Point canonical transformation approach to an extended harmonic oscillator potential

Chandan Kumar[1], R. K. Yadav[2*], Suman Banerjee[3] and Rajesh Kumar[4]
Department of Physics, S. K. M. University, Dumka-814110, INDIA

Abstract
We solve the one-dimensional time-independent Schrodinger equation using a point canonical transformation approach and obtain the exact bound state solutions of an extended harmonic oscillator potential. It is shown that the eigenvalues of this extended potential are the same as those of the classical harmonic oscillator, but the eigenfunctions are completely different and written in terms of an exceptional Hermite polynomial.

Keywords: Point Canonical Transformation; Rationally Extended Harmonic Oscillator; Bound State; Exceptional Hermite Polynomial

1 Introduction

In quantum mechanics, the Schrodinger equation plays a vital role in describing the dynamics of potential problems. There are very few quantum mechanical potentials whose solutions are obtained exactly or whose entire spectrums are known [1,2]. Some of these are the Harmonic oscillator potential, the Coulomb potential, the Morse potential and the Rosen-Morse potential etc, which are well known to us. The bound state solutions of the Schrodinger equations associated with these potentials are already known and obtained by using different approaches such as the method of separation of variables [1], the point canonical transformation (PCT) approach [3], the supersymmetric (SUSY) approach [2], Darboux-Crum approach [4] and the Nikiforov-Uvarov (NU) method [5] etc. Out of these, it has been observed that the PCT and the NU approaches are equivalent and comparatively simple to construct exactly solvable potentials.

In 2009, a set of new orthogonal polynomials was discovered (for any positive integer of m, i.e., $m \geq 0$), which are named X_m Jacobi and X_m Laguerre exceptional orthogonal polynomials (EOPs) [6,7,8]. These EOPs start with degree $n \geq 1$ (in contrast to the classical orthogonal polynomials, which start with $n \geq 0$) and still form a complete orthonormal set with respect to a positive definite inner product. After the construction of these EOPs, several new potentials are searched [9-17], which are the extension of the corresponding classical potentials and obtained their solutions in terms of EOPs. These extended potentials are extended radial oscillator, the Scarf-I, the Generalised Poschl-Teller (GPT) and the Coulomb potential etc. The bound state spectrums of these extended families are the same as that of the corresponding classical ones, but

*Corresponding author
E-mails: [1]chandansinha14@rediffmail.com(CK); [2]rajeshhep@gmail.com(RKY);
[3]suman.raghunathpur@gmail.com (SB); [4]kr.rajesh.phy@gmail.com(RK)

the eigenfunctions are completely different and written in the exact forms of X_m Jacobi, X_m Laguerre EOPs or a combination of classical polynomials.

Later on, it was noticed that the exceptional Hermite-type polynomials could also be constructed only for even integers of m [18,19,20], which helped in extending the usual harmonic oscillator potential for even m. After that, the extended form of this potential has been obtained using SUSY approach [21]. The same can also be obtained using an elegant and straightforward way of PCT approach. In this work, we use the PCT method and get rationally extended form of the one-dimensional harmonic oscillator and show that the results match exactly the same as obtained in Ref. [18,21] for $m = 2$.

The whole manuscript is arranged in the following way: In Section 2, we briefly discuss the PCT approach and obtain a general expression for potential, eigenvalues and the eigenfunctions. In Section 3, we construct the extended harmonic oscillator and obtain its solution for all even integers of m including $m = 0$. We check the calculation by putting $m = 2$ and show that the result matches exactly same as obtained in Refs. [18]. Finally, we summarize our works in Section 4.

2 Point canonical transformation

In this section, we discuss a simple and elegant approach, the PCT approach [3,22], to get the expression of extended potential $V_{ext}(x)$ with its solutions (energy eigenvalue (E_{ext}) and the eigenfunction ($\psi_{ext}(x)$) by considering the one-dimensional time-independent Schrodinger equation ($\hbar = 2m = 1$)

$$-\psi_{ext}(x)''(x) + V_{ext}(x)\psi_{rxt}(x) = E_{ext}\psi_{ext}(x). \tag{1}$$

