Some Fixed Point Results in Tvs- Cone Metric Space using Contractive Conditions

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Abstract:
In this article, we establish common fixed point results for four map- pings in Tvs-conemetric space with contractive conditions and we buildup fixed point result for integral type mappings in Tvs-cone metric space as an application. The inspiration driving this paper is to develop a theory of the results demonstrated by A.K.Dubey et al.[4] for T-contraction mappings. The result here summarizes and expand a portion of the notable outcomes present in the writing (see[1],[5],[6],[7],[8],[9]).

Keywords: Complete metric space, Tvs-Cone Metric Space, Cauchy Sequence, Fixed Point, Continuous Mapping.
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1. Introduction:

The generalization of classical metric space was introduced by Huang and Zhang [11] by replacing an ordered Banach Space for the real numbers set, and proved some fixed point results with contractive mappings on cone metric spaces ful- filling different contractive conditions. Subsequently, numerous authors concentrated on cone metric spaces and obtained some fixed point theorems for cone metricspaces which was an extension of Banach's contraction mapping principle into conemetricspaces. Du [9] has presented the idea of Tvs-conemetric zspace as an improved version of cone metric spaces[16]. Followed by the outcome, Fadailet al. [10] proved the unique fixed results utilizing c-distance in the conemetric spaces. S.K.Tiwari et.al.[18] proved fixed point results for generalized contractive mappings in cone metric spaces. A.K. Dubey et.al.[4] provedfixed point results in Tvs-cone metic space withc-distance.

The purpose of this paper is to prove common fixed point theorems for four map- pings with different contractive conditions in Tvs-conemetricspace. We provide an application to a system of integral equations to validate our results.

2. **Preliminaries:** The accompanying definitions and notations will be needed in continuation. In this paper, we have signified set of real numbers byR, a Banach Space by E.

Definition 2.1([4]): Let *E*be a real Banach space and P ⊂ *E*. ϑbeing the zero element in *E,*then Pis called a cone iff:

a) Pis closed,non-empty,P$\neq\{\vartheta\}$.
b) For all non-negative real numbersa, band y, z∈ Pthen ay + bz∈ P.
c) $y \in P, -y \in P \Rightarrow y = \theta$ iff $P \cap (-P) = \theta$.

GiventhattheconePisasubsetof*E*,apartialordering≤,definedby$y \leq z$
iff$z - y \in P$on Ein which $y \ll z$stands for $z - y \in P$. Pis solid if$int P \neq \phi$.
If $y, z \in E$, then there is a least number K >0such that,

$\theta \leq y \leq z \Rightarrow \| y \| \leqslant K \| z \|$

Then, at thatpoint,conePis called normal and number *K* which fulfills the given condition is known as a normalconstant.

Definition 2.2([3,4,5]): Suppose a vector valued mapping $d_\alpha : X_1 \times X_1 \to E$, where X_1be a non-empty setsatisfies

a) For every $y, z \in X_1$, $d_\alpha(y, z) \geq 0$and $d_\alpha(y, z) = 0$ iff $y = z$,

b) For every $y, z \in X_1$, $d_\alpha(y, z) = d_\alpha(z, y)$,

c) For every $y, z, w \in X_1$, $d_\alpha(y, z) \leq d_\alpha(y, z) + d_\alpha(w, z)$.

Then $d_\alpha$is said to be a tvs-cone metric on X_1, and the pair $(X_1, d_\alpha)$is tvs-cone metric space. 2705

Definition 2.3([2,4,5]): Let Ebe a real Banach space, (X_1, d_α)be a tvs-cone metric space over E and$\{y_n\}_{n\geq1}$be a sequence in X_1.Then

a) For all $c \in E$with $\theta \ll c$,$\{y_n\}_{n\geq1}$converges to X_1,then there exists a natural number *N* such that $d_\alpha(y_n, y) \ll c \ \forall n \geq N$. We represent this by $y_n \to y$ *as* $n \to \infty$.

b) For all $c \in E$with $\theta \ll c$, $\{y_n\}_{n\geq1}$is a Cauchy Sequence if there is a natural number *N* such that $d_\alpha(y_n, y_m) \ll c \ \forall n, m \geq N$.

c) If every Cauchy sequence in $X_1$is convergent, then $(X_1, d_\alpha)$is a complete Tvs-cone metric space.

