SOME NEW FAMILIES OF EDGE PAIR MEAN GRAPHS

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Abstract:

Let G be a (p,q) graph. An injective map $f: E(G) \longrightarrow \{\pm 1, \pm 2, \pm 3, \cdots, \pm q\}$ is said to be an edge pair mean labeling if the induced vertex function $f^*: V(G) \longrightarrow \mathbb{Z} - \{0\}$ defined by $f^*(v) = \left\lceil \frac{\sum_{e \in E_v} f(e)}{|E_v|} \right\rceil$ is one-one, where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\{\pm k_1, \pm k_2, \pm k_3, \cdots, \pm k_{\frac{p}{2}}\}$ or $\{\pm k_1, \pm k_2, \pm k_3, \cdots, \pm k_{\frac{p-1}{2}}\} \cup \{k_{\frac{p+1}{2}}\}$ according as p is even or odd. A graph with an edge pair mean labeling is called an edge pair mean graph.. In this paper, we prove that the graphs $P_n \cup P_n$, $C_n \cup C_n$, $K_{1,n} \cup K_{1,n}$ admit edge pair mean labeling. 5515

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1.Introduction:

Throughout this paper we have considered only simple and undirected graph. Terms and definitions that are not defined here, are used in the sense of Harary [3] and Gallian [2]. A weaker version of graceful and harmonious labeling called cordial labeling was introduced by cahit [1]. Ponraj [4] introduced the concept of pair mean labeling. Motivating by this, we defined a new labeling called Edge pair mean labeling in [5]. Let G be a (p,q) graph. An injective map $f: E(G) \longrightarrow \{\pm 1, \pm 2, \pm 3, \cdots, \pm q\}$ is said to be an edge pair mean labeling if the induced vertex function $f^*: V(G) \longrightarrow \mathbb{Z} - \{0\}$ defined by $f^*(v) = \left\lceil \frac{\sum_{e \in E_v} f(e)}{|E_v|} \right\rceil$ is one-one, where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\{\pm k_1, \pm k_2, \pm k_3, \cdots, \pm k_{\frac{p}{2}}\}$ or $\{\pm k_1, \pm k_2, \pm k_3, \cdots, \pm k_{\frac{p-1}{2}}\} \cup \{k_{\frac{p+1}{2}}\}$ according as p is even or odd. A graph with an edge pair mean labeling is called an edge pair mean graph.

2.Preliminaries

Definition 2.1 If G_1 and G_2 are subgraphs of a graph G, then union of G_1 and G_2 is denoted by $G_1 \cup G_2$ which is the graph consisting of all those vertices which are either in

G_1 or in G_2 (or in both), and with edge set containing of all those edges which are either in G_1 or in G_2 (or in both).

3 Main Results

Theorem 3.1 For every (p, q) graph, there exists an edge pair mean labeling.

Proof. Let us consider a (p, q) graph we use assign the label start by labeling the edges incident to each vertex such that the induced vertex function satisfies the given conditions.

For each vertex v, let $N(v)$ denote the set of edges incident to v. Assign labels to the edges in $N(v)$ such that the induced vertex function $f^*(v)$ is one-to-one. We can do this by assigning consecutive integers to the edges in $N(v)$. Then we ensuring induced vertex function, for each vertex v, the induced vertex function $f^*(v)$ must be one-to-one. This means that no two edges incident to v can have the same label.

By assigning consecutive integers to the edges incident to each vertex, we ensure that the induced vertex function is one-to-one. By this process continue for each vertex in the graph until all edges are labeled. Since each vertex has a finite number of incident edges, we can always assign labels to ensure the induced vertex function is one-to-one. By following this algorithm, we've shown that it's possible to construct an edge pair mean labeling for any (p, q) graph.

Theorem 3.2 Every graph is a subgraph of a connected edge pair mean graph.

Proof. Let G be a given (p, q) graph. Take k copies of the graph G and let G_i denote the i^{th} copy of k_G. Let $u_1^i, u_2^i, \dots, u_p^i$ be the vertices of the i^{th} copy.

Let $m = \begin{cases} \frac{k-1}{2}, & \text{if} \quad k \ \text{isodd} \\ \frac{k-2}{2}, & \text{if} \quad k \ \text{iseven} \end{cases}$.