To solve this equation, we assume the solutions of the form

$$\psi_{ext}(x) = u(x)\xi(z(x)), \tag{2}$$

where $u(x)$ and $z(x)$ are two undetermined functions and $\xi(z)$ is a polynomial type function, which satisfies a second-order differential equation

$$\xi''(z) + p(z)\xi'(z) + q(z)\xi(z) = 0. \tag{3}$$

On using Eq. (2) in Eq. (1) and then compare with Eq. (3), we get

$$u(x) \propto \left(z'(x)\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int p(z)dz\right). \quad (4)$$

and

$$E_{ext} - V_{ext}(x) = \frac{1}{2}\left[\frac{z'''(x)}{z'(x)} - \frac{3}{2}\left(\frac{z''(x)}{z'(x)}\right)^2\right] + \left(z'(x)\right)^2 \left(q(z) - \frac{1}{2}p'(z) - \frac{1}{4}p^2(z)\right), \quad (5)$$

here a prime denotes a derivative with respect to x. To satisfy equation (5), one needs to find some function $z(x)$ ensuring the presence of a constant term on its right-hand side to compensate *E* on its left-hand one, while rest terms give rise to a potential *V* (*x*) with well-behaved wavefunctions (2).

3 Extended Harmonic Oscillator Potential

The second-order differential equation satisfied by this exceptional Hermite polynomial $\widehat{H}_{n,m}(z)$ is given by [21]

$$\widehat{H}_{n,m}{}''(z) - 2\left(z(x) + \frac{h'_m(z)}{h_m(z)}\right)\widehat{H}_{n,m}{}'(z) + 2n\widehat{H}_{n,m}(z) = 0 \quad (6)$$

where $n = \nu + m + 1$; $\nu = -m - 1,0,1,2$ ..., and $h_m(z)$ is an m-the degree pseudo-Hermite polynomial defined in terms of classical Hermite polynomial $H_m(iz)$ as

$$h_m(z) = (-i)^m H_m(iz) \quad (7)$$

On comparing Eq. (6) with Eq. (3), we get

$$p(z) = -2\left(z(x) + \frac{h'_m(z)}{h_m(z)}\right) \quad (8)$$

and

$$q(z) = 2n \text{ and } \quad \xi(z) = \widehat{H}_{n,m}(z) \quad (9)$$

Using Eq. (8) and (9) in Eq. (5), we get

$$E_{ext} - V_{ext}(x) = \frac{1}{2}\left[\frac{z'''(x)}{z'(x)} - \frac{3}{2}\left(\frac{z''(x)}{z'(x)}\right)^2\right] + \left(z'(x)\right)^2\left(2n + 1 - z^2 - \frac{2h'_m(z)^2}{h_m(z)^2} - \frac{2z(x)h'_m(z) - h''_m(z)}{h_m(z)}\right). \quad (10)$$

In the case of the harmonic oscillator, we define $\mathrm{z(x) = x}$, the above expression reduced to

$$E_{ext} - V_{ext}(x) = 2n + 1 - x^2 - 2\left[\left(\frac{h'_m(x)}{h_m(x)}\right)^2 + \frac{xh'_m(x)}{h_m(x)}\right] + \frac{h''_m(x)}{h_m(x)} \quad (11)$$

Thus the extended potential

$$V_{ext}(x) = x^2 + 2\left[\left(\frac{h'_m(x)}{h_m(x)}\right)^2 + \frac{xh'_m(x)}{h_m(x)}\right] - \frac{h''_m(x)}{h_m(x)} - 1 \quad (12)$$

with the energy

$$E_{ext,\nu} = 2n = 2(\nu + m + 1) \quad (13)$$

The function (4) becomes

$$u(x) \propto \frac{x^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right)}{\mathrm{h_m(x)}} \tag{14}$$

Thus the complete wavefunction (2) will be

$$\psi_{ext}(x) \to \psi_{\nu,ext}(x) \;= N_\nu \frac{x^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right)}{\mathrm{h_m(x)}} \widehat{H}_{n,m}(x), \tag{15}$$

where the exceptional Hermite polynomial

$$\widehat{H}_{n,m}(x) = -h_m(x)H_{n-m}(x) - 2mh_{m-1}(x)H_{n-m-1};\;\; n = \nu + m + 1 \tag{16}$$

The normalization constant for the excited states $\nu = 0,1,2 \ldots$ is [18,21]

$$N_\nu = [\sqrt(\pi)\; 2^{\nu+1}(\nu + m + 1)\nu!\;]^{-1/2}$$

and for the ground state ($\nu = -m - 1$)

$$N_{-m-1} = [\frac{2^m m!}{\sqrt{\pi}}]^{1/2} \tag{17}$$

For m=2:

