A Critical Analyzation of the Comparison of Solutions of Matrix Eigenvalue Problems

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Abstract:

Eigenvalues play quite an important role in differential equations and the real life. Several phenomena depend upon the eigenvalues (frequency) such as wobbling and collapsing of the bridges, analyzation of the stable and unstable, study of rotatory bodies, vibrating bodies, and their small oscillations, and many more. In this paper we will be observing different methods and their rate of convergence, algorithms, and efficiency. The focus will be on the QR method being the most efficient method in its converging power, efficiency, computational cost, and algorithms. For this we will be looking at some of the basics of the eigenvalues and the matrices as well.

I.   INTRODUCTION

The focus of this expository paper is to compare the different methods of solving the eigenvalue problems and to consider the most efficient method of all the methods. In this, one can see the different methods initializing from the power method and ending to the QR method, their convergence, computational time, computational cost or complexity, algorithm, and efficiency. These methods lead to the eigenvalue solutions which are quite important in the real life as well as it helps in various applications such as analyzation of stability, study of rotatory bodies, analyzing different population growth models, vibrating systems and their small oscillations and many more mechanical systems [1]. These eigenvalues correspond to the energy levels of a molecule which results in vibrations and because every mechanical system depends on vibrations so such eigenvalues (frequencies) impact in both the manner positively and negatively as it can lead to catastrophic consequences like collapsing of bridges, wobbling of bridges as well [2]. As there are several methods of finding the solution of eigenvalues, these have been divided into two categories; one is a direct method and the other one is an iterative method [3]. Direct methods are generally for computing all the eigenvalues for a particular matrix, either be its sparse, dense, symmetric, asymmetric or of any

form. These methods specifically require $O(n^3)$ computational time to find the eigenvalues for a considerable n * n matrix [3]. Iterative methods mainly focus on the computation of a particular set of eigenvalues and the associated eigenvectors to them [3]. The convergence of the iterative methods is generally based on the properties of a given matrix. These methods generally help to find various arbitrary eigenvalues and their associated eigenvectors in only a set of some iterations even that to a very little error. In 1961, the QR algorithm was brought to light independently by John G. F. Francis and Vera Kublanovskaya [4], it was somehow the continuation of the LR algorithm. In the first attempts of the QR algorithm by Francis, one can find the general proof of the lower triangular matrices along with proving the matrix as singular and considering its different eigenvalues. Also, the connection between the QR algorithm and LR algorithm can be easily reviewed for the positive-definite symmetric matrices. Later on, he did a number of changes such as origin shifts and deflations in the iterated matrices. One of the approaches for applying QR to get the better results was to reduce the given general matrix into a Hessenberg matrix and then to the tridiagonal matrix by using the Gauss elimination criteria [4]. In 1995, D S Watkins considered the QR algorithm and confirmed its forward stability and the transmission of shifts as the tiny entries in the sub diagonals do not cause any hindrance in the stability of the matrix or reaching to the concerned output. Even, he did talk about the multishift or the shifting strategies in the QR algorithm [5]. In 2005, Raf Vandebril et.al give a new approach on QR algorithm that is by reducing the given matrix into a semi separable matrix and not into a tridiagonal matrix which will lead to the reduction in the computational cost of QR from $O(n^3)$ to $O(n)$ flops making it simpler to use as the larger order terms will be less per iteration

[6]. After that a number of authors such as Parlett, Braman, Colbrook, Henry and many others had given their astonishing works on QR algorithm. The most recent work has been given by Nikhil Srivastava with his coauthors on the global convergence of the Hessenberg Shifted QR for the nonsymmetric matrices [7]. We will begin with one of the easiest iterative methods, i.e., the power method and talk about its generalization, its procedure, convergence and algorithm, and then for the further methods. For this to understand we must know the basics that are as under:

II. PRELIMINARIES

A. Eigenvalues:

Eigenvalues are defined as the scalars (constants) or their sets related to the given system of linear equations (i.e., a matrix equation). 'Eigen' - a German word meaning 'proper' or 'characteristic' so these are termed as characteristic roots, characteristic values, proper values, or latent roots as well [8]. The basic use of eigenvalues is to convert the eigenvector.