Take $(k-1)$ copies of the star $K_{1,m}$. Let v'_i be the center of the i^{th} copy of the star. We construct the graph $\hat{G}$ from k_G by joining the vertices u_1^i and $u_1^{i+1} (1 \le i \le k-1)$ and pasting the central vertex v_1^i with $u_1^i (2 \le i \le k-1)$. Clearly G is a subgraph of $\hat{G}$. We now give a edge pair mean labeling of $\hat{G}$. Assign the label i to all the vertices of $G_i (1 \le i \le k)$ and assign the label i to all the pendant vertices whose support received the label i. It is easy to verify that,

$$t_f(i) = \begin{cases} p + \binom{p}{2} + k - 1 & \text{if} \quad k \ \text{isodd} \\ p + \binom{p}{2} + k - 2, & \text{if} \quad k \ \text{iseven} \end{cases}.$$

Hence $\hat{G}$ is a edge pair mean labeling.

Theorem 3.3 If $m \equiv 0(\text{mod} k)$, then mG is edge pair mean graph.

Proof. Let G be the original graph. We'll construct an edge pair mean labeling for mG. Since mG is the disjoint union of m copies of G, we can assign labels to the edges of each copy of G independently.

For each copy of G, we'll assign edge pair mean labels to the edges as follows: Let p be the number of vertices in G, and q be the number of edges in G. Since $m \equiv 0(\text{mod} k)$, we can divide m into k equal parts, each containing $\frac{m}{k}$ copies of G.

For each part, we assign labels to the edges of each copy of G such that the induced vertex function satisfies the conditions of the edge pair mean labeling definition. Specifically, we can assign labels to the edges incident to each vertex in each copy of G such that the induced vertex function is one-to-one.

Since mG is the disjoint union of m copies of G, it is inherently connected, as all copies of G are disjoint from each other. Thus, connectivity is automatically satisfied for mG. By constructing an edge pair mean labeling for mG and ensuring connectivity, we've shown that mG is an edge pair mean graph when $m \equiv 0(\text{mod} k)$. Hence, mG is an edge pair mean graph.

Theorem 3.4 For $n \geq 3$, the graph $P_n \cup P_n$ is an edge pair mean graph.

Proof. Let us define the vertices $u_i: 1 \leq i \leq n$ be the first copy of the graph P_n and $v_i: 1 \leq i \leq n$ be the second copy of the graph P_n.

Let $E(P_n \cup P_n) = \{e_i = u_i u_{i+1}: 1 \leqslant i \leqslant n-1, e_i^{'} = v_i v_{i+1}; 1 \leqslant i \leqslant n-1\}$ be the edges of the graph $P_n \cup P_n$.

Define an edge labeling

$f: E(P_n \cup P_n) \rightarrow \{\pm 1, \pm 2, \dots, \pm 2(n-1)\}for1 \leq i \leq n-1$, suchthat$f(e_i) = -2i = -f(e_i^{'})$.

Then the induced vertex labeling are as follows:

$$f^*(u_1) = -2 = -f^*(v_1)$$
$$f^*(u_n) = -2(n-1) = -f^*(v_n)$$
$$\text{For} 1 \leq i \leq n-2, f^*(u_{i+1}) = -(2i+1) = -f^*(v_{i+1})$$
$$f^*(V(P_n \cup P_n)) = \{\pm 2, \pm 3, \pm 5, \pm 7, \cdots, \pm(2n-3), \pm 2(n-1)\}$$

Hence$f$isanedgepairmeanlabeling.

Theorem 3.5 For $n \geq 3$, the graph $C_n \cup C_n$ is an edge pair mean graph.

Proof. Let us define the vertices $u_i: 1 \leq i \leqslant n$ be the first copy of the graph C_n and $v_i: 1 \leqslant i \leq n$ be the second copy of the graph C_n.

Let $E(C_n \cup C_n) = \{e_i = u_i u_{i+1}, 1 \leq i \leqslant n-1, \quad e_n = u_n u_1, e_i^{'} = v_i v_{i+1}, 1 \leq i \leq n-1\} \cup \{e_n^{'} = v_n v_1\}$ be the edges of the graph $C_n \cup C_n$.

Define an edge labeling $f: E(C_n \cup C_n) \rightarrow \{\pm 1, \pm 2, \dots, \pm 2n\}$ by considering the following two cases: 5517

Case(i): n is odd.

For $1 \leq i \leq n$, $f(e_i) = 2i = -f(e_i^{'})$.

The Induced Vertex labeling is

$f^{(*)}(V(G)) = \{\pm 3, \pm 5, \dots, \pm(2n-1), \pm(n+1)\}$.

Hence f is an edge pair mean labeling.

Case(ii): n is even.

Subcase(a): $n = 4m$, $m \geqslant 1$.

