Dynamical Interpretations Of Sine Logistic Map Using Picard Orbit

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Abstract

The standard logistic map hunt down prominent space in the field of dynamics which is widely used in many scientific domains such as, cryptography, communication, biology, physical and chemical science, and stock market. In this reference, we investigated the chaoticcharacteristics of the sine logistic equationoperating Picard orbit. It is observed thatfor the conventional logistic equation the parameter varies between$r \in [0,4]$ while in the sine logistic map the parameter approaches to $r \in [0,7.5]$. Analytically as well as experimentally analysis is performed like time-series analysis, bifurcation and functional analysis and another important property, Lyapunov exponent is calculated. Hence, the enhanced chaotic region in Sine Logistic Mapwill develop the chaos-based operations.

Keywords: Sine Logistic map, Lyapunov exponent, Bifurcation, Chaos

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1. INTRODUCTION

In formerly decades, thechaos, and the theory of dynamics of difference equations has conspicuous corner in the field of mathematics and engineering. Most of the natural phenomenon can be suitably described by the logistic model which is proposed by the mathematician P.F. Verhuslt [30]. The dynamical natureof the irregular structure was examined by Poincare [23]. He observed that the minute difference in the input of a nonlinear systems may effect a consequential change. Hence, it is impractical to conclude about the wide-ranging behavior of a nonlinear dynamical system. A great meteorologist Lorenz [20] discussed the chaos theory in weather forecasting. In 1978, Mitchell Feigenbaum[15] discovered the mathematical model of chaos which summarize the dynamics of logistic map. R. Dettmer [13] briefly explained the terminology of dynamical systems and then discussed logistic mapping and bifurcation. Different types of measure of complexity have been explained and tested on the logistic map by Wacker-Bauer [31]. He investigated the particular types of behavior random number which are aperiodic, infinite, and not correlated. For a brief knowledge of logistic map, particular can go along withAlligood [1],Holmgren [17], Devaney [16], Elagdi [14], Diamond [12], M. Ausloos [10]. The importance of the logistic equation is that the fundamental difference in the behavior of the function under some tiny changes in the initial conditions. This utmost reactivity to the initial state is the most interesting features of chaotic maps which make chaotic system perfect for numerous applications.

The caprice character of logistic maps made us curious about further analysis of chaotic behavior of nonlinear maps. The concern was further pushed by the improvement of software tools and use of electronic computers in the latter half of the age of modernity. Many authors indicated the importance of logistic maps in chaos, optimization, image encryption and decryption, population dynamics. To observe the stability of logistic equation in 2005, a new technique is applied on the logistics map by M.

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discussed by M. Andrecut [2] to generate a first bifurcation of the map. N. Singh and A.

Sinha [29]used logistic map to demonstrate anew electro-opticaldefend transmission system and chaotic signals are generated by using a customized version of the electronic systems of the logistic equation. In 2012 Ashish et al. [11] examined the logistic equation via a multistep iterative procedure which is known as Noor orbit. A logistic model using Ishikawa iteration method through time-series and Lyapunov exponent analysis was proposed by B. Parsad and K. Katiyar [22] in 2013. In that year, A. G. Radwan [24] compressed the standard logistic equation by introducing different types of generalized logistic maps with random power.

Next year, R. Rak and E. Rak [26] illustrated the nature of the generalized logistic map for the whole spectrum of parameters for the transition from regularity to chaos. They focused on the nature of the analogous bifurcation pattern and gave specific cases for both periodic and chaotic regime. W.S. Sayed et.al [28] designed the positive and negative logistic map, mostly positive and mostly negative logistic map established from their ultimate chaotic range of the outcome. Bifurcation plots for each generalized map and maximum Lyapunov exponent was discussed briefly. In the same year, D. Baleanu et. al. [33]discussed a few descriptions of fractional variations and compared their implementations to fractional equation. They [32] also reported the delayed logistic map which is detached by applying the DFC approach and discussed the coupleddistinct chaos.D. Aniszewska [3] tested the dynamical characteristics of the multiplicative logistic equations that is converted from the standard logistic map. The bifurcation analysis and Lyapunov exponent were discussed. During the same year, Ashish et al. [8]examined the chaotic properties like periodic amplifying, Lyapunov exponent and period-3 window of the conventional logistic equation. They added anadditional parameter of freedom in growth rate parameter which provides an enhanced chaotic characteristic of the map which is useful in many applications. In 2018, the theorized double humped logistic map is designed by A. G. Radwan et al. [18], which is used to produce a deterministic-random number key. Based on the deterministic-random number series generation, an image encryption algorithm is described using this map offering very good secure communication. Following year,[5] they demonstrated the chaotic characteristics of the discrete logistic map using empirical approach. In this description they used an additional degree of freedom applying on the constraint parameters which enhanced the dynamic properties of the equation. In 2020, using Mann orbit, the invariability of generalized logistic map is discussed by Ashish et al. [19].By using extra parameter β the generalization of the conventional logistic equation is done, from this they observed that as the scale of parameter increases the scale of invariability of the generalized logistic map increases.