The basic equation for this:

$Mx = \lambda x$

where λ is an eigenvalue of the matrix M to the corresponding eigenvector x.

B. Eigenvectors:

Eigenvectors are the vectors (not zero) that do not get converted when some transformations (linear) are performed on them. The change can be done only by a scalar factor [9]. In simple words, if M is known to be a linear transformation from a vector space V and x is the vector in V, which is a vector not equal to zero, then V will be considered as an eigenvector of M if M (x) is the scalar multiple of x.

An Eigenspace of a particular vector x is the set of all the eigenvectors with their corresponding eigenvalues including the zero vector, but the zero vector is not considered as an eigenvector.

C. Orthogonal Matrix:

An orthogonal matrix, also known as an orthonormal matrix, is defined as the real square matrix whose rows and columns are orthonormal vectors[10], i.e.,

Transpose (Q)*Q = Q*Transpose (Q) = I where I is known as the Identity matrix.

Or a matrix is supposed to be orthogonal if its transpose came to be equal to its inverse.

D. Bidiagonal Matrix:

A bidiagonal matrix can be defined as the matrix having non-zero elements in the main diagonal along with the non-zero entries either in the just above diagonal or in the just below diagonal [11].

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

OR

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

E. Tridiagonal Matrix:

A tridiagonal matrix is defined as the matrix having non-zero elements in the main diagonal as well as the non-zero entries in the above and the below the main digaonal [11].

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

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F. Error:

The error is estimated as the difference between the exact value and the approximated value [12]. There are several types of errors including the relative error, absolute error, percentage error, random error, and systematic error.

The error can be resolved using different methods basically by using the iterative methods which are described below. After applying the required formula for the different methods, in each iteration, the error becomes small and hence helps in acquiring the required result, i.e., eigenvalue.

III.   TYPES OF METHODS

There are several methods to reach to the solution. Some of these are:

A. Power method:

It is one of the easy methods to find the largest eigenvalue and the associated vector. For a given diagonalizable matrix B, this method produces a number λ which is known to be the greatest eigenvalue of matrix A, and a vector x which is not equal to zero, is a corresponding eigenvector of λ i.e.

$$Bx = \lambda x$$

This method is also termed as Von Mises iteration [13].

It is a simple method, but the convergence rate of this method is very slow; also multiplying of matrix B with vector v

again and again takes much time, so it is mainly effective in large sparse matrices. The rate of convergence can be found for this method as λ2/ λ1. In this we start with a vector x, an approximation, and continues with multiplication of matrix B by this vector x and putting this equal to λ times vector x.

Algorithm [11]:

Given x_0 , we iterate,

n = 0,

repeat

$y_{n+1} = $ B x_n ,

$x_{n+1} = y_{n+1}$ / ‖y_{n+1}‖2 (Approximate eigenvector)

$\tilde{\lambda}_{n+1} = x_{n+1}^T B x_{n+1}$ (Approximate eigenvalue)

$n = n + 1$

until the required convergence is acquired.

B. Inverse Power Method:

This method is quite the same as the power method except that it finds the smallest value instead of the greatest value. It is used when the corresponding eigenvalue is already known. This method is basically a more exact form of the power method to reach the smallest eigenvalue. This method converges linearly. It starts with an approximation μ for the eigenvalue corresponding to the designed eigenvector or a vector x, an approximation [13] i.e.

inverse(A) *x = (1/ λ) x where 1/ λ = μ

Algorithm [11]:

Given x_0 , we iterate,

n = 0,

repeat

$y_{n+1} = (B - \sigma I)^{-1} x_n,$

$x_{n+1} = y_{n+1}$ / ‖y_{n+1}‖2 (Approximate eigenvector)

$\tilde{\lambda}_{n+1} = x_{n+1}^T B x_{n+1}$ (Approximate eigenvalue)

$n = n + 1$

until convergence.

