Solution of Serial Service Channels in steady state With Balking and Feedback

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Abstract: In this article, we generate realistic and practical queuing model with X-servers in series having balking, feedback, Poisson arrivals and exponential service times. Here, more general concept of feedback is used by considering feedback from any server to all its previous servers including itself. Various queue characteristics such as steady-state solutions, marginal probability and mean queue length of the queuing system have been calculated for unlimited waiting space. The model has also been analyzed for finite capacity of the system. Particular case has been derived also.

Keywords and Phrases: Balking, feedback, steady-state, Marginal probability, Mean queue length.

Introduction:

Number of Researchers such as O'Brien (1954), Jackson (1954), Hunt (1955), Finch (1959) have shown their keen interest in the field of queuing theory and so developed the number of queuing models under different conditions and assumptions. Singh (1984) discussed the serial queue model introducing impatient behaviour of the customers. Singh et.al. (2015) solved problem of serial queues with balking and feedback permissible to each service channel to its preceding one. Gupta et.al. (2020) analysed various queue characteristics in a queue process. Here, the queuing model generated by Singh et al. (2015) has changed by presuming that feedback is allowed from each service channel to all of its prior service channels, including the server's own service channel. The main feature of the present work is listed as under:

(i) Consumers may join any serial service channel from the outside and may exit the system at any moment after obtaining service through any of the serial service channels, which are represented by the number X.

(ii) The customer may balk as it hesitates to join the queue due to large number of customers already present there.

(iii) Each service channel is allowed to provide feedback after a service to all of its prior channels, including itself i.e., after getting service at any service channel, the customer may join back all previous channels including itself.

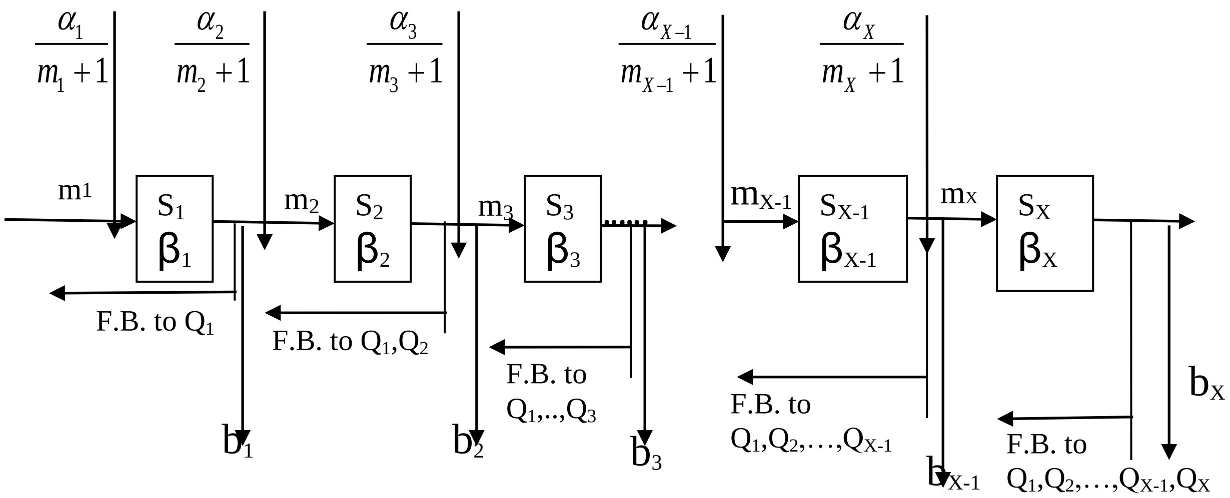
(iv) The queue size is a factor in the Poisson probability distribution of the input process.

(v) The service rate is exponentially distributed and queue pattern is first come first served (FCFS).

(vi) We have considered two models, one with infinite capacity and other for finite capacity.

Case-I

Description of the Model:



Fig(i) Serial Queuing Model

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The serial queuing model is described for attaining the difference-differential equations, steady-state equations and solutions in time-independent stage for unlimited waiting space. We assume that there are $Q_j\left(j=1,2,3,....,X\right)$ service channels in the system with respective servers $S_j\left(1\le j\le X\right)$. The arrival rate follows Poisson distribution with parameters $\alpha_j\left(1\le j\le X\right)$ at $Q_j\left(1\le j\le X\right)$ for receiving different types of service. If the customer, on arrival, finds that there are already large number of customers at service channel $Q_j\left(1\le j\le X\right)$, then he avoids to join the queue because it is not possible for him to spare so much time for the desired service, under such situation, the Poisson input rate α_j may be $\frac{\alpha_j}{m_j+1}$ where m_j is the j[th] channel size $\left(1\le j\le X\right)$. Let $\beta_1,\beta_2,\beta_3,....,\beta_{X-1},\beta_X$ be the respective exponential service rates of the servers $S_1,S_2,S_3,...,S_{X-1},S_X$. Further; it has been assumed that after getting served at S_j, the system is left by customers with probability b_j or the next queue is joined with probability $\frac{d_j}{m_{j+1}+1}$ or all the prior service channels inclusing itself are joined with probabilities $\frac{r_{ji}}{m_i+1}(1 \le i \le j)$ such that

