A Study of Ws- Exact Sequence

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Abstract

In the context of group theory, a sequence The sequence is called exact if it is exact at each {\displaystyle G_{i}} for all {\displaystyle 1\leqi<n}, i.e., if the image of each homomorphism is equal to the kernel of the next. The sequence of groups and homomorphisms may be either finite or infinite.A similar definition can be made for other algebraic structures. For example, one could have an exact sequence of vector spaces and linear maps, or of modules and module homomorphisms.More generally, the notion of an exact sequence makes sense inany category with kernels and cokernels, and more specially in abelian categories, where it is widely used.

KeyWords: Study, Exact, Sequence.

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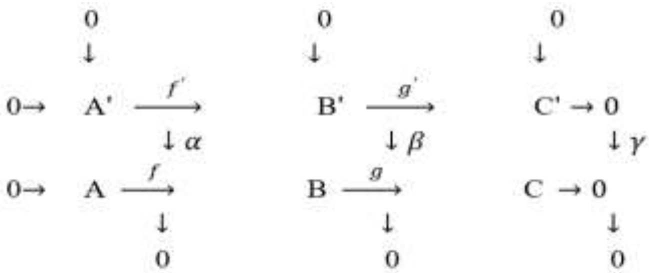
Introduction

All rings in this paper are expected to be commutative with non-zero identity and all modules are unitary. The exact sequence has been used intensively in many disciplines, as commutative algebra. Consider a sequence of R-modules and homomorphism. Then Im f = ker g. It is raising a question that; Standby a submodule U of C in place of the small submodule {0} as superfluous. Davvaz and Parnian[2] introduced the concept of the chain U-complex. U-Homology, chain (U, U') – map, chain (U, U') –homology and U-factor. In this chapter, we introduced Ws – Exact sequence with the help of a particular condition of superfluous. We used means N is a submodule of M. A submodule of N of module M is claimed to be superfluous (small) N<<M, if for any submodule K of M, N + K = M implies that K = M.

Lemma 1.1[2]: Diagram of the module with exact rows and columns:



Proof: (Obviously)

Proposition 1.1[2]: Let M be an R-module; then, M ∈ Cn

There is a small submodule A of M such $\frac{M}{A}$ ∈ Cn.

$\frac{M}{A}$ ∈ Cn for each small submodule A of M.

Proof: (Obviously)

Difinition 1.1[4]: A sequence of R-modules and R-homomorphism. ……..→ Cn+2 $\xrightarrow{\partial n-2}$ Cn+1 $\xrightarrow{\partial n-1}$ Cn $\xrightarrow{\partial n}$ Cn-1 →…………… is claimed to be Up- exact at Cp+1 if, Im ∂n+2 = (∂ ∂n-1)-1 Un-1 ⊂ Cn-1, ((∂n+1)-1(∂n)-1(∂n-1)).

Im ∂n+2 = {…. Up+1, Un, Un-1………}. A U–exact sequence is a sequence Up- exact at each of its point.Module (Cs, Us, ∂s) represent an Us-exact sequence and satisfying the condition (∂n ∂n+1) ⊂ Un-1. So, each Us-exact sequence such Un-1 ⊆ Im ∂n ∂n-1 is chain Us-complex. A chain (C, U, ∂) Us-sequence if and only if Hp (C, U, ∂) = 0, ∀ p ∈ Z.

Definition 1.2[4]: Let Us-exact sequence (C, U, ∂)is claimed to be isomorphic to exact sequence (C, U, ∂) if there is a chain Us U's -map F such Fp is an R-module isomorphism.

Proposition 1.2[4]: If two Us-exact and U's-exact sequences are isomorphic, then Us ≅ U's.

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Proof : Let 0 → A → B → C→ 0 be a zero exact at A, Us-exact at Band zero exact at C.

Proposition 1.3[2]: Let (C, U, ∂) be a chain of R-modules and R-homomorphism and (C, U, ∂) be U's –exact sequence, If in sequence F = {Fn}, each Fn is R-module isomorphism such that F(U) = U' at the subsequent figure is commutative, then is a U's –exact sequence.

$$\begin{array}{ccccccc} (C,U,\partial):\ldots\ldots\to C_{n+2} & \xrightarrow{\partial_{n-2}} & C_{n+1} & \xrightarrow{\partial_{n-1}} & C_n & \xrightarrow{\partial_n} & C_{n-1}\ldots\ldots \\ \downarrow F_{n+2} & & \downarrow F_{n+1} & & \downarrow F_n & & \downarrow F_{n-1} \\ (C,U,\partial):\ldots\ldots\to C'_{n+2} & \xrightarrow{\partial_{n-2}} & C'_{n+1} & \xrightarrow{\partial_{n-1}} & C'_n & \xrightarrow{\partial_n} & C'_{n-1}\ldots\ldots \end{array}$$

Proof: We show that Im ∂n+2 = (∂n ∂n-1)-1 Un-1. Assume that x ∈ Im ∂n+2 ∃ c ∈ Cn+2 Such x = ∂n+2 (C).