Lemma 2.4([2,4,5,16]):

1) Let Ebe a real Banach space Pbe a cone and $\alpha \le \lambda b$, where $b \in$ Pand $\theta \le \lambda < 1$,thenb = θ.

2) If $c \in$ int P, $\theta \le b_n$and $b_n \to \theta$,then there exists a positive integer Nin such a way that $b_n \ll c\ \forall\ n \ge N$.

3. Main Results

Theorem 3(a): Let *A, B, S* and *T* becontinuous mappings on a tvs-valued cone metric space (X_1, d_α),where $A(X_1) \subseteq S(X_1)$and $B(X_1) \subseteq T(X_1)$and suppose there exists mappings $\kappa_1, \kappa_2, \kappa_3 : X_1$ [0, 1)satisfying the following conditions:

a) $2(\kappa_1 + \kappa_2 + \kappa_3)u < 1$ *for* $y \in X_1$.

b) i)$d_\alpha(Ay, Bz) \le \kappa_1(y)[d_\alpha(Sy, Tz)+d_\alpha(Ay, Sy)]+\kappa_2(y)[d_\alpha(Sy, Tz)+d_\alpha(Bz, Tz)]+ \kappa_3(y)[d_\alpha(Sy, Tz) + \frac{d_\alpha(Sy,Bz)+d_\alpha(Ay,Tz)}{2}]$ *and*

ii) $d_\alpha(Bz, Ay) \le \kappa_1(z)[d_\alpha(Tz, Sy)+d_\alpha(Sy, Ay)]+\kappa_2(z)[d_\alpha(Tz, Sy)+d_\alpha(Tz, Bz)]+ \kappa_3(z)[d_\alpha(Tz, Sy) + \frac{d_\alpha(Bz,Sy)+d_\alpha(Tz,Ay)}{2}]$ 2706

for all $y, z \in X_1$, then the mappings *A, B, S* and *T* has a unique common fixed point$y^* \in X$.

Proof. Choose any random $y_n \in X_1$.Considering the sequence $\{y_n\}$and $\{z_n\}$such that

Set $Ay_{2n} = Sy_{2n+1} = z_{2n+1}$,

$By_{2n+1} = Ty_{2n+2} = z_{2n+2}$ for all $n \ge 0$.

Then by putting $y = y_{2n+1}$and $z = y_{2n+2}$in b (i), we have

$d_\alpha(y_{2n+1}, y_{2n+2}) \le d_\alpha(Ay_{2n}, By_{2n+1})$

$\le \kappa_1(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1})+d_\alpha(Ay_{2n}, Sy_{2n})]+\kappa_2(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1})+ d_\alpha(By_{2n+1}, Ty_{2n+1})] + \kappa_3(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1}) + \frac{d_\alpha(Sy_{2n},By_{2n+1})+d_\alpha(Ay_{2n},Ty_{2n+1})}{2}]$

$\le \kappa_1(y_{2n})[d_\alpha(y_{2n}, y_{2n+1})+ d_\alpha(y_{2n+1}, y_{2n})]+ \kappa_2(y_{2n})[d_\alpha(y_{2n}, y_{2n+1})+d_\alpha(y_{2n+2}, y_{2n+1})] + \kappa(y_{2n})[d_\alpha(y_{2n}, y_{2n+1}) + \frac{d_\alpha(y_{2n},y_{2n+2})+d_\alpha(y_{2n+1},y_{2n+1})}{2}]$

$$\le \frac{2\kappa_1(y_{2n}) + \kappa_2(y_{2n}) + \frac{3}{2}\kappa_3(y_{2n})}{1 - \kappa_2(y_{2n}) - \frac{1}{2}\kappa_3(y_{2n})} d_\alpha(y_{2n}, y_{2n+1})$$

Continuing in this way

$$d\alpha(y_{2n+1}, y_{2n+2}) \le \frac{2\kappa 1(y0) + \kappa 2(y0) + \frac{3}{2}\kappa 3(y0)}{1 - \kappa 2(y0) - \frac{1}{2}\kappa 3(y0)} d_\alpha(y_{2n}, y_{2n+1}) \ldots\ldots\ldots (1)$$

