Fixed point theorem for multi-valued condesing signals in Gähler 2-metric spaces.

S. K. Sahani[1*], B. K. Nirala[2] and A. K. Thakur[3]
[*1]*Department of Mathematics, MIT Campus, (T.U.), Janakpurdham, Nepal*
[2]*Department of Mathematics, Janak Hajari Campus (N.S.U), Janakpurdham, Nepal*
[3]*Department of Mathematics, Dr. C.V. Raman University, Bilaspur (C.G.)*
Email: sureshkumarsahani35@gmail.com[1], binaynirala4@gmail.com[2], drakthakurmath@gmail.com[3]
Corresponding author: sureshkumarsahani35@gmail.com[1*]

Abstract:
In this research paper, authors have studied 2-metric spaces, D-metric spaces, G-metric spaces, cone metric spaces, complex valued metric spaces and obtained a new result of 2-metric spaces of fixed point theorem for multi-valued condesing signals, in Gähler theorem. This work is motivated by the works of [1, 17-21].

Keywords: 2-metric spaces, cone metric spaces, complex valued metric spaces, etc.
DOI Number: 10.48047/NQ.2022.20.1.NQ22352 **NeuroQuantology2022;20(1): 726-730**

Introduction:
There have been some generalization of a metric spaces and its fixed point problem such as 2-metric spaces, D-metric spaces, G-metric spaces, cone-metric spaces, complex valued metric spaces. The notion of a 2-metric spaces was introduced by Gähler (see [1]). We notice that a 2-metric is not a continuous function of its variables, where as an ordinary metric is. This led Dhage to introduce the notion of a D-metric spaces (see [2]). In [3], Mustafa and Sims showed that most of topological properties of D-metric spaces were not correct. In [4], they introduce the notion of G-metric spaces many fixed point theorems on G-metric spaces have been obtained. In [5], Jleli and Samet showed that most of the obtained fixed point theorems on G-metric spaces can be deduced immediately from fixed point theorems on metric spaces or quasi metric spaces. In [6], Huang and Zhang defined the notion of a cone metric space which generalized a metric and metric spaces, and proved a some fixed point theorems on metric spaces to cone metric spaces. In [7], Feng and Mao introduced a metric on a cone metric space and then proved that a complete cone metric space is always a complete metric space. They verified that a contractive map on a cone metric space is contractive map on a metric space, then fixed point theorems on a cone metric spaces are essentially, fixed point theorems on a metric space. In [8], Azam et al. introduced the notion of complex valued metric space and some fixed point theorems on this space were stated. In [9], Sastry et al. showed that some fixed point theorems recently generalized to complex valued metric spaces are consequence of their counter parts in the setting of metric spaces and hence are redundant. We notice that in the above generalization, only a 2-metric space is not topologically equivalent to an ordinary metric. Then there was no easy relationship between results obtained in 2-metric spaces and metric spaces. The fixed point theorems on 2-metric spaces and metric spaces may be unrelated easily (see [10-[21]). In this note we prove a fixed point theorem on 2-metric space. This works is motivated by the works of (17, 19).

Let (F,ρ) be a 2-metric space. In [22], Siddiqi et al denote $CB(F)$, the set of all non-empty closed and bounded, subset of F, have studied Gähler 2-metric on $CB(F)$ in the following way:

Let $P,Q,R \in CB(F)$, setting

$$d(P,Q,R) = \sup_{p\in P} \inf_{q\in Q} \rho(p,q,r)$$

where $r \in R$, then Gähler 2-metric L on $CB(F)$ is defined as follows:

$$L(P,Q,R) = \max\{d(P,Q,R), d(Q,R,P), d(R,P,Q)\}.$$

Theorem: If (F,ρ) be a complete 2-metric space and ξ,ζ are two signals which are defined from F to set of all non-empty closed and bounded subset of F in such a way that i.e. (such that)

$$L^n(\xi p,\zeta q,\{r\}) =\leq \delta.\rho \frac{\rho^g(p,\xi p,r)+\rho^g(q,\zeta q,r)}{k^{g-n}(p,\xi p,r)+k^{g-n}(q,\zeta q,r)}$$

hold for all $p,q,r \in F$ in which $k^{g-n}(p,\xi p,r)+k^{g-n}(q,\zeta q,r) \neq 0$ where $o<\delta<1, n\geq 1, g\geq 2$ and $g>n$ then the signals ξ and ζ have common fixed points and $T(\xi)=T(\zeta)$.