We have Fn [∂n+1 ∂n+2 (C)] = (∂′n+1 ∂′n+2) Fn+2 (C). So ∂′n [Fn {∂n+1 ∂n+2}] = ∂′n [∂′n+1 ∂′n+2 Fn+2 (C)] ∈ U′n-1 => Fn-1(∂n ∂n+1 ∂n+2) (c) ∈ U′n-1.

Since Fn-1(Un-1) = U'n-1 and Fn-1 is isomorphism then we get, (∂n ∂n+1 ∂n+2) (c) ∈ Un-1 or (∂n ∂n+1) (x) ∈ Un-1⇒ x ∈ (∂n ∂n+1)-1 (Un-1).

Henceforth, Im ∂n+2 ⊆ (∂n ∂n-1)-1 Un-1. Conversely let, x ∈ (∂n ∂n+1)-1 (Un-1), we have Fn-1(∂n ∂n+1∂) (x) ∈ U'n-1,and so, (Fn-1(∂'n ∂′n+1∂) Fn-1) (x)) ∈ Un-1, also we have Fn-1 (x) ∈ (∂′n ∂′n+1 ∂)-1 Un-1 = Im∂′n+2 .

So ∃ y ∈ C′n+2 such Fn+2 (x) = ∂′n+2 Fn+2 (z) and so Fn+1 (x) = ∂′n+2 Fn+2 (z) = Fn+1 ∂n+2 (z) = Fn+2 (∂n+2) (z).

Since Fn+1 is monic we get x = ∂n+2 (z), henceforth x ∈ Im ∂n+2.

Corollary 1.1[2]: Consider (C, U, ∂) be a chain US-complex & (C’, U’, ∂′) be a U'S- exact sequence. If (C, U, ∂) and (C', U', ∂′) are isomorphic, then (C, U, ∂) maybe a US-exact sequence.

Proof: (Obviously)

Lemma 1.2[5]: Let (Lambek Lemma)

$$\begin{array}{ccccc} A' & \to & A & \to & A'' \\ \downarrow \psi & & \downarrow \varphi & & \downarrow \theta \\ B' & \to & B & \to & B'' \end{array}$$

be a commutative diagram where 1st and 2nd rows are U'S-exact, US-exact respectively.

Let V and W be “submodule” of B and B' respectively, such Imψ ⊇ W and Im φ ⊇ V then φ induced an “isomorphism”.

$$\varphi = \frac{(\theta\alpha_2)^{-1}(U)}{\alpha_2{}^{-1}(U')+\varphi^{-1}(V)} \to \frac{Im\varphi \cap Im\beta_1}{Im(\varphi\alpha_1)}$$

Proof: We prove that φ induces a “homomorphism” of this type

Let x ∈ (θα₂)-1 (U) and φ(x) ∈ Im φ.

Since β2 φ(x) = $\theta\alpha_2(x)$ ∈ U and (β2)-1 (U) = Im (β1).

We get φ(x) ∈ Im (β1). So φ(x) ∈ Im φ ∩ Im (β1).

Now we define φ [x + α1-1 (U') + φ-1 (V)] = φ(x) + Im φα1, first we show that Φ is well defined, then we assume

[x + α2-1 (U') + φ-1 (V)] =[y + α2-1 (U') + φ-1] ⇒ x - y ∈ [α2-1 (U') + φ-1 (V)] and so ∃ a ∈ α2-1 (U') and b ∈ φ-1 (V) with x - y = a – b, hence φ(x) - φ(y) = φ(a) - φ(b). Since a ∈ α2-1 (U').

we have, α ∈ Im(α1) ⇒ (φ)α Im (φα1).S. Since b ∈ φ-1(V) ⟹ φ (b) – ϵβ1(W) Thus ∃ c ∈ W Also ∃ such φ (b) = β1 (c). Also, ∃ d ∈ A' such Ψ(d) = c.

Thus φ (b) = β1Ψ(d) =Ψα1(d) ⟹ φ (b)∈ Im (φα1).

Hence φ (x) – φ(y) = φ(a) – φ(b) ∈ Im (φα1) So Φ is well defined. It is clear that “homomorphism”, φ (x) =y. Soθα2 (x) = β2 φ (x) β2 y.

Also, we have y∈ β2-1 (U') or β2 (y)∈U. So, we obtain θα2 (x)∈U or x∈ (θα2)-1 (U).