$b_j + \frac{d_j}{m_{j+1}+1} + \sum_{i=1}^{j} \frac{r_{ji}}{m_i+1} = 1;\ 1 \le j \le X$ which can be written as $b_j + \frac{d_j}{m_{j+1}+1} + \frac{r_{jj}}{m_j+1} + \sum_{i=1}^{j-1} \frac{r_{ji}}{m_i+1} = 1,$

Thus $1 - \frac{r_{jj}}{m_j+1} = b_j + \frac{d_j}{m_{j+1}+1} + \sum_{i=1}^{j-1} \frac{r_{ji}}{m_i+1};\ 1 \le j \le X.$ (1) where $\frac{d_j}{m_{j+1}+1} = 0$ *for* $j = X$ due to last channel of the system.

Practical Application: The queuing model presented here has its application in the administration set-up of a district.

Formation of Equations:

$P(m_1, m_2, m_3, ...m_{X-1}, m_X; t)$ is defined as the probability such that m_j customers are present waiting at time t (leaving the system after service, joining the next service channel, or rejoining all of the earlier channels, including itself) prior the server $S_j\ (1 \le j \le X)$.

Using the operators $Z_j.,\ Z._j,\ and\ Z._{j,j+1}.$ on vector $\tilde{m} = (m_1, m_2, m_3, ...m_X)$ defined as

$Z_j.(\tilde{m}) = (m_1, m_2, ..., m_j - 1, ...m_X)$, $Z._j(\tilde{m}) = (m_1, m_2, ..., m_j + 1, ..., m_X)$ and

$Z._{j,j+1}.(\tilde{m}) = (m_1, m_2, ..., m_j + 1, m_{j+1} - 1, ..., m_X)$ (1)

for writing the system equations in the compact form as used by Kelly (1979).

Difference-differential Equations: The difference-differential equations of the system are as under

$$\frac{d}{dt}P(\tilde{m};t) = -\left[\sum_{j=1}^{X} \frac{\alpha_j}{m_j+1} + \sum_{j=1}^{X} \delta(m_j)\beta_j\right] P(\tilde{m};t) + \sum_{j=1}^{X} \frac{\alpha_j}{m_j} P(Z_j.(\tilde{m});t) + \sum_{j=1}^{X} \beta_j b_j\, P(Z._j(\tilde{m});t)$$
$$+ \sum_{j=1}^{X-1} \frac{\beta_j\, d_j}{m_{j+1}} P(Z._{j,j+1}.(\tilde{m});t) + \sum_{j=1}^{X} \frac{\delta(m_j)\beta_j\, r_{jj}}{m_j} P(\tilde{m};t)$$
$$+ \sum_{j=2}^{X} \beta_j \left(\sum_{i=1}^{j-1} \frac{r_{ji}}{m_i} P(m_1, m_2, m_3, ..., m_{i-1}, m_i - 1, m_{i+1}, ..., m_j + 1, ..., m_X; t) \right)$$

for $m_j \ge 0; 1 \le j \le X$ (2)

where $\delta(m_j) = \begin{cases} 1 & if \quad m_j \ne 0 \\ 0 & if \quad m_j = 0 \end{cases}$ and $P(\tilde{m};t) = \tilde{0}$ if any argument is less than 0.

Steady–State Equations:

Steady-state equations are written by taking $\frac{d}{dt}P(\tilde{m};t) = 0$ in the equation (2) and reduces it to:

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$$\left[\sum_{j=1}^{X}\frac{\alpha_j}{m_j+1}+\sum_{j=1}^{X}\delta\left(m_j\right)\beta_j\left(1-\frac{r_{jj}}{m_j}\right)\right]P\left(\tilde{m}\right)=\sum_{j=1}^{X}\frac{\alpha_j}{m_j}P\left(Z_j.\left(\tilde{m}\right)\right)+\sum_{j=1}^{X}\beta_j b_j\ P\left(Z._j\left(\tilde{m}\right)\right)$$

$$+\sum_{j=1}^{X-1}\frac{\beta_j\, d_j}{m_{j+1}}P\left(Z._{j,j+1}.\left(\tilde{m}\right)\right)+\sum_{j=2}^{X}\beta_j\left(\sum_{i=1}^{j-1}\frac{r_{ji}}{m_i}P\left(m_1,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_j+1,\ldots,m_X\right)\right)$$

for $m_j \geq 0; 1 \leq j \leq X$. (3)

Solutions:

