MATHEMATICS BEHIND IMAGE COMPRESSION: AN EXPERIMENT OF WAVELETS COMPRESSION OF VARIOUS SIZES AND THEIR RELATIVE COMPRESSION RATIO OF FRUIT IMAGES

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ABSTRACT:
The main aim of this research paper is to introduce wavelet analysis applications and its features that can be applied to specific areas of interest like mathematical physics, information technology, fluid dynamics, optimization theory, biomedical engineering etc. The purpose of this research paper is the use of tools and strategies based on wavelets and wavelet transform to solve problems encountered in the chosen application areas and related fields. This research paper carried out, in general, aims to demonstrate the advantages of wavelet or wavelet based methods by applying them to analyze data, build mathematical models and run simulations arising in science and engineering. This analysis has also been a part of product designs, system prototypes, industrial products and manufacturing processes.

Keywords: Image Compression, Wavelet Transformation, Wavelet Packets.
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1. INTRODUCTION

Wavelets came into limelight in middle of 1980's when they were utilized for signal processing and image compression. It was considered as a tool in these fields and the effectiveness was showcased in experimental results. Wavelets are mostly employed for tasks such as image compression, pattern recognition, and signal denoising that was made possible because of their inbuilt properties. Their main property of localization in space and frequency together with the existence of a fast algorithm makes wavelet an efficient and effective tool for such tasks. Factually wavelets can be used to generate many ideas that are created in pure and applied mathematics, physics and engineering. The wavelet became interesting and useful to people during early 1980's when geophysicist J. Morlet presented his work. He was the one who recognized the need of signal processing techniques going beyond Fourier analysis. He modified Gaussian window used by Gabor and called it a "wavelet" meaning a little wave. The results presented though promising but were not well acknowledged by the mathematical community. In 1984 Grossman realized the potential of Morlet's wavelet and laid a strong groundwork to the theory.

Wavelets is defined by a mathematical function $\psi \varepsilon L_2(¡\,)$ such that $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ here j,kε¢ is an orthonormal basis of the Hilbert space of finite energy function $L_2(¡\,)$. This function space is obtained from a solo function when operators like translation and dilation are performed. This process is an abstract and to explain it to any layman, it could be said that shape of the function is not changed but simply it is shifted to a position/time of interest after squeezing or spreading it. Alternatively, the term wavelet is a function that reveals oscillatory behavior in a some interval I, and then vanishes outside I after exhibiting the behavior and decaying to zero. Symbolically, function

$$\psi : i \to i \text{ be such that i) } \int_{-\infty}^{\infty} \psi(x)dx = 0; \quad \text{ii) } \int_{-\infty}^{\infty} \psi^2(x)dx = 1; \text{ is a wavelet.}$$

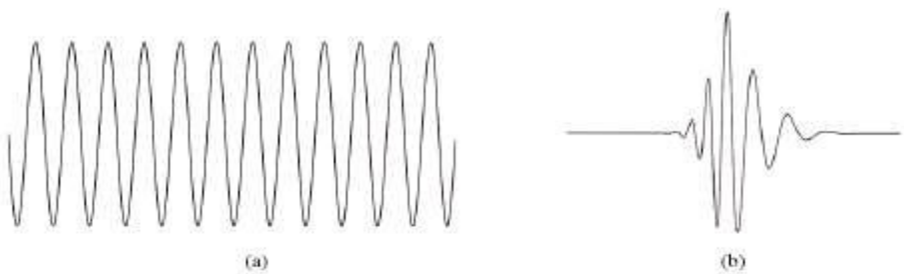

Figure-1: Demonstration of (a) a wave (b) a wavelet

There are several types of wavelets. Wavelets are grouped into families (selected few). These are in order of appearance:

1) Haar wavelet
2) Morlet wavelet
3) Symmlet
4) Meyer Wavelet
5) Daubeachies Wavelet.
6) Coiflet