Now we show that Φ is monic, assume that we have Φ [x + α2-1 (U') + φ-1(V)] = φ (x) + Im (φα1) = Im (φα1)

⟹ φ (x)∈ Im (φα1) ∃ z ∈A such φ (x) = (φα1 (z))

⟹ φ (x -α1 (z)) =0 ⟹ (x-α1 (z)) ∈ Kerφ ∃ t ∈ Ker φ, such (x = t + α1 (z)).

Where α1 (z)∈ Im (φα1) =α2-1 (U') and t ∈ φ-1(V).

So x ∈ α2-1 (U') + φ-1(V).

Lemma 1.3[3]: (Snake lemma) Let

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \to 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0\to A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

be a “commutative diagram” with is small such

The 1st row is “Us-exact” and the 2nd row is “U's-exact”

W ⊆ Imα

U⊆ Imγ

g(V) ⊆ U, f (W) =V

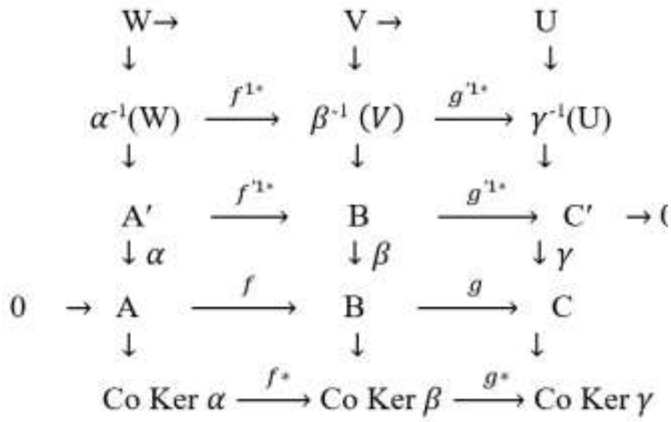
U' ⊆ γ-1(U)

Then there is a connecting “homomorphism”

ω: $\frac{\gamma^{-1}(U)}{U'}$ → Co Ker α Such the subsequent is “exact”.

$\alpha\text{-}1(W) \xrightarrow{f1'*} \beta\text{-}1 \xrightarrow{\gamma-1*} \frac{\gamma^{-1}(U)}{U'} \xrightarrow{\omega}$ Co Ker$\alpha \xrightarrow{f*}$ Co Ker$\beta \xrightarrow{g*}$ Co Kerγ.

Proof: Consider the diagram:



We have g' [β-1(V)] ∈ γ-1(U) and f' [α-1(W)] ∈ β-1(V); so, we have map

$\alpha\text{-}1(W) \xrightarrow{f1'*} \beta\text{-}1\ (V) \xrightarrow{g'1'*} \gamma\text{-}1(U)$

……………………… (I)

Suppose π is the "canonical homomorphism" π: γ-1(U)→ $\frac{\gamma^{-1}(U)}{U'}$

We denote f '* = f ' $|_{\alpha^{-1}(W)}$ and g' *= g' $|$ β-1 (V)

Then

$\alpha\text{-}1(W) \xrightarrow{f'*} \beta\text{-}1\ \frac{\gamma^{-1}(U)}{U'}$

……………………… (II)

On the other hand, f and g induce the subsequent maps;

f* :$\frac{A}{Im\alpha} \rightarrow \frac{B}{Im\beta}$, g*:$\frac{B}{Im\beta} \rightarrow \frac{C}{Im\gamma}$, hence the subsequent sequence

Co Ker α $\xrightarrow{f*}$ Co Kerβ $\xrightarrow{g*}$ Co Kerγ, now we show that ∃ a "homomorphism" ω: $\frac{\gamma^{-1}(U)}{U'}$ → Co Ker α.

Connecting the (I) and (II)

$\alpha\text{-}1(W) \xrightarrow{f1'*} \beta\text{-}1(V) \xrightarrow{g'1'*} \frac{\gamma^{-1}(U)}{U'} \xrightarrow{\omega}$CoKer α $\xrightarrow{f*}$ Co Kerβ $\xrightarrow{g*}$ Co Kerγ

Assume Z+ U' ∈ $\frac{\gamma^{-1}(U)}{U'}$ that, choose b' ∈ B with g' (b') =z.

Since gβ(b') = γg' (b') = γ(z)∈U. We Get β(b') = g-1(b') and β(b') ∈ Imf.

Since f is monic f: A →Im f is "bijective". So ∃ a unique element a ∈ A then ∃ β(b') = f (a) ⟹ a = f-1β(b') defines ω(z +U') =a + Imα We show that ω is well described that is b' ∈ B and b" ∈ B". With g'(b") =z. Then β(b") = β(a').