The steady-state solutions of equations (3) are verified as

$$P(\tilde{m})=P(\tilde{0})\left(\frac{1}{m_1!}\frac{\left(\alpha_1+\sum_{j=2}^{X}\frac{r_{j1}\rho_j}{(m_j+1)\left(1-\frac{r_{jj}}{m_j+1}\right)}\right)^{m_1}}{(\beta_1)^{m_1}\prod_{j=1}^{m_1}\left(1-\frac{r_{11}}{j}\right)}\right)\left(\frac{1}{m_2!}\frac{\left(\alpha_2+\frac{d_1\rho_1}{(m_1+1)\left(1-\frac{r_{11}}{m_1+1}\right)}+\sum_{j=3}^{X}\frac{r_{j2}\rho_j}{(m_j+1)\left(1-\frac{r_{jj}}{m_j+1}\right)}\right)^{m_2}}{(\beta_2)^{m_2}\prod_{j=1}^{m_2}\left(1-\frac{r_{22}}{j}\right)}\right).$$

$$\cdots\left(\frac{1}{m_{X-1}!}\frac{\left(\alpha_{X-1}+\frac{d_{X-2}\rho_{X-2}}{(m_{X-2}+1)\left(1-\frac{r_{X-2,X-2}}{m_{X-2}+1}\right)}+\frac{r_{X,X-1}\rho_X}{(m_X+1)\left(1-\frac{r_{X,X}}{m_X+1}\right)}\right)^{m_{X-1}}}{(\beta_{X-1})^{m_{X-1}}\prod_{j=1}^{m_{X-1}}\left(1-\frac{r_{X-1,X-1}}{j}\right)}\right).\left(\frac{1}{m_X!}\frac{\left(\alpha_X+\frac{d_{X-1}\rho_{X-1}}{(m_{X-1}+1)\left(1-\frac{r_{X-1,X-1}}{m_{X-1}+1}\right)}\right)^{m_X}}{(\beta_X)^{m_X}\prod_{j=1}^{m_X}\left(1-\frac{r_{X,X}}{j}\right)}\right)$$

for $m_j \geq 0; 1 \leq j \leq X$ (4)

Where the unknown parameters $\rho_1, \rho_2, \rho_3, \ldots, \rho_{X-1}, \rho_X$ are related to each other in the following form

$$\rho_1=\alpha_1+\sum_{j=2}^{X}\frac{r_{j1}\rho_j}{k_j};\ \rho_2=\alpha_2+\frac{d_1\rho_1}{k_1}+\sum_{j=3}^{X}\frac{r_{j2}\rho_j}{k_j};\ \rho_3=\alpha_3+\frac{d_2\rho_2}{k_2}+\sum_{j=4}^{X}\frac{r_{j3}\rho_j}{k_j} \quad (5)$$

$$\ldots\ldots\ldots\ldots\ldots\ \rho_{X-1}=\alpha_{X-1}+\frac{d_{X-2}\rho_{X-2}}{k_{X-2}}+\frac{r_{X,X-1}\rho_X}{k_X};\ \rho_X=\alpha_X+\frac{d_{X-1}\rho_{X-1}}{k_{X-1}}$$

Where

$$k_j=\left(m_j+1\right)\left(1-\frac{r_{jj}}{m_j+1}\right)$$

We calculate ρ_X from these X-equations of (5) with the help of determinants,

where,

$$\rho_X = \left(\frac{\alpha_X \Lambda_{X-1} + \frac{d_{X-1}}{k_{X-1}} \alpha_{X-1} \Lambda_{X-2} + \frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \alpha_{X-2} \Lambda_{X-3} + \dots + \left(\frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{d_{X-3}}{k_{X-3}} \dots \frac{d_3}{k_3} \cdot \frac{d_2}{k_2} \cdot \alpha_2 \Lambda_1 \right) + \left(\frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{d_{X-3}}{k_{X-3}} \dots \frac{d_3}{k_3} \cdot \frac{d_2}{k_2} \cdot \frac{d_1}{k_1} \cdot \alpha_1 \right)}{\Delta_{X-1} - \frac{d_{X-1}}{k_{X-1}} \cdot \frac{r_{X,X-1}}{k_X} \Lambda_{X-2} - \frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{r_{X,X-2}}{k_X} \Lambda_{X-3} - \frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{d_{X-3}}{k_{X-3}} \cdot \frac{r_{X,X-3}}{k_X} \Lambda_{X-4} - \dots - \left(\frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{d_{X-3}}{k_{X-3}} \dots \frac{d_3}{k_3} \cdot \frac{d_2}{k_2} \cdot \frac{r_{X,2}}{k_X} \Lambda_1 \right) - \left(\frac{d_{X-1}}{k_{X-1}} \cdot \frac{d_{X-2}}{k_{X-2}} \cdot \frac{d_{X-3}}{k_{X-3}} \dots \frac{d_3}{k_3} \cdot \frac{d_2}{k_2} \cdot \frac{d_1}{k_1} \cdot \frac{r_{X,1}}{k_X} \right)} \right)$$