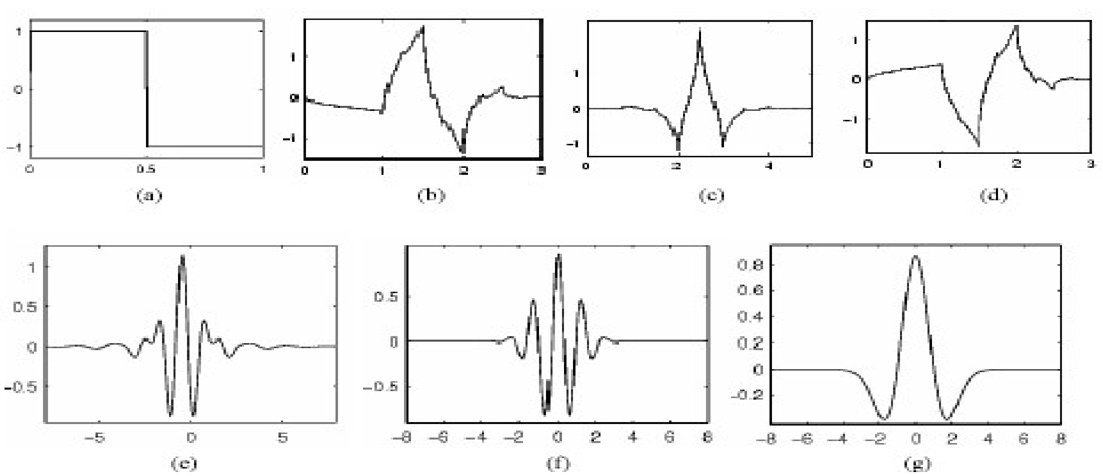

Figure-2: Demonstration of wavelet families: (a) Haar (b) Daubechies (c) Coiflet (d) Symlet (e) Morlet (f) Meyer (g) Mexican Hat

Haar wavelet transform have less computational complexity which in turn requires less hardware, and memory.