We obtain g' (b') = g' (b") ⟹ g' (b' - b") = 0

⟹ b' - b" ∈ ker g'

⟹ b' - b" ∈ ker g' ⊆ g'-1 (U') imf '

Hence ∃ a ∈ A' with b '–b" = f ' (a)

Since, βf ' $\overline{(a)}$ = fα $\overline{(a)}$and β(b' – b") = fα$\overline{(a)}$ ⟹ β(b' – b") = fα(a) so (a' – a") =α$\overline{(a)}$ ∈ Imα.

So a + Im α = a' + Imα ω is a "homomorphism". Proof of "exactness" is rather long, so that it will be divided into several steps.

Step1: Let b' ∈ ker g '* ⟹ g'*(b') =0 then g' (b') + U' =U' and so b' ∈ g'-1 (U') Hence ∃ x ∈A' such b' = f ' (x) .

Now it is enough to show that x∈ α-1(W) or α(x)∈W, we have

f α(x) = β f '(x) = β'(b') since b' ∈ β-1(V) ⟹ β(b') ∈V.

Which implies fα(x)∈V. Since f(W) =V and f is monic we have α(x)∈W or x∈ α-1 = Imf-1*, so, ker g ' * ⊆f ' * . Conversely, it is clear that f '* g'* = 0 ⟹ Imf '* ⊆Ker g' Hence Imf '* ⊆Ker g'*.

Step2:Imf '* = Ker g'* Suppose that g '(b') + U ' ∈ Im g'*.

Where b' ∈ β-1(V) Definition of connecting map, ω(g' (b') +U') =f'β g'-1(g' (b')) + Imα Since b' ∈ β-1(V)⟹ β(b')I ∈ V, since f(W) =V and f is monic ⟹f-1 β(b') ∈ W⟹ f-1 β(b') ∈Imα. Hence ω(g' (b') +U') = Imα ⟹ g' (b') =U ' ∈ ker ω. So Im g'* ⊆Ker ω. Conversely, Let t' +u' ∈Ker ω with t' ∈ γ-1(U) Then ω(t' +u') = f-1 βg-1(t') = α(x) ⟹ βg-1(t') = f (α(x)). We have βh-1(t') = βf ' (x) ⟹ β{g-1(t') – f ' (x)} = 0.

⟹g-1(t') – f ' (x) ∈Kerβ ⊆ β-1(V)

⟹ g-1(t') – f ' (x) ∈ β-1 (V)

Hence g'* { g'-1 (t') – f ' (x)} = g' g'-1 (t') – g ' f ' (x) + U' = t' +U' So g'* { g'-1 (t') – f '(x) } = t' +U' ⟹ t' +U' ∈ Img'*.

Thus kerω ⊆ Img*

Step 3: Imω = ker f*. Consider ω'(t' +u') ∈ Imω.

Then g' (b') ∈ γ-1 + imα ∈ Imω, hence f* { f-1 β g'-1 (t') + Imα } = ff-1β g'-1 (t') + f(Imα) = β g'-1 (t') + f(Imα) = β g'-1 (t) + Imβ = sinβ ⟹ ω(t' + u')∈ kerf*.

So, Imω ⊆ kerf*, Conversely, suppose that α + Imα ∈ kerf* ⟹ f*{ a + Imα }=Imβ ⟹ f(a)∈Imβ and ∃ b' ∈ B' such β(b') =f(a) . Since gf(a) ∈ U and gβ(b')∈ U it is clear that γg' (b') ∈ U and so g' (b') ∈ γ-1 (U). So,

ω{g' (b') + U'} = f-1βg'-1 (g' (b'))+Imα= f-1 β(b') + Imα = α + Imα

⟹ α + Imα ∈ Imω ⟹ kerf*⊆ Imω. Hence Imω= kerf*.

Step 4: Imf* = ker g*

Let f* {a + Imα} ∈ Imf*. Then g*f* {a + Imα} = gf(a) + Imγ.

Since gf(a)∈ U⊂ Imγ, gf "{a + Imα = Imγ} and f "{a + Imα} ∈ ker g*. Now let b+Imβ ∈ ker g*.

Then g(b) ∈ Imγ and ∃ P' ∈ C' such g(b) = γ(p').

Since g' is epic, ∃ b' ∈B' with g'(b') = p. Hence g(b)

=γ g'(b') and so g(b) = gβ(b')⟹ g(b) - gβ(b') = 0 ⟹ g{b - β(b') }= 0 ⟹ {b - β(b') } ∈ ker g ⊆g-1(U) = Imf. So ∃ a ∈ A such b- β(b') = f (a). So b+Imβ =f(a) + Imβ = f*(a +ker γ) . Thus b+Imβ ∈ Imf*.

Conclusion

In this paper Ws – Exact sequence has been introduced with the help of a particular condition of superfluous in different steps. We used means N is a submodule of M. A submodule of n of module M is claimed to be superfluous N<<M, if for any submodule K of M, N+K=M implies that K=M.

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