And

$$\Lambda_X = \begin{vmatrix} 1 & \frac{-r_{21}}{k_2} & \frac{-r_{31}}{k_3} & \dots & \frac{-r_{X-2,1}}{k_{X-2}} & \frac{-r_{X-1,1}}{k_{X-1}} & \frac{-r_{X,1}}{k_X} \\ \frac{-d_1}{k_1} & 1 & \frac{-r_{32}}{k_3} & \dots & \frac{-r_{X-2,2}}{k_{X-2}} & \frac{-r_{X-1,2}}{k_{X-1}} & \frac{-r_{X,2}}{k_X} \\ 0 & \frac{-d_2}{k_2} & 1 & \dots & \frac{-r_{X-2,3}}{k_{X-2}} & \frac{-r_{X-1,3}}{k_{X-1}} & \frac{-r_{X,3}}{k_X} \\ 0 & . & . & \dots & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & . & . & . \\ 0 & 0 & 0 & \dots & \frac{-d_{X-2}}{k_{X-2}} & 1 & \frac{-r_{X,X-1}}{k_X} \\ 0 & 0 & 0 & \dots & 0 & \frac{-d_{X-1}}{k_{X-1}} & 1 \end{vmatrix}$$

Continuing in this way, we get

$$\Lambda_3 = \begin{vmatrix} 1 & \frac{-r_{21}}{k_2} & \frac{-r_{31}}{k_3} \\ \frac{-d_1}{k_1} & 1 & \frac{-r_{32}}{k_3} \\ 0 & \frac{-d_2}{k_2} & 1 \end{vmatrix}, \qquad \Lambda_2 = \begin{vmatrix} 1 & \frac{-r_{21}}{k_2} \\ \frac{-d_1}{k_1} & 1 \end{vmatrix}, \qquad \Lambda_1 = |1| = 1 \qquad (6)$$

Since ρ_X has been calculated, by substituting the values of ρ_X, value of ρ_{X-1} is calculated. Proceeding in this way, we find $\rho_{X-3}, \rho_{X-4}, ---, \rho_3, \rho_2$, and ρ_1.

Since $\rho_1, \rho_2, \rho_3, ---, \rho_{X-1}, \rho_X$ are known, therefore we can write the steady-state solutions (4) of the system as under:

$$P(\tilde{m}) = P(\tilde{0}) \left(\frac{1}{m_1!} \frac{(\rho_1)^{m_1}}{(\beta_1)^{m_1}} \frac{1}{\prod_{j=1}^{m_1} \left(1 - \frac{r_{11}}{j}\right)} \right) \left(\frac{1}{m_2!} \frac{(\rho_2)^{m_2}}{(\beta_2)^{m_2}} \frac{1}{\prod_{j=1}^{m_2} \left(1 - \frac{r_{22}}{j}\right)} \right) \cdot \left(\frac{1}{m_3!} \frac{(\rho_3)^{m_3}}{(\beta_3)^{m_3}} \frac{1}{\prod_{j=1}^{m_3} \left(1 - \frac{r_{33}}{j}\right)} \right) \quad (6)$$

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$$\cdots\left(\frac{1}{m_{X-1}!}\frac{(\rho_{X-1})^{m_{X-1}}}{(\beta_{X-1})^{m_{X-1}}}\frac{1}{\prod_{j=1}^{m_{X-1}}\left(1-\frac{r_{X-1,X-1}}{j}\right)}\right)\cdot\cdot\left(\frac{1}{m_X!}\frac{(\rho_X)^{m_X}}{(\beta_X)^{m_X}}\frac{1}{\prod_{j=1}^{m_X}\left(1-\frac{r_{X,X}}{j}\right)}\right)$$

By applying normalizing condition $P(\tilde{0})$ is found out.

$$\sum_{\tilde{m}=\tilde{0}}^{\infty} P(\tilde{m}) = 1 \tag{7}$$

With the help of (6) and (7), we obtain;

$$1 = P(\tilde{0})\prod_{j=1}^{X}(H_j) \tag{8}$$

Where

$$H_j = 1 + \frac{\rho_j}{\beta_j}\frac{1}{(1-r_{jj})} + \frac{1}{2!}\left(\frac{\rho_j}{\beta_j}\right)^2\frac{1}{(1-r_{jj})\left(1-\frac{r_{jj}}{2}\right)} + \frac{1}{3!}\left(\frac{\rho_j}{\beta_j}\right)^3\frac{1}{(1-r_{jj})\left(1-\frac{r_{jj}}{2}\right)\left(1-\frac{r_{jj}}{3}\right)} + \ldots to\ \infty$$

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If $H_j \ (j=1,2,3,\ldots,X)$ is a divergent series then $P(\tilde{0})=0$ which is meaningless.

$P(\tilde{0})$ will not vanish and have some non-zero value provided H_j is convergent which is possible under the condition that the utilization factor is less than unity for service channels.

