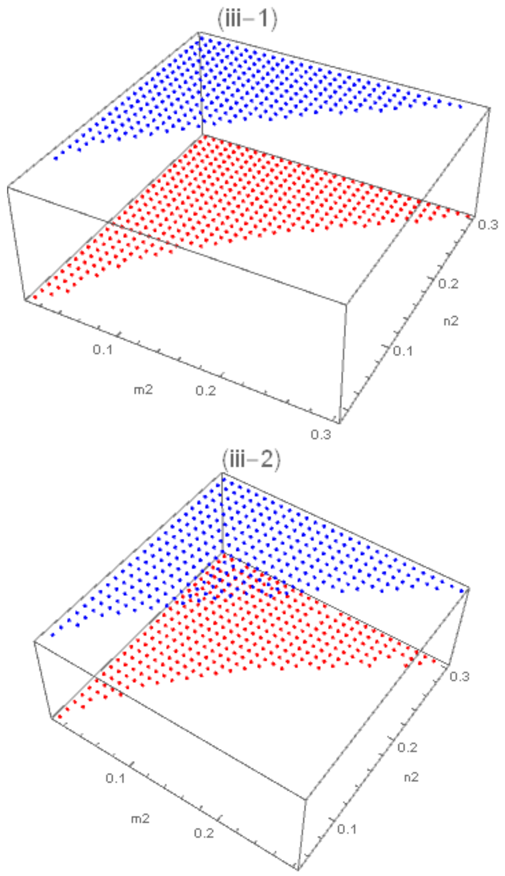

Figure 8. A similar result of optimality versus near-optimality is shown in Fig. 6 for m_2 in (6) and n_2 in (7). The vertical difference between optimal (blue) and near-optimal cases (red) is to be ignored.

Remarks

When the idea for the quantum computer was first introduced in 1980s, very few people thought this brilliant idea could be realized in the real world. Even when it was shown that the quantum computer might break the most widely used cryptosystem, many people were still skeptical of the machine actually being built due to the extreme difficulty of scalability at the quantum level. Nevertheless, it is now being reported that large-scale (i.e., large enough to claim quantum supremacy) quantum computers have been realized in laboratories performing certain calculations unparalleled in classical counterparts.

Moreover, Bell's idea of examining non-locality, which seems to be at the heart of quantum theory, has been realized by extremely precise experiments over a long-distance range (Tittel *et al*, 1998; Ma *et al*, 2012; Yin *et al*, 2017). In the era of rapid advancement in quantum technology, it is useful to consider some characteristics of entanglement, particularly using numerical means. Indeed, in recent years, new developments in machine learning and the deep-learning fields have introduced the possibility that data analytic computational approaches can be useful in scientific endeavors that traditionally emphasize analytic approaches in mathematics.

This work was financially supported by the Research Year of Chungbuk National University in 2021.

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