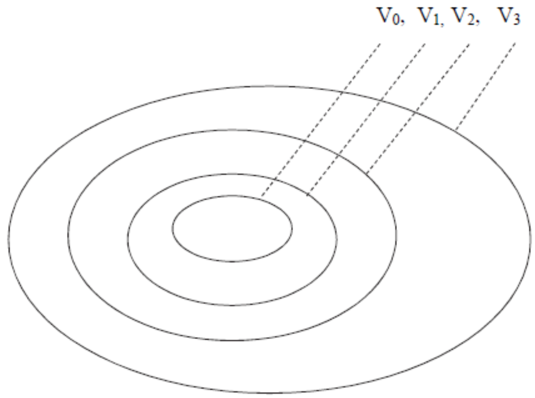

Figure-3: Nested vector spaces spanned by the scaling functions

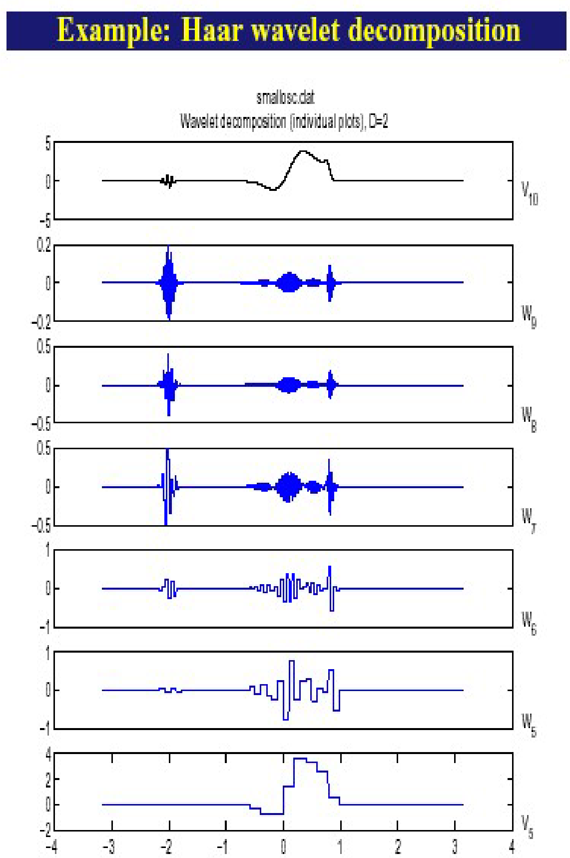

Figure-4: Wavelet decompositions of one dimensional signal illustrating the advantages of one wavelet over the other.

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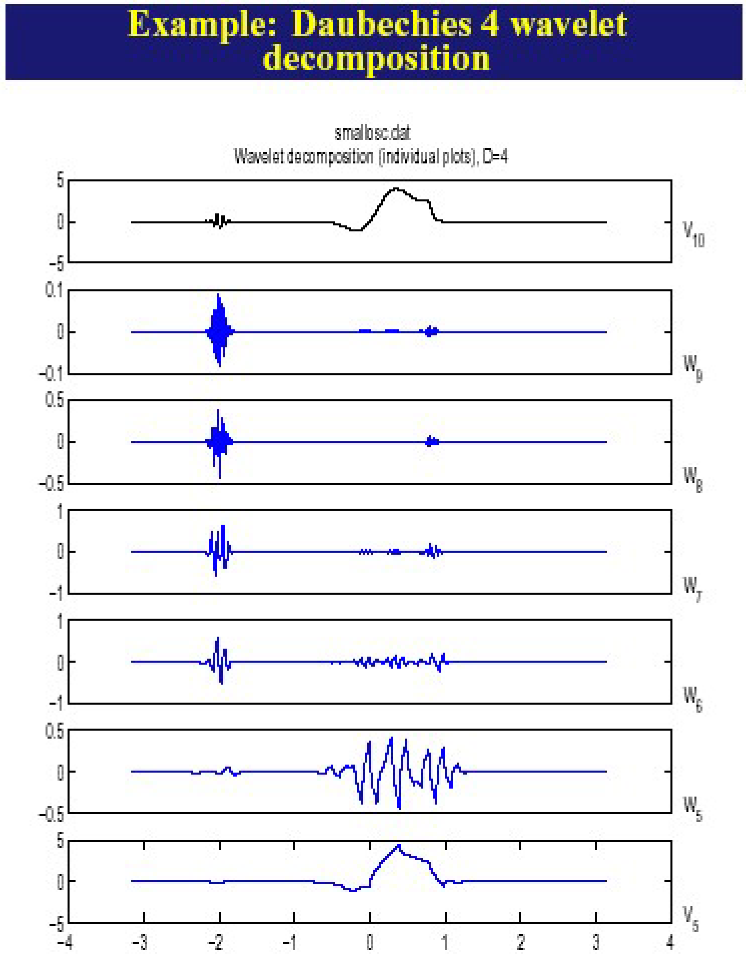

Figure-5: Wavelet decompositions of one dimensional signal illustrating the advantages of one wavelet over the other.

Orthogonal wavelets obtained from MRA are categorized into two forms, wavelets that include and doesnot include feature of compact support. Orthogonal wavelets characterized by Daubechies come under wavelets that include feature of compact support. She demonstrates that Haar wavelet is the only wavelet with compact support that is either symmetric or skew symmetric at any point. With the help of arbitrary regularity, selection of compactly supported wavelet can be made. However, support width varies directly with respect to regularity. Wavelet's feature of compact support, given by Daubechies and wavelets defined by Meyer and Lemarie described with feature of rapid decay provide efficient computation, time localization ability for wavelet transform. All said and done from the standpoint of real-world applications MRA is really an efficient and operative mathematical tool for ordered decomposition of a signal into modules of different scales (frequencies) as illustrated below.