Therefore, $P(\tilde{m})$ has been defined completely.

Steady-State Marginal Probabilities:

$P(m_1)$ = (service channel marginal probability having m_1 customers before the server S_1)

$$P(m_1) = \sum_{m_2,m_3,\ldots,m_{N=0}}^{\infty} P(\tilde{m})$$

With the help of (9) and (11);

$$P(m_1) = \frac{\frac{1}{m_1!}\left(\frac{\rho_1}{\beta_1}\right)^{m_1}\frac{1}{\prod_{j=1}^{m_1}\left(1-\frac{r_{11}}{j}\right)}}{H_1} \tag{9}$$

Similarly

$$P(m_i) = \frac{\frac{1}{m_i!}\left(\frac{\rho_i}{\beta_i}\right)^{m_i}\frac{1}{\prod_{j=1}^{m_i}\left(1-\frac{r_{ii}}{j}\right)}}{H_i};\ 2 \le i \le X$$

Mean Queue Length:

Similarly, marginal mean queue length before S_i (i=1,2,3,4,…,X) are

$$L_i = \sum_{m_1,m_2,m_3,\ldots,m_{i-1=0}}^{\infty} m_i P(m_i) = \frac{\frac{\rho_i}{\beta_i(1-r_{ii})}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\rho_i}{\beta_i}\right)^j \frac{1}{\prod_{k=1}^{j}\left(1-\frac{r_{ii}}{k+1}\right)}\right]}{H_i}$$

Thus, mean queue length of the system

$$= L(say) = \sum_{i=1}^{X} L_i = \sum_{i=1}^{X}\left\{\frac{\frac{\rho_i}{\beta_i(1-r_{ii})}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\rho_i}{\beta_i}\right)^j \frac{1}{\prod_{k=1}^{j}\left(1-\frac{r_{ii}}{k+1}\right)}\right]}{H_i}\right\}$$

Case-II

The queuing model described in case-I for infinite waiting space, is discussed here for finite waiting -space i.e. $\sum_{j=1}^{X} m_j = K$, thereafter, the system will suffer a loss as consumers who arrive at that time are denied access. Under such situation, the difference-differential equation (2) and steady-state equation (3) would be modified and are written in the given form. 2490

$$\frac{d}{dt}P(\tilde{m};t) = -\left[\sum_{j=1}^{X} t_j\left(\frac{\alpha_j}{m_j+1}\right) + \sum_{j=1}^{X}\delta(m_j)\beta_j\right]P(\tilde{m};t) + \sum_{j=1}^{X}\frac{\alpha_j}{m_j}P(Z_j.(\tilde{m});t)$$

$$+\sum_{j=1}^{X} t_j\beta_j b_j\, P(Z._j(\tilde{m});t) + \sum_{j=1}^{X-1}\frac{\beta_j d_j}{m_{j+1}}P(Z._{j,j+1}.(\tilde{m});t)$$

$$+\sum_{j=1}^{X}\frac{\delta(m_j)\beta_j r_{jj}}{m_j}P(\tilde{m};t) + \sum_{j=2}^{X}\beta_j\left(\sum_{i=1}^{j-1}\frac{r_{ji}}{m_i}P(m_1,m_2,m_3,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_j+1,\ldots,m_X;t)\right)$$

where $\delta(m_j) = \begin{cases} 1 & if \quad m_j \neq 0 \\ 0 & if \quad m_j = 0 \end{cases}$

$$t_j = \begin{cases} 1 & if \quad \sum_{j=1}^{X} m_j < K \\ 0 & if \quad \sum_{j=1}^{X} m_j = K \end{cases}$$

$$\left[\sum_{j=1}^{X} t_j\left(\frac{\alpha_j}{m_j+1}\right)+\sum_{j=1}^{X}\delta\left(m_j\right)\beta_j\right]P(\tilde{m})=\sum_{j=1}^{X}\frac{\alpha_j}{m_j}P\left(Z_j.(\tilde{m})\right)+\sum_{j=1}^{X}t_j\beta_j b_j\,P\left(Z._j(\tilde{m})\right)$$

$$+\sum_{j=1}^{X-1}\frac{\beta_j\,d_j}{m_{j+1}}P\left(T._{j},_{j+1}.(\tilde{m})\right)+\sum_{j=1}^{X}\frac{\delta(m_j)\beta_j\,r_{jj}}{m_j}P(\tilde{m})$$

$$+\sum_{j=2}^{X}\beta_j\left(\sum_{i=1}^{j-1}\frac{r_{ji}}{m_i}P\left(m_1,m_2,m_3,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_j+1,\ldots,m_X\right)\right) \tag{10}$$