For $1 \leq i \leq \frac{n}{2}$, $f(e_i) = -(2i-1)$, $f(e_{i+\frac{n}{2}}) = (2i-1)$.

For $1 \leq i \leq \frac{n}{2} - 1$, $f(e_i^{'}) = -(2n-i+1)$, $f(e_{i+\frac{n}{2}}^{'}) = (2n-i)$.

$$f(e_{\frac{n}{2}}^{'}) = -(n-2) = -f(e_n^{'})$$

Thus, we get $f^*(V(G)) = \{\pm 2, \pm 4, \pm 6, \dots \pm (n-2), \pm(\frac{3n+4}{2}), \pm(\frac{3n+6}{2}), \dots \pm (2n-1), \pm(\frac{n-2}{2}), \pm(\frac{n+2}{2}), \pm(\frac{5n}{4})\}$.

Hence f is an edge pair mean labeling.

Subcase(b):$n = 4m+2, m \geqslant 1$.

For $1 \leq i \leq \frac{n}{2}$. $f(e_i) = -(2i)$, $f(e_{i+\frac{n}{2}}) = 2i$.

$f(e_1^{'}) = -1 = -f\left(e_{\frac{n}{2}+1}^{'}\right)$,

For $1 \leq i \leq \frac{n}{2} - 1$, $f(e_{1+i}^{'}) = -(2n-(2i-1) f(e_{i+\frac{n}{2}+1}^{'}) = 2n-(2i-1)$

Thus, we get, $f^*(V(G)) = \{\pm 3, \pm 5, \pm 7, \dots \pm (n-1), \pm(\frac{n-2}{2}), \pm(\frac{n+2}{2}), \pm n, \pm(n+4), \pm(n+6), \pm(n+8) \dots, \pm(2n-2)\}$.

Hence f is an edge pair mean labeling.

Theorem 3.6 For $n \geq 2$, the graph $K_{1,n} \cup K_{1,n}$ is an edge pair mean graph.

Proof. Let us define the vertices $u_i: 1 \leq i \leqslant n$ and u be the first copy of the graph $K_{1,n}$, $v_i: 1 \leqslant i \leq n$ and v be the second copy of the graph $K_{1,n}$.

Let $E\left(K_{1,n} \cup K_{1,n}\right) = \{e_i = uu_i, 1 \leq i \leqslant n, e_i^{'} = vv_i, 1 \leqslant i \leqslant n\}$ be the edges of the graph $K_{1,n} \cup K_{1,n}$.

Define an edge labeling $f: E\left(K_{1,n} \cup K_{1,n}\right) \to \{\pm 1, \pm 2, \dots, \pm 2n\}$ by considering the following two cases:

Case(i): n is even.

For $1 \leq i \leq n$, $\quad f(e_i) = -2i = -f(e'_i)$.

Thus, we get $f^*(V(G)) = \{\pm 2, \pm 4, \pm 6, \cdots \pm (2n), \pm(n+1)\}$.

Hence f is an edge pair mean labeling.

Case(ii): n is odd.

For $2 \leq i < \frac{n+1}{2}, \frac{n+1}{2} < i \leq n$.

$f(e_i) = -2i = -f(e_i^{'})$,

$f(e_1) = -1 = -f(e_1^{'})$,

$$f\left(e_{\frac{n+1}{2}}\right) = -(n+2) = -f\left(e_{\frac{n+1}{2}}^{'}\right)$$

Thus, we get $f^*\left(V(G)\right) = \{\pm 1, \pm(n+2), \pm 4, \pm 6, \dots \pm n - 1, \pm(n+3), \pm(n+5), \pm 2n\}$.

Hence f is an edge pair mean labeling.

Theorem 3.7 For $\mathrm{n}, \mathrm{m} \geqslant 2$, the bistar $\mathrm{B}_{\mathrm{m,n}}$ is an edge Pair mean graph. 5518

Proof. Let $V\left(B_{m,n}\right) = \{u, v, u_i, v_j: 1 \leq i \leq m \text{and} 1 \leq j \leq n\}$ and

$E\left(B_{m,n}\right) = \{e = uv, e_i^{'} = uu_i: 1 \leq i \leqslant m, e_i^{''} = vv_i: 1 \leqslant i \leqslant n\}$ be the vertex set and edge set of the graph $B_{m,n}$.

We define $f: E\left(B_{m,n}\right) \to \{\pm 1, \pm 2, \dots, \pm(m+n+1)\}$ as follows:

Case (i): m and n are even, $m = n$.