Similarly by putting $z = y_{2n+1}$ and $y = y_{2n+2}$ in b) ii), we have

$d_\alpha(y2n+2, y2n+1) \le d_\alpha(By2n+1, Ay2n)$

$\le \kappa_1(y_{2n+1})[d_\alpha(Ty_{2n+1}, Sy_{2n}) + d_\alpha(Ay_{2n}, Sy_{2n})] + \kappa_2(y_{2n+1})[d_\alpha(Ty_{2n+1}, Sy_{2n}) + d_\alpha(Ty2n+1, By2n+1)] + \kappa 3(y2n+1)[d_\alpha(Ty2n+1, Sy2n) + \frac{d\alpha(By2n+1,Sy2n)+d\alpha(T y2n+1,Ay2n)}{2}]$

$\le \kappa_1(y_{2n+1})[d_\alpha(y_{2n+1}, y_{2n}) + d_\alpha(y_{2n}, y_{2n+1})] + \kappa_2(y_{2n+1})[d_\alpha(y_{2n+1}, y_{2n}) + d_\alpha(y2n+1, y2n+2)] + \kappa 3(y2n+1)[d_\alpha(y2n+1, y2n) + \frac{d\alpha(y2n+2,y2n)+d\alpha(y2n+1,y2n+1)}{2}]$

$$\le \frac{2\kappa 1(y2n+1) + \kappa 2(y2n+1) + \frac{3}{2}\kappa 3(y2n+1)}{1 - \kappa 2(y2n+1) - \frac{1}{2}\kappa 3(y2n+1)} d\alpha(y2n+1, y2n)$$

Continuing in the same manner

$$d\alpha(y2n+2, y2n+1) \le \frac{2\kappa 1(y0) + \kappa 2(y0) + \frac{3}{2}\kappa 3(y0)}{1 - \kappa 2(y0) - \frac{1}{2}\kappa 3(y0)} d\alpha(y2n+1, y2n) \ldots\ldots\ldots\ldots (2)$$

Denote $\rho_n = d_\alpha(y_{2n+1}, y_{2n+2}) + d_\alpha(y_{2n+2}, y_{2n+1})$

Adding equations (1) and (2), we get

$\rho_n \le p\rho_{n-1}$ with

$$0 \le p = \frac{2k_1(y_0) + k_2(y_0) + \frac{3}{2}k_3(y_0)}{1 - k_2(y_0) - \frac{1}{2}k_3(y_0)} < 1$$

Since $2(k_1 + k_2 + k_3)u < 1$ *for* $y \in X_1$ for all $x \in X$. By induction we get

$\rho_n \le p^n \rho_0$ and $d_\alpha(y_{2n+1}, y_{2n+2}) \le \rho_n \le p^n[d_\alpha(y_0, y_1) + d_\alpha(y_1, y_0)]$

Suppose $m > n \ge 1$. Then it follows that

$$D\alpha(yn,ym) \leq \frac{p^n}{1-p}[d_\alpha(y_0,y_1) + d_\alpha(y_1,y_0)] = \rho_n$$

Hence, from Lemma 2.3 it is obvious that $\{\rho_n\}$ is a Cauchy sequence inX_1. Since the given set X_1as complete, then $y_n \rightarrow y^* as n \rightarrow \infty$.

Since A, B, S, T are continuous mappings then $y_{n+1}= Sy_{n+1}= Ay_n \rightarrow Ay^*$and since the limit of a sequenceint vs-conemetric space is unique,w eget$Ay^*=y^*$ *i.e.*$y^*$is the unique common fixed point of$A$ and S.Similarly we can prove that $y^*$is unique common fixed point of B and T.Hence $y^*$is the common fixed point of $A, B, S,$ &T .

Suppose that $Ax = x$,Then b (i) implies that

$d_\alpha(\rho, \rho) = d_\alpha(A\rho, B\rho)$

$\leq \kappa_1(\rho_0)[d_\alpha(S\rho, T\rho) + d_\alpha(A\rho, S\rho)] + \kappa_2(\rho_0)[d_\alpha(S\rho, T\rho)+$

$d_\alpha(B\rho, T\rho)] + \kappa_3(\rho_0)[d_\alpha(S\rho, T\rho) + \frac{d\alpha(S\rho,B\rho)+d\alpha(A\rho,T\rho)}{2}]$ 2708