Currently,in 2021, the dynamical property of the logistic type difference equation was described by Ashish et al. [6] by using Mann orbit.In comparison with previous logistic maps, it gives more well-structuredand practical dynamical properties. That year, theyalso observed the hyperbolicity which is used to introduce the periodic and non-periodic behavior of the dynamical system [7]. They examined magnification and equalization of fixed orbits through a superior two step iteration method. In the same time, (2021) R. Chugh et. al. [9] presented period-doubling bifurcation, Lyapunov exponent and period-3 window of a regulated logistic complexby using a superior chaos.In year 2022, a great effort put by Ashish et al. [27] studied the discrete difference map using Mann orbit. Moreover, Ashish et. al. [21]investigatedthe dynamical properties for a simple one-dimensional map which is dependent on three control parameters and exhibits a reverse bifurcation plot. The chaotic characteristics of the logistic equation using Euler's algorithm is established theoretically and numerically by Ashish et. al.[4].

This article concerns with dynamical properties of such one-dimensional sine logistic map using Picard orbit. Dynamical characteristics like fixed –orbit, period doubling, period-3 window are analyzed.In this article, there are five major segments. 1st segment is introductory section in which an introduction and brief literature review is given. Segment 2nd contains some basic definitions like fixed-point, periodic point etc. In segment 3rd, chaotic analysis of sine logistic map using Picard orbit is done through time – series analysis, bifurcation analysis. Lyapunov exponents is calculated and explained in segment 4th. At last, the result and discussion of the article is explained in segment 5.

2. PRELIMINARIES

This division enclosed by some descriptions and outputs about chaos theory and iterations.

Definition 2.1(Fixed point). Let us define a function f from the non-void set X to the non-void set Y. Then, for any arbitrary point $a \in X$ which fulfills$f(a) = a$, is known as the fixed or stationary point of the map f.

Definition 2.2 (Periodic points). For the function f, if one can find a smallest non-negative integer p for any point $a \in X$ such that the point a holds the assumption $f^p(a) = a$ is defined as the periodic point. This least positive integer p is called as the prime period of point a .

Definition 2.3(Picard Orbit). Let us define a function f from a non-void set X to X. Now we define a sequence $\{a_n\}$ for every $a \in X, a_{n+1} = f(a_n)$
where n = 0,1,2.....termed as the Picard iteration method. Primary input of the sequence is denoted by a_0 and is termed as Picard orbit of a_0.

Definition 2.4 (Stability of fixed point). A fixed point x_0 is known to be a stable if $|f'^{(x_0)}| < 1$ and unstable if$|f'^{x_0}| > 1$.

Definition 2.5 (Lyapunov Exponent). Let us define a operator$f$ on R. The Lyapunov exponent of f is given as

$$\alpha(x_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log|f(x_i)|.$$

3. DYNAMICS OF SINE LOGISTIC EQUATION

This division enclosed the ordered and dis-ordered nature of the sine logistic equation in Picard's orbit by examining their functional plots and bifurcationplots. Let us define a nonlinear equation,

$$x_{n+1} = x_n r(1 - \sin(x_n)), \qquad (1)$$

where $x_n \in [0,2]$, $r \in [0,7.5]$ and $n \in N$.Let us take x_1 as the new outcome then from (1) we get, $x_1 = f(x_0) = rx_0(1 - \sin x_0)$. At this place, we know that the dynamical properties of the above system based only on the growth rate parameter r. Simply, we denote

$$x_1 = f(x) = rx(1 - \sin x) = M_r(x)(2)$$

Here, we are familiar from the analysis of unidimensional equations that the control parameter r always influences the dynamical nature of the nonlinear system. In this study, we have taken the specific range for the parameter r and we set the initial point as $x_0 = 0.3$.