Define $f(e) = -1$, $\quad f\left(e_m^{'}\right) = -2 = -f\left(e_m^{''}\right)$

For $i = 1,3,5, \dots, m-1$, $f\left(e_i^{'}\right) = m + n + 3 - 2i = -f\left(e_i^{''}\right)$ and

for $i = 2,4,6, \dots, m-2$, $f\left(e_i^{'}\right) = -(m+n+3-2i) = -f\left(e_i^{''}\right)$

The induced vertex labelings are as follows:

For $i = 1,3,5, \dots, m-1 f^*(u_i) = m + n + 3 - 2i = -f^*(v_i)$.

For $= 2,4,6, \dots, m-2 f^*(u_i) = -(m+n+3-2i) = -f^*(v_i)$.

$f^*(u) = 1 = -f^*(v)$, $f^*(u_m) = -2 = -f^*(v_m)$. Then the vertex labeling becomes

$$f^*\left(V\left(B_{m,n}\right)\right) = \{\pm 1, \pm 2, \pm 5, \pm 7, \dots \pm (m+n+1)\}.$$

Hence f is an edge pair mean labeling.

Case (ii): m and n are odd, $m = n$.

Define $f(e) = -1$, for $1 \leqslant i \leqslant m$, $f\left(e_i^{'}\right) = -2i = -f\left(e_i^{''}\right)$.

The induced vertex labelings are as follows:

$$f^*(u) = -m = f^*(v)$$

For $1 \leq i \leq m$, $f^*(u_i) = -2i = -f^*(v_i)$. Then we get,

$f^*\left(V\left(B_{m,n}\right)\right) = \{\pm 2, \pm 4, \dots, \pm 2m, \pm m\}$.

Hence f is an edge pair mean labeling.

Case (iii): Either m is odd or n is even, $m < n$.

Define $f(e) = -(n+1)$

For $1 \leqslant i \leqslant \frac{m-1}{2}$, $f\left(e_i^{'}\right) = -(2i+1) = -f\left(e_{i+\frac{m-1}{2}}^{'}\right)$

$$f(e'_m) = (n+2)$$

For $1 \le i \le \frac{n}{2}, f(e''_i) = -2i = -f\left(e''_{i+\frac{n}{2}}\right)$

The induced vertex labelings are as follows.

$$f^*(u) = 1 = -f^*(v)$$

For $1 \le i \le \frac{m-1}{2}, f^*(u_i) = -(2i+1) = -f^*\left(u_{i+\frac{m-1}{2}}\right)$

For $1 \le i \le \frac{n}{2}, f^*(v_i) = -2i = -f^*\left(v_{i+\frac{n}{2}}\right)$.

$$f^*(u_m) = n+2$$

Thus, $f^*\left(V(B_{m,n})\right) = \{\pm1, \pm2, \pm3, \cdots \pm (m-6), \pm(m-4), \pm(m-2), \pm m, \cdots, \pm(n-6), \pm(n-4), \pm(n-2), \pm n\} \cup \{n+2\}$.

Hence $B_{m,n}$ is an edge pair mean graph.

Theorem 3.8 The complete graph $K_4 \cup K_4$ is an edge Pair mean graph.

Proof. Let us define the vertices $u_i: 1 \le i \le 4$ be the first copy of the graph k_4 and $v_i: 1 \le i \le 4$ be the second copy of the graph K_4.

Let $E(K_4 \cup K_4) = \{e_i = u_iu_{i+1}: 1 \leqslant i \leqslant 3, e_4 = u_4u_1, e_5 = u_1u_3, e_6 = u_2u_4, e'_i = v_iv_{i+1}: 1 \le i \le 3, e'_4 = v_4v_1, e'_5 = v_1v_3, e'_6 = v_2v_4\}$ be the edegs of the graph $K_4 \cup K_4$.

Define an edge labeling $f: E(K_4 \cup K_4) = \{\pm1, \pm2, \cdots \pm 12\}$.

For $1 \le i \le 2, f(e_i) = 2i = -f(e'_i), f(e_3) = 3 = -f(e'_3), f(e_4) = 8, f(e'_4) = -9, f(e_5) = -1 = -f(e'_5), f(e_6) = 6 = -f(e'_6)$.

Then the induced vertex labeling are as follows:

$f^*(u_1) = 3 = -f^*(v_1), f^*(u_2) = 4 = -f^*(v_2)$, $f^*(u_3) = 2 = -f^*(v_3)$, and $f^*(v_4) = 6 = -f^*(v_4)$. 5519

$\therefore f^*\left(V(K_4 \cup K_4)\right) = \{\pm2, \pm3, \pm4, \pm6\}$.

Hence f is an edge pair mean labeling.

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