$\leq 2(\kappa_1+ \kappa_2+ \kappa_3)(\rho_0)d_\alpha(\rho, \rho) <1 for\ \rho \in X_1$

then by property (P_5) it shows that $d_\alpha(\rho, \rho) = \vartheta$

Finally let us assume that there is another fixed point$z^*$of $A, B, S,$ &T then we have

$d_\alpha(y^*, z^*) = d_\alpha(Ay^*, Bz^*)$

$\leq \kappa_1(y^*)[d_\alpha(Sy^*, T\rho) + d_\alpha(Ay^*, Sz^*)] + \kappa_2(y^*)[d_\alpha(Sy^*, Tz^*)+$

$d_\alpha(Bz^*, Tz^*)] + \kappa_3(y^*)[d_\alpha (Sy^*, Tz^*) + \frac{d_\alpha(Sy^*,Bz^*)+d_\alpha(Ay^*,Tz^*)}{2}]$

$2(\kappa_1+ \kappa_2+ \kappa_3)(y^*)d_\alpha(y^*, z^*)\ for\ \rho \in X_1$

$2(\kappa_1+ \kappa_2+ \kappa_3) <1$ and by property (P_5)shows that $d_\alpha(y^*, z^*) = \vartheta$ and also we have $d_\alpha(y^*, y^*) = \vartheta$ Hence by lemma (1) $y^*= z^*$. The fixed point is unique.

Theorem 3(b): Let *A, B, S* and *T* be continuous mappings on a tvs-valued cone metric space (X_1, d_α), where $A(X_1) \subseteq S(X_1)$ and $B(X_1) \subseteq T(X_1)$ and suppose there exists mappings $\kappa_1, \kappa_2, \kappa_3, \kappa_4: X_1[0, 1)$ satisfying the following conditions:

a) $2(\kappa_1+ \kappa_2+ \kappa_3+ \kappa_4)u<1$ *for* $y \in X_1$.

b) i) $d_\alpha(Ay, Bz) \leq \kappa_1(y)[d_\alpha(Sy, Tz)+d_\alpha(Ay, Sy)]+\kappa_2(y)[d_\alpha(Sy, Tz)+d_\alpha(Bz, Tz)]+$

$\kappa_3(y)[d_\alpha(Ay, Sy)+ d_\alpha(Bz, Tz)]+ \kappa_4(y)[d_\alpha(Bz, Tz)+ \frac{d_\alpha(Sy,Bz)+d_\alpha(Ay,Tz)}{2}]$ *and*

ii) $d_\alpha(Bz, Ay) \leq \kappa_1(z)[d_\alpha(Tz, Sy)+d_\alpha(Sy, Ay)]+\kappa_2(z)[d_\alpha(Tz, Sy)+d_\alpha(Tz, Bz)]$

$+\kappa_3(z)[d_\alpha(Sy, Ay) + d_\alpha(Tz, Bz)] + \kappa_4(z)[d_\alpha(Tz, Bz) + \frac{d_\alpha(Bz,Sy)+d_\alpha(Tz,Ay)}{2}]$

for all $y, z \in X_1$, then the mappings *A, B, S* and *T* has a unique common fixed point $y^* \in X$.

Proof. Choose any random $y_* \in X_1$. Considering the sequence $\{y_n\}$ and $\{z_n\}$ such that

Set $Ay_{2n}= Sy_{2n+1}= z_{2n+1}$,

$By_{2n+1}= Ty_{2n+2}= z_{2n+2}$ *for all* $n \geq 0$.

Then by putting $y = y_{2n+1}$ and $z = y_{2n+2}$ in b) i), we have

$d_\alpha(y_{2n+1}, y_{2n+2}) \leq d_\alpha(Ay_{2n}, By_{2n+1})$

$\leq \kappa_1(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1}) + d_\alpha(Ay_{2n}, Sy_{2n})]+$

$\kappa_2(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1}) + d_\alpha(By_{2n+1}, Ty_{2n+1})] + \kappa_3(y_{2n})[d(Ay_{2n}, Sy_{2n})$

$+d(By_{2n+1}, Ty_{2n+1})]+\kappa_4(y_{2n})[d_\alpha(Sy_{2n}, Ty_{2n+1})+\frac{d_\alpha(Sy_{2n},By_{2n+1})+d_\alpha(Ay_{2n},Ty_{2n+1})}{2}$