2. WAVELET TRANSFORMATIONS

The continuous wavelet transform (CWT) is defined as the integration of the signal multiplied by the wavelet function ψ (scale, position, time) which is scaled and shifted, within the limits taken over the time spanned by the signal. The Continuous Wavelet Transform (CWT) of the signal $f \epsilon L_2(I)$ is defined mathematically as

$$W_{\psi}[F](a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\psi^{*}\left(\frac{t-b}{a}\right)dt$$

Where $\psi_{a,b}(t) = \psi\left(\frac{t-b}{a}\right)$ is the kernel. The Inverse Wavelet Transform (IWT) for the recovery of signal f(t) is given by:

$$f(t) = \frac{1}{c}\int\int_{-\infty}^{\infty} W_{\psi}[F](a,b)\psi\left(\frac{t-b}{a}\right)\frac{dadb}{a^2}$$

Where $c = \int_{-\infty}^{\infty} \frac{\left|\overline{\psi}(\omega)\right|^2}{|\omega|} dw < +\infty$

The wavelet expansion/series is a discretized version of CWT. Thus,

$$f \varepsilon L_2(¡\ ) \Rightarrow f(x) = \sum_i \sum_j \left\langle f, \psi_{i,j} \right\rangle \psi_{i,j}$$

The Fourier transforms and wavelet transform both have discrete versions. This is obtained by replacing the integral with a finite sum. The Discrete Fourier Transform (DFT) is obtained from Fourier transform and in a similar way Discrete Wavelet Transform is obtained from wavelet transform. A Fast Fourier transform algorithm is generally used to compute DFT, reason being computation time which is reduced from power of two to O(n log n). The DWT consists of matrix formed by wavelet coefficients. A specific family of wavelets is stated by a particular set of numbers, called wavelet filter coefficients. Interestingly, unlike DFT, to compute DWT of a signal, we need neither scaling function nor wavelet, just very simple digital filters. Here we shall confine ourselves to wavelet filters Db4 discovered by Daubechies. This is the simplest and most confined member of the whole family. It has only four coefficients c_0 ,c_1,c_2 and c_3. There are two filters used in the operation, namely a low pass filter and a high pass filter. Consider the following transform matrix:

$$\begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & & & & & & & & \\
c_3 & -c_2 & c_1 & c_0 & & & & & & & & \\
 & & c_0 & c_1 & c_2 & c_3 & & & & & & \\
 & & c_3 & -c_2 & c_1 & -c_0 & & & & & & \\
\cdot & \cdot & & & & & \cdot & \cdot & & & & \\
\cdot & \cdot & & & & & \cdot & \cdot & & & & \\
 & & & & & & & & c_0 & c_1 & c_2 & c_3 \\
 & & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\
c_2 & c_3 & & & & & & & & & c_0 & c_1 \\
c_1 & -c_0 & & & & & & & & & c_3 & -c_2
\end{bmatrix}$$

Blank entries here represent zeroes. The DWT consists of applying a wavelet transform matrix like the one above, recursively. When all is said and done, it is useful to remark that, CWT is a mathematical tool employed for theoretical research and scientific investigations, whereas DWT is exclusively meant for practical applications, mainly in engineering and computer science.

3. WAVELET PACKETS

Extension of wavelets and MRA which are simple yet powerful are wavelet packets. It is well known that the traditional MRA is found by splitting V_j into V_{j-1} and W_{j-1}. Then repeating the same procedure for V_{j-1} recursively, results in the decomposition L_2 (i). The wavelet packets are the basis functions that we obtain if we also use the "splitting trick" on the W_j spaces. For example: If $V_3 = V_0 \oplus W_0 \oplus W_1 \oplus W_2$, we obtain after using the splitting trick three times, a wavelet packet - basis functions:

$$\begin{Bmatrix} \psi_0^1(4x-k), \psi_{1,1}^2(2x-k), \\ \psi_{0,0,1}^3(x-k), \psi_{1,0,1}^3(x-k), k\varepsilon ¢ \end{Bmatrix}$$

The flexibility and effectiveness of the DWT can be increased by using the comprehensive form, wavelet packet transform (WPT). Advantage of WPT is that it utilizes both components, low frequency known as approximation and high frequency component known as details.