Note: Inverse power method is basically a more advanced approach to the power method. In this method, we just modify the formula to reach the smallest eigenvalue instead of the greatest one, using inverse of the given matrix instead of its actual value as well as the inverse of the eigenvalue is used in modified approach.

C. Shifted power method:

This method is basically used to find the eigenvalue close to a vector. It is a modified version of the power method to reach not only to the number or root of the problem but to the vector specified solution. It is much better than the power method and inverse power method as it converges quadratically, whereas these two converge linearly. The computational cost for this method is quite expensive as it uses O (n^2) flops. Let s be a scalar magnitude so to find its closest eigenvalue

(A- s * I) x = (λ - s) x

D. Jacobi Method:

Jacobi method is one of the oldest methods for finding the solution of eigenvalue problems, as it was initiated in 1846[9]. It is very slow as compared to the other methods in reaching to the solution. But this method is interesting only because of

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its accuracy as many times it is helpful in finding the small eigenvalues and their associated eigenvectors much more efficiently and accurately than any other method [14]. This method is very helpful in obtaining the eigenvalues of the dense matrix.

Algorithm [1]: Jacobi rotation to B in the coordinates j, k:

Proc Jacobi Rotation (B, j, k)

if | b_{jk} | is comparatively not trivial

Y = (b_{jj} − b_{kk}) / (2. b_{jk})

x = sign (T) / (|T| +$\sqrt{1 + T^2}$)

c = 1 / $\sqrt{1 + x^2}$

s = c. x

B = R^Y (j, k, θ). B. R (j, k, θ)

if the required eigenvectors are found

J = J. R (j, k, θ)

end if

end if

Main command is of the following form:

repeat

pick a pair (j, k)

call the above-mentioned Jacobi rotation (B, j, k)

until B is desirably diagonalized

E. Rayleigh Quotient:

It is an eigenvalue method by which one can extend the inverse iteration method by using Rayleigh Quotient to find the increasingly accurate eigenvalue solutions. This is one of the fastest methods of getting the solution as it converges very rapidly and a very few iterations are required to reach the limit. It converges cubically that implies in every iteration the number of correct digits will be tripled automatically and the error will become very small, as well as the eigenvalue will come to be very simple [9].

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Algorithm [1]:

Given x_0, with $\|x_0\|2 = 1$ and a stopping tolerance tol given by user, we iterate

$$\rho_0 = \rho(x_0, B) = \frac{x_0^T B x_0}{x_0^T x_0}$$

$n = 0$

repeat

$y_n = (B - \rho_{n-1}I)^{-1} x_{n-1}$

$x_n = y_n / \|y_n\|2$

$\rho_n = \rho(x_n, B)$

$n = n + 1$

until the convergence ($\|B x_n - \rho_n x_n\|2 < tol$

And when the stopping criteria is fulfilled for the given matrix then ρ_n will be within the tol of an eigenvalue of B.

F. QR Algorithm:

This method is solved by using the principle of matrix decomposition. This method mainly takes O (n^2) computational cost [9]. It converges cubically. The QR decomposition is initiated by producing the matrix as the product of an orthogonal (orthonormal) matrix and an upper triangular matrix and then multiplying the factors in the reverse order and repeat these steps [8].

i.e., B = Q*R, where Q is known as the orthogonal matrix and R as the upper triangular matrix.

If the diagonal elements of R are non-negative, then the eigenvalue so obtained is unique. An orthogonal matrix, also known as an orthonormal matrix, is a real square matrix in which the rows and colums are orthonormal vectors[8], i.e.,

Transpose $(Q)*Q = Q*$Transpose $(Q) = I$ where I is the Identity matrix.