Equation (10) reduces to

$$\left[\sum_{j=1}^{X} t_j\left(\frac{\alpha_j}{m_j+1}\right)+\sum_{j=1}^{X}\delta\left(m_j\right)\beta_j\left(1-\frac{r_{jj}}{m_j}\right)\right]P(\tilde{m})=\sum_{j=1}^{X}\frac{\alpha_j}{m_j}P\left(Z_j.(\tilde{m})\right)+\sum_{j=1}^{X}t_j\beta_j b_j\,P\left(Z._j(\tilde{m})\right)$$

$$+\sum_{j=1}^{X-1}\frac{\beta_j\,d_j}{m_{j+1}}P\left(Z._{j},_{j+1}.(\tilde{m})\right)+\sum_{j=2}^{X}\beta_j\left(\sum_{i=1}^{j-1}\frac{r_{ji}}{m_i}P\left(m_1,m_2,m_3,\ldots,m_{i-1},m_i-1,m_{i+1},\ldots,m_j+1,\ldots,m_X\right)\right)$$

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Steady-State Solutions:

The solutions to the equation may be confirmed to be

$$P(\tilde{m})=P(\tilde{0})\left(\frac{1}{m_1!}\frac{\left(\alpha_1+\sum_{j=2}^{X}\frac{r_{j1}\rho_j}{(m_j+1)\left(1-\frac{r_{jj}}{m_j+1}\right)}\right)^{m_1}}{(\beta_1)^{m_1}\prod_{j=1}^{m_1}\left(1-\frac{r_{11}}{j}\right)}\right)\cdot\left(\frac{1}{m_2!}\frac{\left(\alpha_2+\frac{d_1\rho_1}{(m_1+1)\left(1-\frac{r_{11}}{m_1+1}\right)}+\sum_{j=3}^{X}\frac{r_{j2}\rho_j}{(m_j+1)\left(1-\frac{r_{jj}}{m_j+1}\right)}\right)^{m_2}}{(\beta_2)^{m_2}\prod_{j=1}^{m_2}\left(1-\frac{r_{22}}{j}\right)}\right)$$

$$\cdots\left(\frac{1}{m_{X-1}!}\frac{\left(\alpha_{X-1}+\frac{d_{X-2}\rho_{X-2}}{(m_{X-2}+1)\left(1-\frac{r_{X-2,X-2}}{m_{X-2}+1}\right)}+\frac{r_{X,X-1}\rho_X}{(m_X+1)\left(1-\frac{r_{X,X}}{m_X+1}\right)}\right)^{m_{X-1}}}{(\beta_{X-1})^{m_{X-1}}\prod_{j=1}^{m_{X-1}}\left(1-\frac{r_{X-1,X-1}}{j}\right)}\right)\cdot\left(\frac{1}{m_X!}\frac{\left(\alpha_X+\frac{d_{X-1}\rho_{X-1}}{(m_{X-1}+1)\left(1-\frac{r_{X-1,X-1}}{m_{X-1}+1}\right)}\right)^{m_X}}{(\beta_X)^{m_X}\prod_{j=1}^{m_X}\left(1-\frac{r_{X,X}}{j}\right)}\right) \tag{11}$$

For $m_j\geq 0;1\leq j\leq X$ *and* $\sum_{j=1}^{X}m_j\leq K$

The outcome shown above may be expressed as

$$P(\tilde{m}) = P(\tilde{0})\left(\frac{1}{m_1!}\frac{(\rho_1)^{m_1}}{(\beta_1)^{m_1}}\frac{1}{\prod_{j=1}^{m_1}\left(1-\frac{r_{11}}{j}\right)}\right)\left(\frac{1}{m_2!}\frac{(\rho_2)^{m_2}}{(\beta_2)^{m_2}}\frac{1}{\prod_{j=1}^{m_2}\left(1-\frac{r_{22}}{j}\right)}\right)$$

$$\left(\frac{1}{m_3!}\frac{(\rho_3)^{m_3}}{(\beta_3)^{m_3}}\frac{1}{\prod_{j=1}^{m_3}\left(1-\frac{r_{33}}{j}\right)}\right)\cdots\left(\frac{1}{m_{X-1}!}\frac{(\rho_{X-1})^{m_{X-1}}}{(\beta_{X-1})^{m_{X-1}}}\frac{1}{\prod_{j=1}^{x_{X-1}}\left(1-\frac{r_{X-1,X-1}}{j}\right)}\right)$$

$$\cdot\left(\frac{1}{m_X!}\frac{(\rho_X)^{m_X}}{(\beta_X)^{m_X}}\frac{1}{\prod_{j=1}^{m_X}\left(1-\frac{r_{X,X}}{j}\right)}\right);\ for\ m_j \geq 0; 1 \leq j \leq X\ and\ \sum_{j=1}^{X} m_j \leq K \qquad (12)$$

Where the values of $\rho_1, \rho_2, \rho_3 .....\rho_X$ are the same as derived in case-I. 2492

$P(\tilde{0})$ would be evaluated from the normalizing condition with the relation $\sum_{j=1}^{X} m_j \leq K$ and with the constraint that each service channel's usage factor cannot be more than one.