Let $f(r, x) = rx\,(1 - \sin x)$ be the one-dimensional sine logistic map. Then in Picard's orbit fixed point can be evaluate as,

$$f(x) = x$$
$$rx(1 - \sin x) = x$$
$$\Rightarrow x - rx(1 - \sin x) = 0$$
$$x(1 - r(1 - \sin x) = 0$$
$$\text{Either } x = 0 \text{ or } \left(1 - r(1 - \sin x)\right) = 0$$
$$1 - r(1 - \sin x) = 0$$
$$\Rightarrow \quad 1 - \sin x = \frac{1}{r}$$
$$\sin x = 1 - \frac{1}{r}$$
$$\Rightarrow \quad x = \sin^{-1}(1 - \frac{1}{r})$$

Thus, the point $x = 0$ and $x = \sin^{-1}(1 - \frac{1}{r})$, where r >0 is the stationary point of the sine logistic map . For periodic points,

$$M_{r,x}^2 = r[rx(1 - \sin x)][1 - \sin(rx(1 - \sin x))] = x$$

$$r^2x - r^2x\sin x - r^2x\sin(rx(-\sin x)) + r^2x\sin x\sin(rx(1 - \sin x)) - x = 0$$

$$r^2x[1 - \sin x - \sin(rx(1 - \sin x)) + \sin x\sin(rx(1 - \sin x))] - x = 0$$

either$x = 0$ or $r^2[1 - \sin x - \sin(rx(1 - \sin x)) + \sin x\sin(rx(1 - \sin x))] - 1 = 0$

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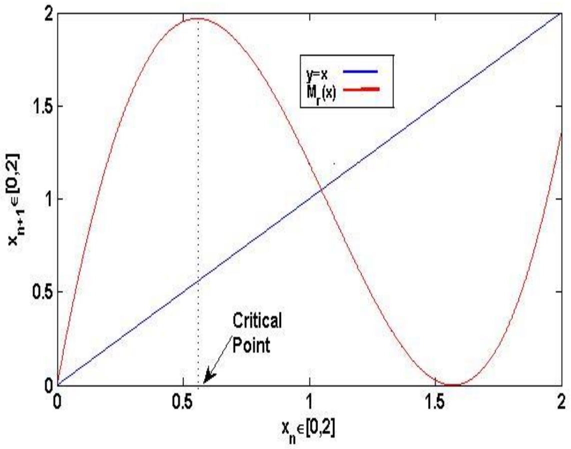

Figure 1. Functional plot of the map $M_r(x)$ and $y = x$

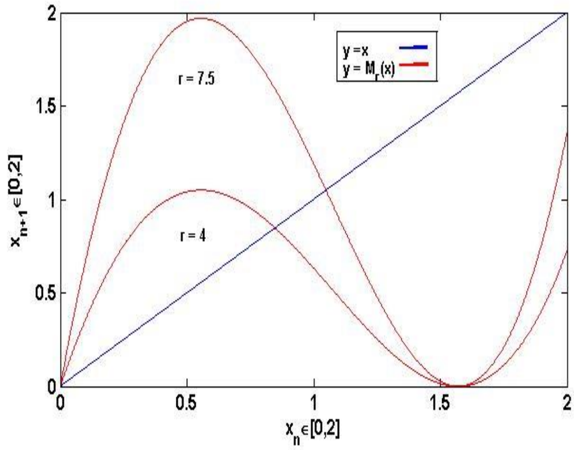

Figure 2. Functional plot of the map $M_r(x)$ for r = 4 and r = 7.5

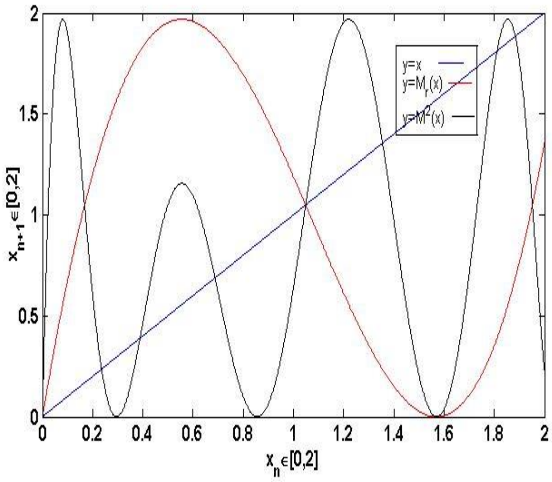

Figure 3. Comparative representation of plots of $M_r(x), M_r^2(x)$ and $y = x$

3.1 Time-series Analysis

When the growth rate parameter r varies in [1,3.5605], it shows stable behavior and converges to fixed point $\sin^{-1}(1-\frac{1}{r})$ of the map which is displayed in fig.4. In this graphic, we observed the convergence of the equation for three distinct values of control parameter. As we increase the value of r from 3.5605< r ≤ 4.4234, the nature of the map changes and the boundary values of the map bounded by two steady solutions which are fixed solution of the map$M_r^2(x)$.