For the proof of our theorems, we need the following lemma:

Lemma: If ξ and ζ are two signals defined from F to the set of all non-empty closed and bounded, subsets of F in such a way that $L^n(\xi p,\zeta q,\{r\}) \leq \delta$.

$$\frac{\rho^g(p,\xi p,r)+\rho^g(q,\zeta q,r)}{k^{g-n}(p,\xi p,r)+k^{g-n}(q,\zeta q,r)}$$

holds bitterly for all $p,q,r \in F$ in which

$k^{g-n}(p,\xi p,r)+k^{g-n}(q,\zeta q,r)$

≥ 0 where $0<\delta<1$,

$n\geq 1$ and $g\geq 2$ and $T(\xi)\neq \emptyset$ then

$T(\xi)=T(\zeta)$.

Proof: $\rightarrow$ Let $a\in T(\xi)$

$\Rightarrow a \in \zeta a$

and let $\rho(q,\zeta a,r)\neq 0$

then by hypothesis of definition

$$\rho^n(a,\zeta a,r) \leq L^n(\xi a,\zeta a,\{r\})$$

$$\leq \delta \frac{\rho^g(a,\xi a,r)+\rho^g(a,\zeta a,r)}{k^{g-n}(a,\xi a,r)+k^{g-n}(a,\zeta a,r)}$$

$$\leq \delta \frac{\rho^g(a,\zeta a,r)}{\rho^{g-n}(a,\xi a,r)+\rho^{g-n}(a,\zeta a,r)}$$

$\Rightarrow \rho(a,\zeta a,r)=0$.

Also, since ζa is closed then

$a\in \zeta a \Rightarrow T(\xi)\subset T(\zeta)$.

Similarly, we can show that
$T(\zeta)\subset T(\xi)$.

Proof of the theorem:

Let b be real number such that $1<b<(\delta)^{-3^{-1}}$. (1)

Also, let $c_0\in F$ and $c_1\in\xi c_0$ then $c_2\in\xi c_1$
and
$\rho(c_1,c_2,r)\le b\ L(\xi c_0,\zeta c_1,\{r\})$.
Let $c_3,c_4,\ldots,c_{2n},c_{2n+1},\ldots$
could be choosen so that $c_{2n}\in\xi c_{2n-1},c_{2n+1}\in\zeta c_{2n}$
and
$\rho(c_{2n},c_{2n+1},r)\le b\ H(\xi c_{2n-1},\zeta c_{2n},\{r\})$
$\rho(c_{2n+1},c_{2n+2},r)\le\delta.H(\xi c_{2n+1},\zeta c_{2n},\{r\})$.
We consider
$k^{g-n}(c_{2n},\xi c_{2n},r)+k^{g-n}(c_{2n+1},\xi c_{2n+1},r)=0$
$\Rightarrow C_{2n+1}=C_{2n}$ is a
common fixed point of ξ and ζ.
Again, if
$k^{g-n}(c_{2n},\xi c_{2n},r)+k^{g-n}(c_{2n+1},\zeta c_{2n+1},r)\neq 0$
and $k^{g-1}(c_{2n+1},\zeta c_{2n+1},r)+k^{g-n}(c_{2n},\xi c_{2n},r)\neq 0$
then

$$\rho^n(c_{2n},c_{2n+1},r)\le b^nL^n(\xi c_{2n-1},\zeta c_{2n},\{r\})$$
$$\le s.b^n.\frac{[\rho^g(c_{2n-1}),\xi c_{2n-1},r]+\rho^g(c_{2n},\zeta c_{2n},r)]}{k^{g-n}(c_{2n-1},\xi c_{2n-1},r)+k^{g-n}(c_{2n},\zeta c_{2n},r)}$$
$$s.\ b^n\frac{[\rho^g(c_{2n-1},c_{2n},r)+\rho^g(c_{2n},c_{2n+1},r)]}{\rho^{g-n}(c_{2n-1},c_{2n},r)+\rho^{g-n}(c_{2n},c_{2n+1},r)}$$
$$\Rightarrow\rho^g(c_{2n},c_{2n+1},r)(1-sb^n)+\rho^g(c_{2n},c_{2n+1},r)\rho^{g-n}(c_{2n},c_{2n+1},r)$$
$$-sb^n\rho^g(c_{2n-1},c_{2n-1},r)\le 0$$
and
$$\ell^g(1-sb^n)+\ell^n-sb^n\le 0$$
where
$$\ell=\frac{\rho(c_{2n},c_{2n+1},r)}{\rho(c_{2n-1},c_{2n-1},r)}.$$
Again, we define another new function J from $[0,\infty]$ to R be function b g
$$J(\ell)=\ell^g(1-sb^n)+\ell^n-s\ell^n)$$
then $J(\ell)>0$ and $J(0)<0$
thus $J(\ell)\le 0$

$\Rightarrow \rho(c_{2n}, c_{2n+1}, r) \leq$
$v\rho(c_{2n}, c_{2n+1}, r)$

Similarly, we obtain

$\rho(c_{2n+1}, c_{2n+2}, r) \leq$
$v\ \rho(c_{2n}, c_{2n+1}, r)$

and in the same way,
we obtain

$\rho(c_{2n+1}, c_{2n}, r) \leq v^n \rho(c_0, c_1, r)$

which show that $\{c_n\}$ is a Cauchy sequence and F is complete then $\lim c_n = v_0$ for some $v_0 \in F$.