Or a matrix is considered to be orthogonal if the transpose of the matrix is equal to the inverse of the matrix.

Algorithm [11]:

Given B_0, we iterate,

$n = 0$

repeat

Factor $B_n = Q_n R_n$ (the QR decomposition)

$B_{n+1} = R_n Q_n$

$n = n + 1$

until the convergence is acquired.

Algorithm: QR iteration including a shift [11]

Given B_0, we iterate

$n = 0$

repeat

Pick a shift σ_n close to the eigenvalue of B

Factor $B_n - \sigma_n I = Q_n R_n$ (the QR decomposition)

$B_{n+1} = R_n Q_n + \sigma_n I$

$n = n + 1$

until the required convergence occurs.

Example using commands:

There has been several software to get the required eigenvalues of a matrix problem using the different commands. Here, one can see the commands for finding the solution of the problem using QR algorithm with the help of MATLAB software [15].

A=vander(1:4)

for k=1:20, [Q,R]=qr(A); A=Q'*A*Q;

end

A

eig(A)

This is the procedure to find the eigenvalue for a 4*4 matrix. Similarly, just by applying a change in the value of A i.e., the matrix, one can find the solution or the eigenvalues of the different matrix problems.

Method	Computational Complexity (FLOPS)	Rate of Convergence
Power Method	O (n^2)	Linear
Inverse Power Method	O (n^3)	Linear
Shifted Method	O (n^2)	Quadratic
Jacobi Method	O (n^2)	Quadratic
Rayleigh Quotient Method	O (n^3)	Cubic
QR Method (Without shift)	O (n^2)	Cubic
QR Method (With shift/ tridiagonal decomposition)	O (n)	

Table 1: Comparison

Here, FLOPS denotes the floating-point operations per second i.e., number of operations needed to be performed in one iteration by a particular method. Lesser the complexity per iteration, higher the accuracy and output.

IV.   CONCLUSION

We have studied several methods starting from the power method to the QR method previously by which we can reach our eigenvalue solution. Out of all these methods, the most effective method for the purpose of solution is: QR algorithm method. The QR algorithm is the fastest method of all the iterative and the direct methods as it helps to obtain all the required solutions to a high relative accuracy. This implies that all the digits of the solutions are correct up to the three approximated places per iteration. Also, it is backwardly stable, as Watkins has proved in his works. In the QR method, the reduced bidiagonal matrix is much more accurate than the reduced tridiagonal matrix. The latter approaches by Vandebril in the semiseparable matrices makes it more efficient, reducing the higher order terms in each iteration, making it less time consuming. The QR algorithm has been considered a jewel in the crown of matrix decomposition by Higham for solving the nonsymmetric eigen problems. And, Parlett considered it a complete original idea and not a refinement for finding solutions with satisfactory experience. One more thing which we have taken into consideration is its computational cost, as many of the methods have the same convergence rate but high computational cost. The computational complexity in the basic QR method is of approximately O (n^2) flops whereas the one with the shifts or decompositions is only O (n) flops. For this, a number of modifications have been done after the basic QR, such as the transmission of shifts, deflation techniques which helps in reducing the computational cost. The upgraded QR method takes only 10% flops of the general method for a general matrix leading to the decrement of computational cost. The QR method in comparison to other methods converges faster (cubically) and accurately along with its low computational cost. So, in comparison to the other methods, the QR algorithm came to be the most trustable, less time consuming and of less computational cost. One of the recent works by Nikhil Srivastava (2021) has given a completely new view to the QR algorithm in the nonsymmetric matrices opening new paths to further research. Another great work by Vandebril (2022) leads to the upgradation of the QR method to the QZ method helping in choosing the concerned eigenvalue and moving to the next conditions without any further delay in the process. Through the further studies on QR algorithm, the solutions of the eigenvalue problems will be easily achieved by using new approaches and methods which will help to decrease the consummation of time to get the better and accurate results and solutions.

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