Further, as the r increases the vibrations in the trajectories approaches stable solutions of order 2^n, and it will have more and more vibrations in trajectory which is the chaos in the system. For r= 5.3 we get three stable solution of the equation which is displayed in fig.6. In fig.5 periodic stability is seen for r = 4 and fig.7 displaysthe chaotic performance of the map for r = 6.

When r = 3, the fixed-point value approaches to 0.723. As the value of parameter decreases i.e., for r = 2, the fixed-point value approaches to 0.523 and for r = 1.5, it reaches to 0.339 as shown in figure 4. In figure 5, for r = 4, there are two solutions of the map x = 0.5565 and x = 1.0502.

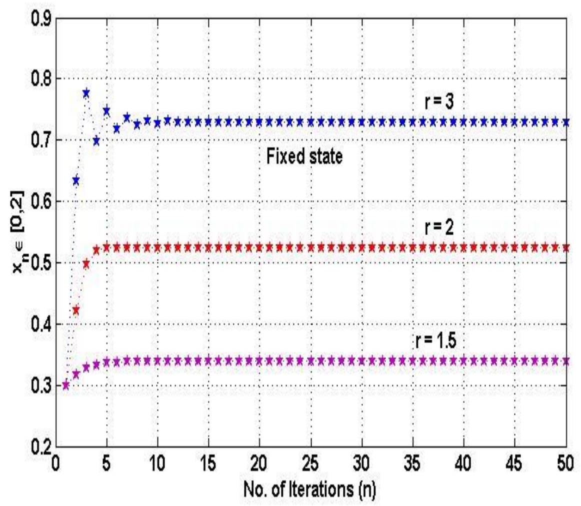

Figure 4. Fixed state of the map $M_r(x)$ for r = 3 and x_0 = 0.3

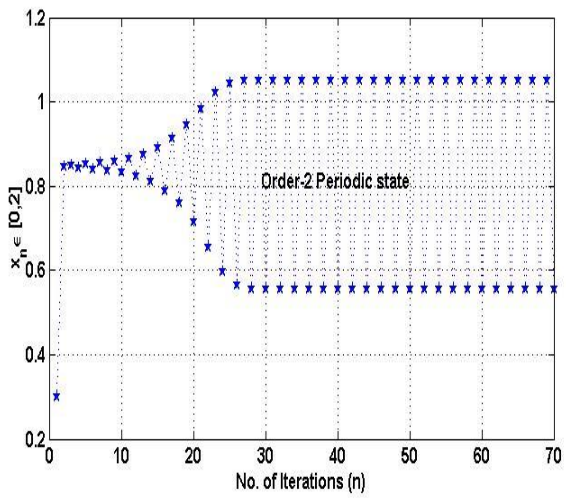

Figure 5. Period-2 solution of the map $M_r(x)$ for r = 4 and x_0 = 0.3

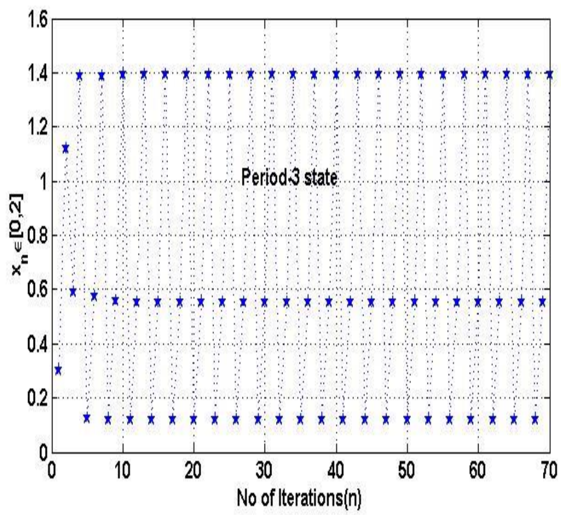

Figure 6. Three stable solutions of the map $M_r(x)$ for r = 5.3 and x_0 = 0.3

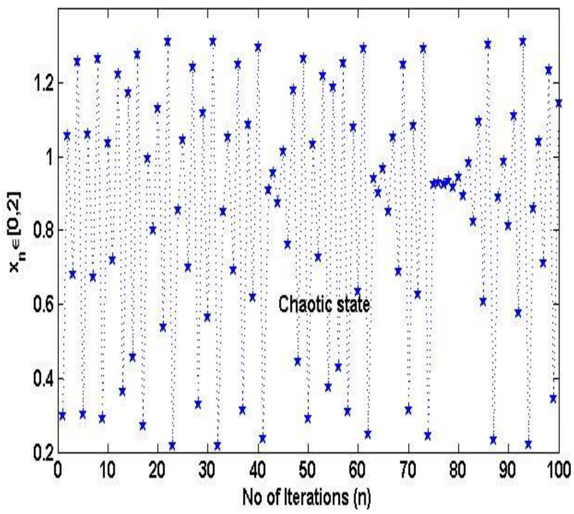

Figure 7. Unstable solution of the map $M_r(x)$ for r = 6 and x_0 = 0.3

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3.2 Experimental Observation