$\leq \kappa_1(y_{2n})[d_\alpha(y_{2n}, y_{2n+1}) + d_\alpha(y_{2n+1}, y_{2n})] + \kappa_2(y_{2n})[d_\alpha(y_{2n}, y_{2n+1})+$

$d_\alpha(y_{2n+2}, y_{2n+1})] + \kappa_3(y_{2n})[d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})]+ \kappa_4(y_{2n})[d_\alpha(y_{2n}, y_{2n+1}) + \frac{d_\alpha(y_{2n},y_{2n+2})+d_\alpha(y_{2n+1},y_{2n+1})}{2}]$

$\leq 2\kappa_1(y_{2n}) + \kappa_2(y_{2n}) + \kappa_3(y_{2n}) + \frac{1}{2}\kappa_4(y_{2n})d_\alpha(y_{2n}, y_{2n+1})$

$$\therefore d_\alpha(y_{2n+1}, y_{2n+2}) \leq \frac{2\kappa1(y0) + \kappa2(y0) + \kappa3(y0) + \frac{1}{2}\kappa4(y0)}{1 - \kappa2(y0) - \kappa3(y0) - \frac{3}{4}\kappa4(y0)} d_\alpha(y2n, y2n + 1)$$

Continuing in this way

$$d_\alpha(y_{2n+1}, y_{2n+2}) \leq \frac{2\kappa1(y0)+\kappa2(y0)+\kappa3(y0)+\frac{1}{2}\kappa4(y0)}{1-\kappa2(y0)-\kappa3(y0)-\frac{3}{4}\kappa4(y0)} d_\alpha(y_{2n}, y_{2n+1}) \quad \text{.......................}(3)$$

Similarly by putting $z = y_{2n+1}$ and $y = y_{2n+2}$ in (b) ii), we have

$d_\alpha(y_{2n+2}, y_{2n+1}) \leq d_\alpha(By_{2n+1}, Ay_{2n})$

$\leq \kappa_1(y_{2n+1})[d_\alpha(Ty_{2n+1}, Sy_{2n}) + d_\alpha(Ay_{2n}, Sy_{2n})]+$

$\kappa_2(y_{2n+1})[d_\alpha(Ty_{2n+1}, Sy_{2n}) + d_\alpha(Ty_{2n+1}, By_{2n+1})] + \kappa_3(y_{2n+1})[d_\alpha(Sy_{2n}, Ay_{2n})+$

$d_\alpha(Ty_{2n+1}, By_{2n+1})] + \kappa_4(y_{2n+1})[d_\alpha(Ty_{2n+1}, Sy_{2n}) + \frac{d_\alpha(By_{2n+1},Sy_{2n})+d_\alpha(Ty_{2n+1},Ay_{2n})}{2}]$

$\leq \kappa_1(y_{2n+1})[d_\alpha(y_{2n+1}, y_{2n}) + d_\alpha(y_{2n}, y_{2n+1})]+$

$\kappa 2(y2n+1)[d\alpha(y2n+1, y2n) + d\alpha(y2n+1, y2n+2)] + \kappa 3(y2n+1)[d\alpha(y2n, y2n+1)+$

$d\alpha(y2n+1, y2n+2)] + \kappa 4(y2n+1)[d\alpha(y2n+1, y2n) + \underline{d\alpha(y2n+2,y2n)+d\alpha(y2n+1,y2n+1)}]\ _2$

$\leq 2\kappa 1(y2n+1) + \kappa 2(y2n+1) + {}_2\ \kappa 4(y2n+1)d\alpha(y2n+1, y2n)^{\underline{1}}$

$$d\alpha(y2n+2, y2n+1) \leq \frac{2\kappa 1(y2n) + \kappa 2(y2n) + \kappa 3(y2n) + \frac{1}{2}\kappa 4(y2n)}{1 - \kappa 2(y2n) - \kappa 3(y2n) - \frac{3}{2}\kappa 4(y2n)} d\alpha(y2n+1, y2n)$$

Continuing in the same manner

$$d_\alpha(y_{2n+2}, y_{2n+1}) \leq \frac{2\kappa_1(y_0)+\kappa_2(y_0)+\kappa_3(y_0)+\frac{1}{2}\kappa_4(y_0)}{1-\kappa_2(y_0)-\kappa_3(y_0)-\frac{3}{2}\kappa_4(y_0)} d_\alpha(y_{2n+1}, y_{2n}) \ldots\ldots\ldots\ldots (4)$$