4. MULTIDIMENSIONAL WAVELETS

There are also wavelets in higher dimensions. Images, for example, are two dimensional signals. A modest way to obtain higher dimensional signal is to use tensor products. Let us consider the case of two dimensional extension:

$$\phi(x,y) = \phi(x).\phi(y) = \phi \otimes \phi(x,y) \text{ and } \mathrm{V}_0 = \left\{ f \,;\, f(x,y) = \sum_{i,j} \lambda_{i,j} \phi(x-i, y-j), \lambda \varepsilon I^2(¢) \right\}$$

If $If \ \{\phi(x-k)\}_{k\varepsilon ¢}$ is an orthonormal set, then $\phi(x-i, y-j)$ form an orthonormal basis of V_0. The complement W_0 of V_0 in V_1 is generated similarly by translating of the three functions:

$$\psi^{(1)} = \phi \otimes \psi, \quad \psi^{(2)} = \psi \otimes \phi, \ \psi^{(3)} = \psi \otimes \psi$$

Wavelet decomposition in two dimensions is:

$$f(x,y) = \sum_{i,j} \sum_{k,l} \left\langle f, \psi_{i,j} \otimes \psi_{k,l} \right\rangle \psi_{i,j} \otimes \psi_{k,l}$$

5. WAVELETS IN NUMERICAL ANALYSIS

Historically, the solution of differential and integral equations was one of the motivations for the development of wavelet theory from the onset. Wavelets are nowadays used as a tool in scientific computing and numerical analysis of differential, integral and integrodifferential equations. Main fields where wavelets are gaining currency are the numerical treatment of ordinary, delayed and partial differential equations. Numerical methods using wavelets for the resolutions of evolutionary partial differential equations were investigated in Fluid mechanics. The Haar wavelets have been used in the computation of eigenvalues and solution of Regular Sturm-Liouville's eigenvalue problems (SLEP) with Dirichlet's boundary conditions. Currently, the most effective algorithm for solving stiff differential equations encountered in Chemical engineering is the combination of implicit Runge-Kutta method. These methods have not been, to some extent, successful in reducing the calculation effort and time. However, Single Term Haar Wavelet Series (STHWS) method has turned out to be more effective in its ability to solve systems ranging from mildly to highly stiff differential equations. Problems arising in fluid dynamics in the form of partial differential equation are better solved by wavelets, reason being its hierarchical nature that has been appealing to other fields as well. The suitability and applicability of wavelet methods to problems typically arising in fluid dynamics are investigated by several researchers. Haar wavelet packets have been used in the solution of linear and nonlinear integral equations with separable kernel. Multigrid method, despite its ability to solve partial differential equations efficiently, suffers from drawbacks such as easy implement ability and rapidity of convergence. An attractive alternative to this is, wavelet-multigrid scheme proposed recently. The research work to be carried out, in general, aims to demonstrate the advantages of wavelet or wavelet based methods over traditional numerical methods.

6. WAVELETS IN OPTIMIZATION AND CONTROL THEORY

Optimization is a necessary part in every field to get optimum result. Several methods in textbooks and literature are available for computation of optimum results and optimal control. These methods are although sophisticated but computation part related to optimal control becomes difficult as it involves two point boundary value problem's numerical solution. For optimization of dynamic systems, requirement of analysis is solution of differential equations. Freshly wavelets have been used for the solution of problems in calculus of variations, optimization theory, control theory etc. Using techniques of optimization theory and wavelets, numerical solutions of SLEP's have been obtained trading off between speed and accuracy.