This section presents the experimental study for chaotic behavior of the sine logistic map using Picard's Iterative method. Let

$$x_{n+1} = rx_n(1 - \sin x_n)$$

is the sine logistic map for all $x_n \in [0,2]$ and r ∈ [0,7.5].

The performance of the map based on the parameter r. For $x_0 = 0.3$, from the bifurcation plot it is noted that, the equation shows stable behavior for 1≤r≤3.5605 as shown in figure 8. For 3.5605 < r ≤ 4.4234, it bifurcates into periodicity of order two; i.e., P_1 and P_2 . Further as the parameter r increases through 4.4234, the periodicity of order four i.e., P_{11} , P_{12}, P_{21}, and P_{22} has been seen upto r ≤ 4.6298.

Furthermore, as the parameter r increases from 4.6298 i.e., for 4.6298< r ≤ 4.6711 the periodicity of order eight has been seen and continuing as follows, as we farther increase the parameter r, the arrangement of period implying bifurcation of order 2^n approaches upto $r_\infty \cong$ 4.6823 that is the place of infinite bifurcation.

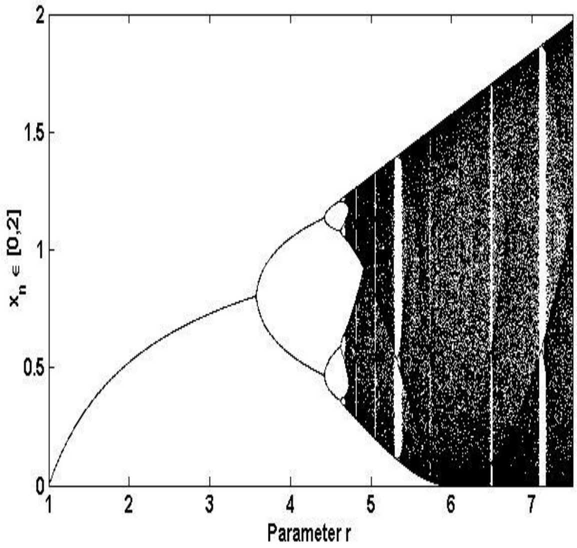

Figure 8. Bifurcation plot of the map $M_r(x)$ for x_0 = 0.3 and $r \in [1,7.5]$

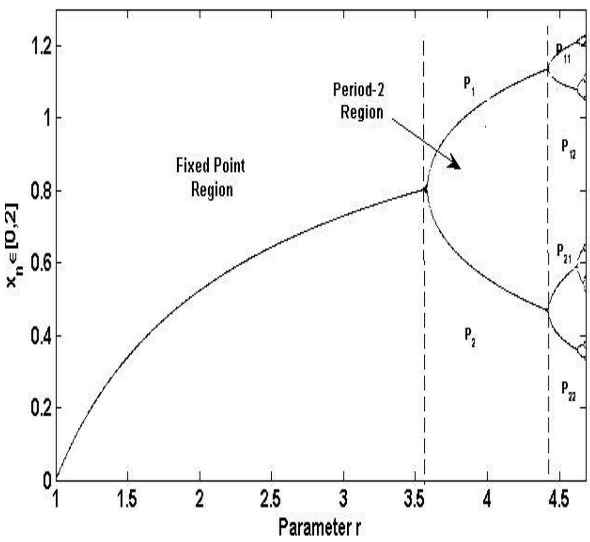

Figure 9. Periodic representation of the map $M_r(x)$ for x_0 = 0.3 and $r \in [1,4.6823]$