Denote $\rho_n = d_\alpha(y_{2n+1}, y_{2n+2}) + d_\alpha(y_{2n+2}, y_{2n+1})$

Adding equations (3) and (4), we get

$\rho_n \leq p\rho_{n-1}$ with

$$0 \leq p = \frac{2\kappa_1(y_0) + \kappa_2(y_0) + \kappa_3(y_0) + \frac{1}{2}\kappa_4(y_0)}{1 - \kappa_2(y_0) - \kappa_3(y_0) - \frac{3}{2}\kappa_4(y_0)} < 1$$

Since $2(\kappa_1+\kappa_2+\kappa_3+k_4)<1$ *for* $y\in X_1$ for all $y\in X_1$. By induction we get

$\rho_n \leq p^n\rho_0$ and $d_\alpha(y_{2n+1}, y_{2n+2}) \leq \rho_n \leq p^n[d_\alpha(y_0, y_1) + d_\alpha(y_1, y_0)]$

Suppose $m > n \geq 1$.Then it follows that

$$d_\alpha(y_n, y_m) \leq \frac{p^n}{1-p}[d_\alpha(y_0, y_1) + d_\alpha(y_1, y_0)] = \rho_n$$

Hence, from Lemma 2.3 it is obvious that $\{\rho_n\}$ is a Cauchy sequence in X_1. Since the given set X_1 as complete, then $y_n \to y^\star$ as $n \to \infty$.

Since *A, B, S, T* are continuous mappings then $y_{n+1} = \to Sy_{n+1} = Ay_nAy^\star$ and sincethelimitofasequenceintvs-conemetricspaceisunique,weget $Ay^\star=y^\star$ *i.e* $y^\star$ is the unique common fixed point of *A* and *S*.Similarly we can prove that $y^\star$ is unique common fixed point of *B* and *T* .Hence $y^\star$ is the common fixed point of *A, B, S, T* .

Soppose that $Ax = x$,Then b (i) implies that

$d_\alpha(\rho, \rho) = d_\alpha(A\rho, B\rho)$ 2712

$\leq \kappa_1(\rho_0)[d_\alpha(S\rho, T\rho) + d_\alpha(A\rho, S\rho)] + \kappa_2(\rho_0)[d_\alpha(S\rho, T\rho) + d_\alpha(B\rho, T\rho)]+$

$\kappa_3(\rho_0))[d(A\rho, S\rho) + d(B\rho, T\rho,)] + \kappa_4(\rho_0)[d_\alpha(S\rho, T\rho) + \frac{d\alpha((S\rho,B\rho)+d\alpha(A\rho,T\rho)}{2}]$

$2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)(\rho_0)d_\alpha(\rho, \rho) < 1$ for $\rho \in X_1$

then by property (P_5) it shows that $d_\alpha(\rho, \rho) = \theta$

Finally let us assume that there is another fixed point$z^\star$of *A, B, S,* &*T* then we have

$d_\alpha(y^\star, z^\star) = d_\alpha(Ay^\star, Bz^\star)$

$\leq \kappa_1(y^\star)[d_\alpha(Sy^\star, T\rho) + d_\alpha(Ay^\star, Sz^\star)] + \kappa_2(y^\star)[d_\alpha(Sy^\star, Tz^\star)$

$+d_\alpha(Bz^\star, Tz^\star)] + \kappa_3(y^\star)[d(Ay^\star, Sy^\star) + d(Bz^\star, Tz^\star)]+$

$\kappa_4(y^\star)[d_\alpha(Sy^\star, Tz^\star) + \frac{d_\alpha(Sy^\star,Bz^\star)+d_\alpha(Ay^\star,T z^\star)}{2}]$

$\leq 2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)(y^\star)d_\alpha(y^\star, z^\star)$ for $\rho \in X_1$

$2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) < 1$and by property ($P_5$)shows that $d_\alpha(y^\star, z^\star) = \theta$and also we have $d_\alpha(y^\star, y^\star) = \theta$Hence by lemma 1 1) $y^\star = z^\star$.The fixed point is
unique.