Particular case:

When the input process does not depend upon the queue size i.e., there is no balking at any stage in the system and customers may only join the system once, at its initial phase, and must go via each service channel before leaving it, having finite capacity where feedback is permissible from the terminal service channel to all its previous channels including itself, then

(i) $\frac{\alpha_1}{m_1+1} = \alpha_1; \frac{\alpha_j}{m_j+1} = \alpha_j = 0\ for\ j = 2,3,4,...,X\ and\ k_1 = k.$

(ii) $b_j = 0, \frac{d_j}{m_{j+1}} = 1; \frac{r_{jj}}{m_j+1} = r_{jj} = 0\ for\ j = 1,2,3,...,X-1$

$\frac{r_{ji}}{m_i} = r_{ji} = 0\ for\ j = 2,3,4,...,X-1;\ i = 1,2,...,j-1\ and\ i \neq j$

$\frac{r_{Xj}}{m_j+1} = r_{Xj} = r_j;\ j = 1,2,3,...,X\ and\ b_j = b\ for\ j = X$

(iii) $\sum_{j=1}^{X} m_j = K$

Under these conditions, the steady-state equations (10) will reduce to

$$\left[k\alpha_1 + \sum_{j=1}^{X}\delta(m_j)\beta_j\right]P(\tilde{m}) = \alpha_1 P(m_1 - 1, m_2, m_3, ..., m_X) + kb\beta_X P(m_1, m_2, m_3, ..., m_{X-1}, m_X + 1)$$

$$+\sum_{j-1}^{X-1}\beta_j P(m_1, m_2, m_3, ..., m_j + 1, m_{j+1} - 1, ..., m_{X-1}, m_X) + \delta(m_X)\beta_X r_X P(m_1, m_2, m_3, ..., m_X)$$

$$+\sum_{j=1}^{X-1}\delta(m_j)\beta_X r_j P(m_1, m_2, ..., m_j - 1, m_{j+1}, ..., m_{X-1}, m_X + 1). \qquad (13)$$

for $m_j \geq 0;\ and\ \sum_{j=1}^{X} m_j \leq K$.

and $P(\tilde{m}) = \tilde{0}$ if any condition is negative.

$$t = \begin{cases} 1 \ \ if \ \ \sum_{j=1}^{X} m_j < K \\ 0 \ \ if \ \ \sum_{j=1}^{X} m_j = K \end{cases}$$

The above system of equations (13) is satisfied by

$$P(\tilde{m}) = P(\tilde{0})\prod_{j=1}^{X}(\rho_j)^{m_j} \qquad (14)$$

2493

Where $\rho_j = \dfrac{\alpha_1(1 - r + r_1 + r_2 + r_3 + ... + r_j)}{\beta_j(1 - r)}$

$for\ \ j = 1, 2, 3, ..., X\ \ such\ that\ \ \ r = \sum_{j=1}^{N} r_j\ \ and\ \ p + r = 1$

The steady-state equations (13) and solutions (14) are in agreement with the corresponding general equations (1) and solution (2) given by Finch (1959).