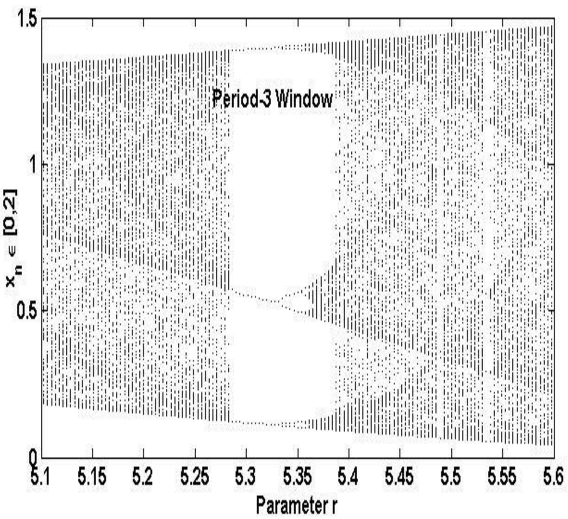

Figure 10. Period-3 window of the map $M_r(x)$ for $r \in [5.28,5.33]$ and x_0 = 0.3

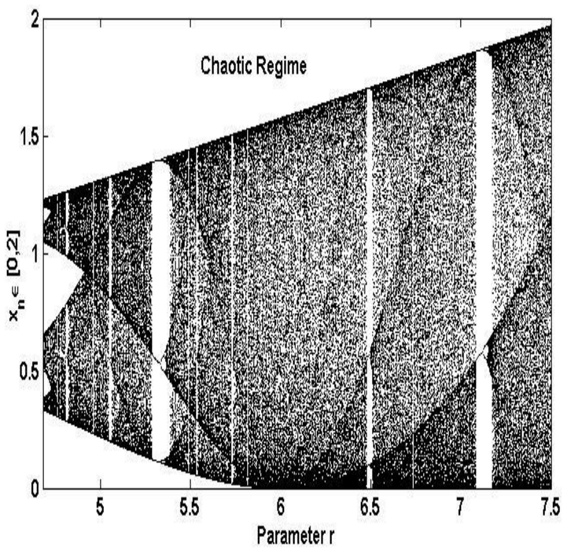

Figure 11. Chaotic region of the map $M_r(x)$ forr $\in [4.68,7.5]$and x_0 = 0.3

3.3 Period – Three Condition

This section, deals with the period-three interval which is the most important characteristics of chaology.The presence of periodic window of order three implies the continuation of periodic windows of higher orders, i.e., Period –three window indicates chaos, and conclusively proceeding towardschaos. The mathematical as well as computational observations are noticed. For initial value$x_0 = 0.3$, period -3 window is observed for $r \in (5.2863,5.3350]$. Period-3 windowis shown in figure 10.

We have made a comparison between the sine logistic map and the standard logistic map from their bifurcation plots as shown in figure 12. It is notified that the sine logistic map has greater chaotic regime as compared to standard logistic map. It is notified that the equation displays the dynamical characteristics for large range of parameter r in comparison with the standard logistic map. So, it is useful in many applications which are based on chaos theory such as in engineering and science as secure communication and cryptography.

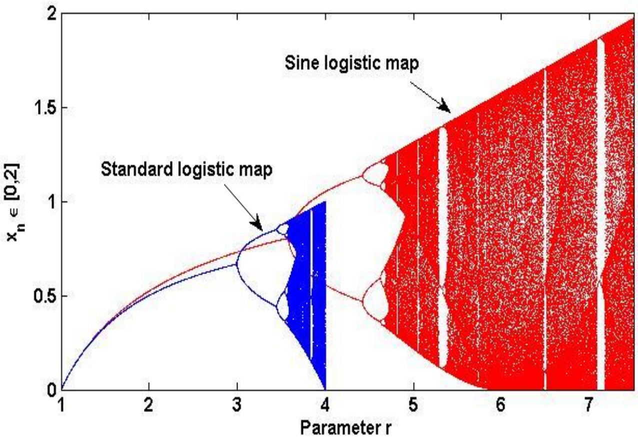

Figure 12. Comparative representation of bifurcation plots of the Sine logistic map and Standard logistic map

4.LYAPUNOV EXPONENT CALCULATION

We accomplished the chaotic properties of the sine logistic equation using Picard orbit by examining time series plots and experimental analysis using bifurcation plots. Now, another way to determine the properties of the dynamical system of sine logistic map is Lyapunov Exponent, in which we compute the susceptible relativity of two orbit originating from very close basic points. Between the orbits of the system the rate of convergence and divergence are measured by Lyapunov Exponent.