7. MULTIPLE IMAGE COMPRESSIONS USING WAVELETS

Pictures/images are the mode of transmitting information from ancient days. In olden days these were presented in physical manner like carving on wall of caves, engraving in stones. Gradually and progressively this method changes and nowadays pictures and videos are represented through electronic mode; however its storage and transmission is totally different from its display. Storage of images is done digitally that increases the image representation possibilities. Numerous algorithms are available to store an image digitally and to convert it back for proper display. This process of changing the representation of image, so that its storage and representation can be obtained with minimum data is image coding. If storage of image representation takes less space in comparison to original, then it is termed as image compression. Many algorithms for encoding and decoding are discussed in this chapter based on compression of an image using multi-wavelets and multi-resolution with increasing accuracy, and efficiency. There is a type of encoding termed as progressive encoding or embedded encoding. In progressive encoding, as and when more bits are added more precision is attained. It is akin to finding the value of π, when more digits after decimal point are added for better accuracy of its value. Adding more bits in a bit stream implies that decoded image will have more information to display.

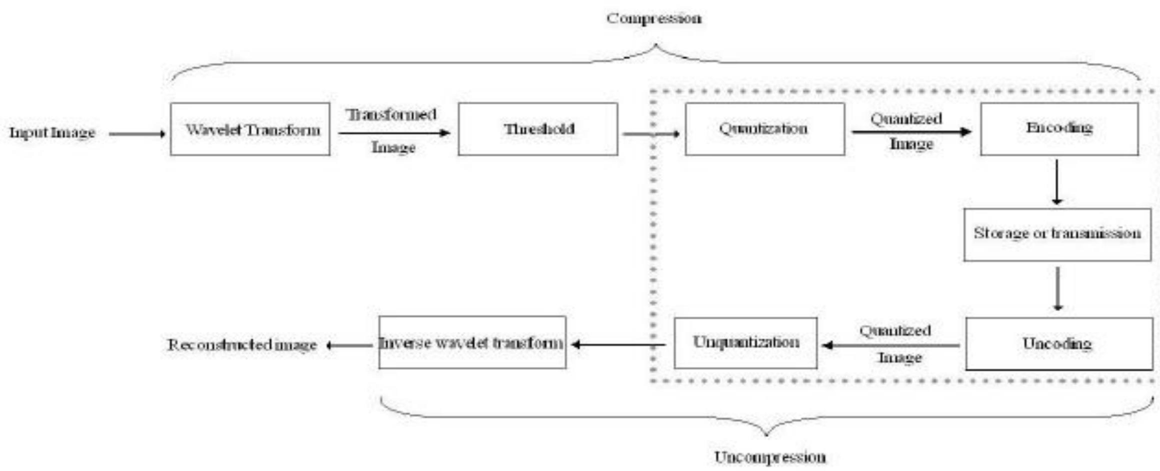

Figure-6: Image compression Methodology

The DWT plays an important role in transforming image into matrix of wavelet coefficients. For this purpose different wavelet family members are chosen like Haar, Daubechies and many more to get the entries of matrix. These entries are then arranged into specified pattern. Generally, four sub-bands are formed after decomposition of data: LL- low pass vertical and flat filter, HH- high pass upright and flat filter, LH- high pass flat and low pass upright filter, HL- high pass upright and low pass flat filter. These orders of horizontal and vertical can be exchanged to get different representation however it gives resultant image unchanged. A two level representation of decomposition (analysis) and reconstruction (synthesis) of a signal are provided in Figures-7 and Figure-8.