<u>Program to calculate mean queue length:</u>

```
n=9;
d[1]=0.01; α[1]=30;        β[1]=35;          m[1]=5;
d[2]=0.15;    α[2]=35;     β[2]=40;          m[2]=6;
d[3]=0.21;    α[3]=40;           β[3]=45;          m[3]=7;
d[4]=0.13;    α[4]=45;           β[4]=50;          m[4]=8;
d[5]=0.14;    α[5]=50;           β[5]=55;          m[5]=9;
d[6]=0.08;    α[6]=55;           β[6]=60;          m[6]=10;
d[7]=0.05;    α[7]=60;           β[7]=65;       m[7]=11;
d[8]=0.17;    α[8]=65;           β[8]=70;          m[8]=12;
d[9]=0.25; α[9]=70;        β[9]=75;          m[9]=13;
r=({
  {0.13, 0, 0, 0, 0, 0, 0, 0, 0},
  {0.1, 0.19, 0, 0, 0, 0, 0, 0, 0},
  {0.14, 0.12, 0.1, 0, 0, 0, 0, 0, 0},
  {0.1, 0.05, 0.13, 0.04, 0, 0, 0, 0, 0},
```

```
  {0.02, 0.11, 0.14, 0.12, 0.13, 0, 0, 0, 0},
  {0.07, 0.03, 0.04, 0.17, 0.24, 0.04, 0, 0, 0},
  {0.15, 0.14, 0.13, 0.23, 0.06, 0.24, 0.05, 0, 0},
  {0.03, 0.01, 0.02, 0.07, 0.15, 0.13, 0.23, 0.16, 0},
  {0.08, 0.07, 0.12, 0.13, 0.09, 0.08, 0.1, 0.07, 0.3}
 });
For[x=1,x⍰n,x++,For[j=1,j⍰n,j++,g[x,j]=r[[x,j]]]]
For[x=1,x⍰ n,x++,k[x]=m[x]+1-g[x,x]]
For[x=1,x⍰ n,x++,For[j=1,j⍰ n,j++,If[x⍰ j,f[x,j]=1,If[j>x, f[x,j]=-g[j,x]/k[j],If[x⍰ j+1,f[x,j]=-q[j]/k[j],f[x,j]=0]]]]]
mat=Table[f[x,j],{x,1,n},{j,1,n}];
(*Calculating Λ_n*)
Λ[0]=1;
For[a=1,a⍰n,a++,Λ[a]=Det[Table[f[x,j],{x,1,a},{j,1,a}]]; Print[ "Λ_",a," is   ", Λ[a]]]
(*Calculating ⍰_n*)
⍰[n]=(α[n] Λ[n-1]+Sum[(Product[d[x]/k[x], {x, j, n-1}]) Λ[j-1] α[j],{j,n-1,1,-1}])/   ( Λ[n-1]-Sum[(Product[d[x]/k[x], {x, j, n-1}]) Λ[j-1] (g[n,j]/k[n]),{j,n-1,1,-1}]);
⍰[n-1]=(⍰[n]-α[n])(k[n-1]/d[n-1]);
For[a=n-2,a⍰1,a+=-1,⍰[a]=(⍰[a+1]-α[a+1]-Sum[(g[j,a+1](⍰[j]/k[j])),{j,a+2,n}]) (k[a]/d[a])]
For[a=1,a⍰ n,a++,Print[ "⍰_",a," is   ", ⍰[a]]]
(*Calculating ⍰_n/β_n*)
For[a=1,a⍰ n,a++,s[a]=⍰[a]/β[a]]
(*Calculating H_n*)
For[a=1,a⍰n,a++,h[a]=1+Sum[(Product[1/(1-g[a,s]/j),{j,1,x}]) (1/(x!))(s[a])^x,{x,1,Infinity}];Print[ "H_",a,"  is   ", h[a]]]
(*Calculating M_n*)
For[a=1,a⍰n,a++,l1[a]=((s[a]/(1-g[a,a]))Sum[(1/(j!))  ((s[a])^j) 1/(Product[1/(1-(g[a,a]/(x+1))),{x,1,j}]),{j,0,Infinity}])/(h[a]); Print["M_",a,"  is   ",l1[a]]]
(*Calculating M*)
l=Sum[l1[a],{a,1,n}];
Print["M is  ",l]
```


Λ_1 is 1	ρ_1 is 100.59	H_1 is 22.2575	M_1 is 2.31061
Λ_2 is 0.999975	ρ_2 is 38.014	H_2 is 3.06183	M_2 is 0.916019
Λ_3 is 0.99964	ρ_3 is 43.9321	H_3 is 2.88787	M_3 is 0.955846
Λ_4 is 0.999251	ρ_4 is 49.9455	H_4 is 2.80581	M_4 is 0.989828
Λ_5 is 0.999069	ρ_5 is 53.5407	H_5 is 2.96052	M_5 is 0.947036
Λ_6 is 0.998755	ρ_6 is 58.0852	H_6 is 2.71847	M_6 is 0.960492
Λ_7 is 0.998608	ρ_7 is 62.1167	H_7 is 2.7061	M_7 is 0.946843
Λ_8 is 0.998533	ρ_8 is 65.622	H_8 is 2.92865	M_8 is 0.911365
Λ_9 is 0.998465	ρ_9 is 70.8688	H_9 is 3.44591	M_9 is 0.88961

Mean queue length: M= 9.82765

Conclusion:

(i) Balking is used in the current model has a bearing effect on the indirect loss to the business.
(ii) When feedback is limited from each server to its prior servers, the outcomes of this research would be the same as the model mentioned in reference 17.
(iii) When the input process is independent of queue size i.e., balking is not occurring at any point in the system, and joining only happens at the beginning of the system and requires passing via every service channel before leaving the system having finite capacity where feedback is permissible from the terminal service channel to all its previous channels including itself then the results would remain in agreement with results in reference 3.

(iv) This model can be extended to non-serial service channels showing balking and reneging while the concept of impatient behaviour of customers can be used in serial service channels.

Conflict of Interest Statement:

I hereby disclose all of my conflicts of interest and other potentially conflicting interests, including specific financial interests and relationships and affiliations relevant to NeuroQuantology Journal (eg, employment/affiliation, grants or funding, consultancies, honoraria, stock ownership or stock options, expert testimony, royalties, or patents filed, received, or pending). This applies to the past 5 years and the foreseeable future.

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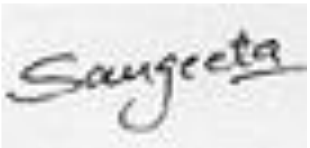

Signature

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Date: 11-12-2022

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