Consider $x$and $x+\epsilon$be two initial points, where 0 $<x+\epsilon<$ 1. The exponential growth $e^{n\alpha}$ is computed as the divergence γ in the two orbits, where α denotesthe Lyapunov Exponent of the equation andn denotes the counts of iterations. Now, we have

$$M_r^n(x+\varepsilon) - M_r^n(x) = \gamma = \varepsilon e^{n\alpha}$$

i.e. $\frac{M_r^n(x+\varepsilon)}{\varepsilon} - \frac{M_r^n(x)}{\varepsilon} = e^{n\alpha}$ (4.1)

now, taking limit $\varepsilon \to 0$ on both side, we get

$$\lim_{\varepsilon\to 0}(M_r^n\,(x+\varepsilon) - M_r^n(x))/\varepsilon = e^{n\alpha}$$

$(M_r^n)'(x) = \; e^{n\alpha}$ (4.2)

Now, by taking logarithm on both side we get,

$$\alpha = \frac{1}{n}\log|(M_r^n)'(x)|\,. \qquad (4.3)$$

Where $r \in [0,7.5]$and $(M_r^n)'(x)$ denotes the initial derivative of $M_r^n(x)$.

The differentiation of nth – degree polynomial using chain rule is given by,

$$(M_r^n)'(x_1) = M_r{}'(x_n)\,.\,M_r{}'(x_{n-1})\,\ldots\ldots M_r{}'(x_2)\,.\,M_r{}'(x_1\,). \qquad (4.4)$$

From (4.3) and (4.4), we obtain the claimed Lyapunov exponent as,

$$\alpha = \frac{1}{n}\log|M_r'(x_n).\; M_r'(x_{n-1}).\ldots\ldots M_r'(x_2).\; M_r{}'(x_1)|,$$
$$=\frac{1}{n}[\log|M_r{}'(x_n)| + \log|M_r{}'(x_{n-1})| + \cdots + \log|M_r{}'(x_2)| + \log|M_r{}'(x_1)|],$$
$$=\frac{1}{n}\sum_{i=1}^{n}\log|M_r{}'(x_i)|,\ (4.5)$$

Now, for fixed orbit of$M_r(x)$, equation (4.5) changes to $\alpha = \log|M_r{}'(x_1)|$.

Lyapunov Exponent for the periodic orbit of order-p is

$$\alpha = \frac{1}{p}\sum_{i=1}^{p}\log|M_r'(x_i)|. \qquad (4.6)$$

Remark 4.1 The stability and non-stability of the orbits of the system is determined by the Lyapunov exponent. The trajectory of the system is called stable if α < 0 and non- stable or chaotic if α > 0.

Example 4.2 Let $f_r(x) = rx(1-\sin x)$, where $x \in [0,2]$ and $r \in [0,7.5]$. For the given map$M_r(x)$, calculate the Lyapunov exponent when$r = 3$.

Solution. It is clear from the section 3.1 and 3.2, that the parameter r belongs between 1≤ r ≤ 3.56, then the trajectory of the map is stable for each $x \in [0,2]$ and hence, the stationary point in the orbit for r = 3 is $\sin^{-1}(1-\frac{1}{r}) =$ 0.723. Hence, to find out LE of the stable orbit, we should solve the equation (4.6).

We have $M_r(x) = rx(1-\sin x)$

Then,$M_r'(x) = r(1-\sin x) - rx\cos x$

$= r[1-\sin x - x\cos x]$

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$$M_3'(0.723) = 3[1 - \sin(0.723) - 0.723 \times (\cos(0.723))] = 0.79331$$

From eq.(4.6), we get $\alpha = \log|0.79331| = -0.1005$

Therefore, the Lyapunov exponent for $r = 3$ is -0.1005, which is a negative quantity and hence represents the stability of the fixed orbit.

Example 4.3 Let $f_r(x) = rx(1 - \sin x)$, where $x \in [0, 2]$and $r \in [0, 7.5]$. Then for the map $M_r(x)$, find out the Lyapunov exponent when $r = 4$.

Solution. For $r = 4$, the orbit of the map is periodic orbit for each $x \in [0,2]$ and the value of periodic points for $r = 4$ are noted as $x_1 = 1.0502$ and $x_2 = 0.5565$.

Then, we can calculate, $M_4'(1.0502) = -0.2734$

And $M_4'(0.5565) = 0.4338$

From eq. (4.6), we get $\alpha = \frac{1}{2}[\log|-0.2734| + \log|0.4338| = \frac{1}{2}[-0.5632 + (-0.3627)] = -0.4629$

The Lyapunov exponent are negative in this case also. Therefore, periodic points are stable. Now, for the system $M_r(x)$ we examined the behavior of the Lyapunov exponent (α) for $r \in [0,7.5]$ and $x_0 = 0.3$. The system acquiresa negative Lyapunov exponent value i.e. α < 0 for 0< r < 4.6823 which indicates the stability ofperiodic and fixed orbits of the system as depicted in figure 13. Further, for r > 4.6823, the Lyapunov exponent α > 0, which indicates the dynamic behavior of the system. The magnified chaotic behavior of the system is shown in fig. 14. The maximum value of the Lyapunov exponent approachesup to +0.9.