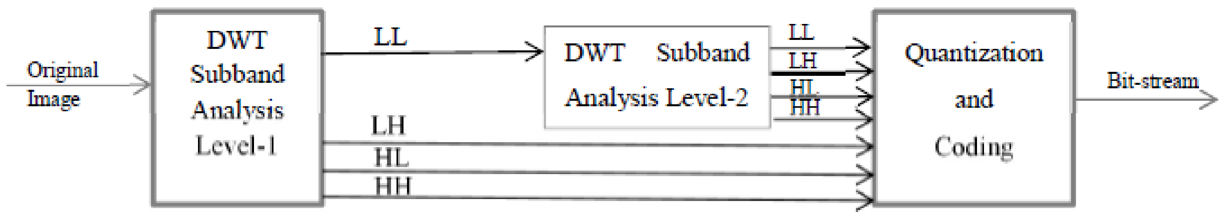

Figure-7: Image compression scheme based on DWT

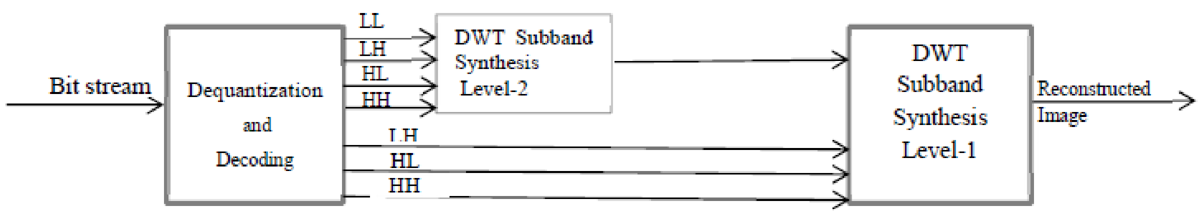

Figure-8: Image decoding scheme based on DWT

8. RESULTS AND DISCUSSION

To show the qualitative and quantitative effect of multiple compression let us take a true color image purely for illustrative purpose. Consider an image of fruit given below in Figure-9.

Figure-9: True color image of fruit (www.pngtree.com)

Using Matlab software application, different algorithm of compression are applied on the image that employs Haar wavelet transform with maximum 12 loops and 4 decomposition level at initial stage. Resultant quality of image in terms of different quantitative measure for intermediate frequency of compression using every algorithm is used to compress the image. Figure-10 shows the compressed image of the fruit.

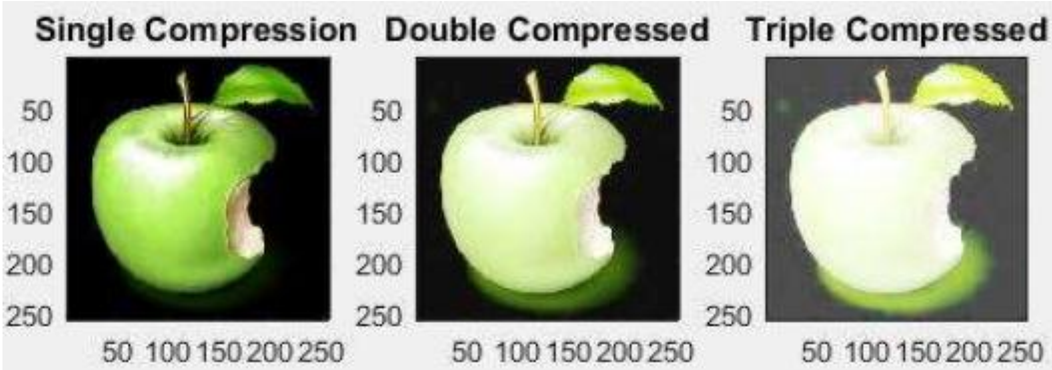

Figure-10: Images of fruit using different compression frequencies

9. CONCLUSION

In this chapter multiple compression were carried out on true color image (fruit image) for demonstrative purpose and gray color image (MRI image) from medical field for comparative study of efficiency and reliability. Many algorithms on compression techniques based on wavelets and wavelets packet were employed and measuring parameters computed and compared. This comparative and investigative study concludes that double/triple compression frequencies can be applied positively for better performance in the direction of storing and retrieval/ transmission of images.

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