Again, we suppose

$\rho(v_0, \xi v_0, r) \neq 0$ then

$\rho^n(c_{2n+1}, \xi v_0, r) \leq$
$L^n(\zeta c_{2n}, \xi v_0, \{r\})$
$\leq \delta \frac{[\rho^g(c_{2n}, \zeta c_{2n}, r) + \rho^g(v_0, \xi v_0, r)]}{\rho^{g-n}(c_{2n}, c_{2n+1}, r) + \rho^{g-n}(v_0, \xi v_0, r)}$
$\rightarrow \rho(v_0, \xi v_0, r) \leq \delta\ \rho(v_0, \xi v_0, r)$ as $n \rightarrow \infty$

Which gives a contradiction

$\Rightarrow \rho(v_0, \xi v_0, r) = 0.$

Thus by using lemma,

$\mathrm{v}_0 \in \xi v_0$ and $v_0 \in \zeta v_0$

and $T(\xi) = T(\zeta)$.

This completes the proof of the theorem.

References:

[1] V. S. Gähler, 2-metrishe Raume und ihre topologische struktur, Math. Nachr. 26, 115-118 (1963/64).

[2] B. C. Dhage, A study of some fixed point theorems, Ph.D. thesis, Marathwada, Aurangabd, India (1984).

[3] Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, In Proceedings of the international conferences on fixed point theorems and Applications, Valencia, Spain, 189-198 (2003).

[4] Z. Mustafa and B. Sims, A New approach to generalized metric spaces, J. Nonlinear convex Anal. 7 (2), 289-297 (2006).

[5] M. Jleli and B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl. 2012, Article ID 201 (2012).

[6] L. G. Huang and X. Zhang, Cone metric spaces and Fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332, 1468-1476 (2007).

[7] Y. Feng and W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, 11 (2), 259-264 (2010).

[8] A Azam et Pat., Common Fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim, 32 (3), 243-253 (2011).

[9] K. P. R. Sastry et al., Metrizability of complex valued metric spaces and some remarks on Fixed point theorems in complex valued metric spaces, Int. J. Pure Appl. Math. 3 (7), 2686-2690 (2012).

[10] A. Aliouche and C. Simpson, Fixed point and lines in 2-metric spaces, Adv. Math., 229, 668-690 (2012).

[11] B. Deshpande et al., Common Fixed point theorems for hybrid pairs of mappings with some weaker conditions in 2-metric spaces, Fase. Math. 46, 37-55 (2011).

[12] R. W. Freese et al., Strictly 2-convex linear 2-normed spaces, J. Korean Math. Soc, 29 (2), 391-400 (1992).

[13] V. S. Gähler, Lineare 2-normlerte Raume, Math. Nachr., 28, 1-43 (1965).

[14] V. S. Gähler, Über die uniformisie rbarkeit 2-metrischer Raume, Math. Nachr., 28, 235-244 (1965).

[15] K, Iseki, Fixed point theorems in 2-metric spaces, Math. Sem. Notes Kobe Univ., 3, 133-136 (1975).

[16] K Iseki, Mathematics on 2-normed spaces, Bull. Korean Math. Soc., 13 (2), 127-135 (1976).

[17] B. K. Lahari et al., Cantor's theorems in 2-metric spaces and its application to fixed point problems, Taiwan, J. Math., 15, 337-352 (2011).

[18] S. N. Lai and A. K. Singh, An Analogue of Banach's Contraction Principle of 2-metric spaces, Bull. Aust. Math. Soc., 18, 137-143 (1978).

[19] S. L. Singh et al., Suzuki Contraction theorem on a 2-metric space, J. Adv. Math. Stud., 5, 71-76 (2012).

[20] S. L. Singh et al., Erratum to Suzuki Contraction theorem on a z-metric space, J. Adv. Math. Stud., 5 (2), 138 (2012).

[21] V. D. Nguyen et al., Remarks on the Fixed Point Problems of 2-metric spaces, Fixed Point Theory and Appl, 2013, 2013: 167.