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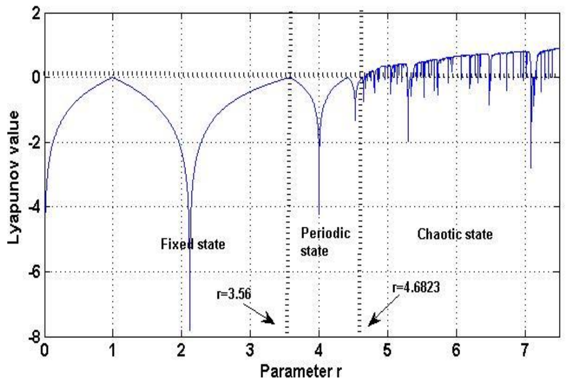

Figure 13. Lyapunov exponent for the system $M_r(x)$ for x_0 = 0.3 and $r \in [0,7.5]$

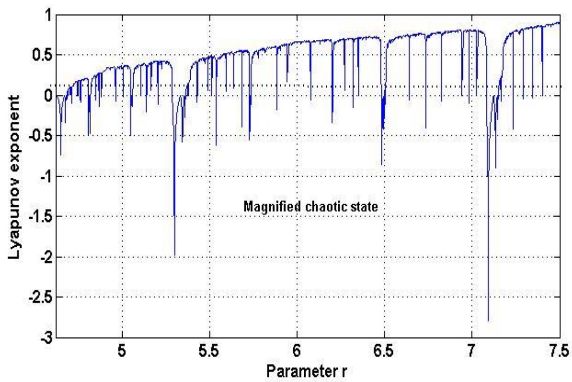

Figure 14. Modified chaotic state for the system $M_r(x)$ for x_0 = 0.3 and $r \in [4.682,7.5]$

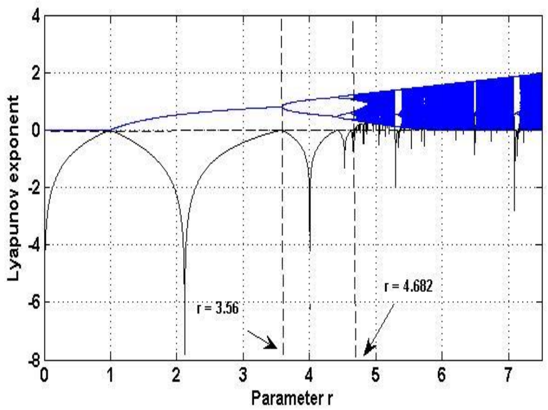

Figure 15. Comparison graph of bifurcation plot and Lyapunov exponent plot for the map $M_r(x)$

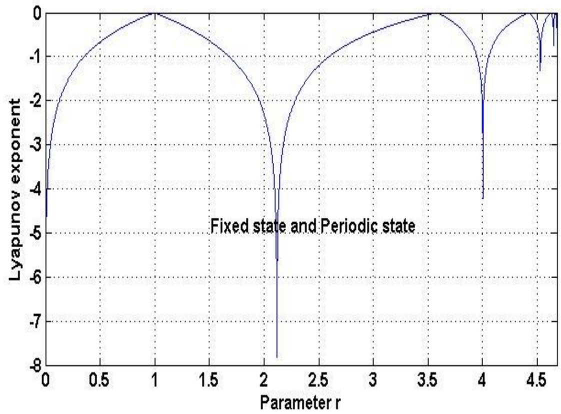

Figure 16. Magnified fixed and periodic state for the map $M_r(x)$ for x_0 = 0.3 and $r \in [0,4.682]$

5. CONCLUSION

This article deals with the observation of dynamical properties of the sine logistic map using one step method i.e., Picard orbit. The whole dynamics depends on the control parameter r. Hence, the subsequent outcomes are concluded.

1. The dynamical properties are determined analytically and experimentally and fixed points are calculated.
2. In the empirical study, the chaotic properties of sine logistic map are examined using bifurcation graphs and time-series, parametric range are described for period-3 window, period doubling, and chaotic regime.
3. This is noticed that, the chaotic characteristicsof the sine logistic map displays for the considerable range of parameter r in comparison with the standard logistic map. Therefore, it improves the chances of using sine logistic map in various applications in engineering, science, secure communication, and cryptography.
4. In section 4 Lyapunov exponent is calculated and graphical analysis is explained briefly.

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