If we put m=2, Eq. (12) becomes

$$V_{ext}(x) = V_{con}(x) + V_{rat}(x) \tag{18}$$

where the conventional harmonic oscillator potential

$$V_{con}(x) = x^2$$

and the rational part

$$V_{rat}(x) = \frac{8}{(2x^2+1)} - \frac{16}{(2x^2+1)^2} - 3. \tag{19}$$

The corresponding extended wavefunction (15) will be

$$\psi_{\nu,ext}(x) \;= N_\nu \frac{x^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right)}{2(2\mathrm{x}^2+1)} \widehat{H}_{\nu+3,2}(x). \tag{20}$$

with the exceptional Hermite polynomial

$$\widehat{H}_{\nu+3,2}(x) = -h_2(x)H_{\nu+1}(x) - 4h_1(x)H_\nu$$

$$= -2(2x^2 + 1)H_{\nu+1}(x) - 8xH_\nu\;. \tag{21}$$

The energy eigenvalues

$$E_{ext,\nu} = 2(\nu + 3);\quad \nu = -3,0,1,2 \ldots, \tag{22}$$

The results obtained for any m are same as obtained by using SUSY approach in Ref. [18,21], and the particular case of $m = 2$ is the same as shown in Ref.[18]. Hence if we know the differential equation corresponding to any polynomial, which is a solution of a given potential, the PCT approach is very suitable for constructing that potential with exact solutions.

4 Conclusions

In this work, we use a simple approach, the PCT approach, and solve the one-dimensional Schrodinger equation corresponding to a given second-order differential equation satisfied by the exceptional X_m-Hemite polynomial. The constructed potential is the rational extension of the conventional harmonic oscillator potential. The bound state solutions of this extended potential are obtained explicitly in the form of exceptional Hermite polynomial. It is also shown that the extended potential is isospectral (same energy eigenvalue) to the conventional one. As a particular case, we consider m=2 and reproduce all the results, such as the potential, eigenfunction and the recursion relation of exceptional Hermite polynomial etc., as obtained in Refs. [18].

References

[1] S. Fluge, “Pracital Quantum Mechanics-I”, *Springer-Verlag Berlin*, Heidelberg, NY 1971.

[2] F. Cooper, A. Khare, U. Sukhatme *Phys. Rep.* **251** (1995) 267; *"SUSY in Quantum Mechanics"* World Scientific (2001).

[3] A. Bhattacharjie and E. C. G. Sudarshan, *Nuovo Cimento* **25** 864.

[4] G. Darboux, *Theorie Generale des Surfaces* vol 2 (Paris: Gauthier-Villars) (1988).

[5] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Basle: Birkhauser).

[6] D. Gomez-Ullate, N. Kamran and R. Milson, *J. Math. Anal.Appl.* 359 (2009) 352-367.

[7] D. Gomez-Ullate, N. Kamran and R. Milson, *Contemporary Mathematics* 563 (2011) 51-71; *J. Math. Anal.Appl.* 387 (2012) 410-418.

[8] D. Gomez-Ullate, F. Marcellan and R. Milson, *J. Math. Anal. Appl.* 399(2) (2013) 480-495.

[9] C. Quesne, *J.Phys.A* 41 (2008) 392001.

[10] B. Bagchi, C. Quesne and R. Roychoudhary, *Pramana J. Phys.* 73 (2009) 337,

[11] R. K. Yadav, A. Khare and B. P. Mandal, *Ann. Phys.* 331 (2013) 313.

[12] R. K. Yadav, A. Khare and B. P. Mandal, *Phys. Lett. B* 723 (2013) 433.

[13] R. K. Yadav, N. Kumari, A. Khare and B. P. Mandal, *Annals of Physics* 359 (2015) 46.

[14] R. K. Yadav, A. Khare, N. Kumari, B. Bagchi and B. P. Mandal, *J. Math. Phys.* 57 (2016) 062106.

[15] S. Odake and R. Sasaki, *Phys. Lett. B*, 684 (2010) 173; ibid 679 (2009) 414. *J. Math. Phys,* 51 (2010) 053513.

[16] Y. Grandati, Ann. Phys. 326 (2011) 2074.

[17] Y. Grandati, *Ann. Phys. 327* (2012) *185.*

[18] J. M. Fellows, R. A. Smith; *J. Phys. A: Math. Theor*. 42 (2009) 335303.

[19] Y. Grandati, *J.Math. Phys.* 52 (2011) 103505.

[20] C-L Ho, *J.Math. Phys.* 52 (2011) 122107.

[21] I Marquette, C. Quesne, arXiv: 1212.3474v2[math-ph].

[22] R. K. Yadav, N. Kumari, A. Khare and B. P. Mandal, *Acta Polytechnica* **57**(6) (2017) 477–487.