4.Applications: 2713

Let *A, B, S* and *T* be continuous mappings on a tvs-valued cone metric space (X_1, d_α), where $A(X_1) \subseteq S(X_1)$ and $B(X_1) \subseteq T(X_1)$ and suppose there exists real numbers $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in [0, 1)$ satisfying the following conditions:

a) $2(\kappa_1+ \kappa_2+ \kappa_3)u<1$ *for* $y \in X_1$.

b) (i)$\int_0^{d\alpha(Ay,Bz)} \psi(t)dt \leq \kappa_1 [\int_0^{d_\alpha(S_y,T_z)+d_\alpha(A_y,S_y)} \psi(t)dt + k_2 \int_0^{d_\alpha(S_y,T_z)+d_\alpha(B_z,T_z)} \psi(t)dt + k_3 \int_0^{[d_\alpha(S_y,T_z)+\frac{d_\alpha(S_y,B_z)+d_\alpha(A_y,T_z)}{2}]} \psi(t)dt$ *and* (ii)$\int_0^{d_\alpha(B_z,A_y)}\psi(t)dt \leq \kappa_1 \int_0^{d_\alpha(T_z,S_y)+d_\alpha(S_y,A_y)}\psi(t)dt + k_1 \int_0^{d_\alpha(T_z,S_y)+d_\alpha(T_z,B_z)}\psi(t)dt +$

$k_3 \int_0^{[d_\alpha(T_z,S_y)+\frac{d_\alpha(S_y,B_z)+d_\alpha(T_z,A_y)}{2}]}\psi(t)dt$

for all *y, z* ∈ X_1, then the mappings *A, B, S* and *T* has a unique common fixed pointy^*∈ *X*.

$$\int_0^{d_\alpha(y_{2n+1},y_{2n+2})}\psi(t)dt = \int_0^{d_\alpha(A_y,B_z)}\psi(t)dt$$

≤

$k_1 \int_0^{d_\alpha(S_{y_{2n}},T_{y_{2n+1}})+d_\alpha(A_{y_{2n}},S_{y_{2n}})}\psi(t)dt + k_2 \int_0^{d_\alpha(Sy2n,Ty2n+1)+d_\alpha(By2n+1,Ty2n+1)}\psi(t)dt +$

$k_3 \int_0^{[d_\alpha(S_{y_{2n}},T\,y_{2n+1})+\frac{d_\alpha(S_{y_{2n}},B\,y_{2n+1})+d_\alpha(A_{y_{2n}},T_{y_{2n+1}})}{2}]}\psi(t)dt$

$\leq k_1 \int_0^{d_\alpha(y_{2n},y_{2n+1})+d_\alpha(y_{2n+1},y_{2n})}\psi(t)dt + k_2 \int_0^{d_\alpha(y_{2n},y_{2n+1})+d_\alpha(y_{2n+2},y_{2n+1})}\psi(t)dt +$

$k_3 \int_0^{d_\alpha(y_{2n},y_{2n+1})+\frac{d_\alpha(y_{2n},y_{2n+1})+d_\alpha(y_{2n+1},y_{2n+1})}{2}]}\psi(t)dt$

$$\leq 2k_1 + k_2 + \frac{3}{2}k_3 \int_0^{d_\alpha(y_{2n}+y_{2n+1})}\psi(t)dt$$

$$\int_0^{d_\alpha(y_{2n+1},y_{2n+2})}\psi(t)dt \leq \frac{2k_1 + k_2 + \frac{3}{2}k_3}{1 - k_2 - \frac{1}{2}k_3}\int_0^{d_\alpha(y_{2n},y_{2n+1})}\psi(t)dt$$

Similarly by putting $z = y_{2n+1}$and $y = y_{2n+2}$ in (b)(ii), we have

$$\int_0^{d_\alpha(y_{2n+2},y_{2n+1})}\psi(t)dt = \int_0^{d_\alpha(B_z,A_y)}\psi(t)dt$$

≤

$k_1 \int_0^{d_\alpha(T_{y_{2n+1}},S_{y_{2n}})+d_\alpha(S_{y_{2n}},A_{y_{2n}})}\psi(t)dt + k_2 \int_0^{d_\alpha(T_{y_{2n+1}},S_{y_{2n}})+d_\alpha(T_{y_{2n+1}},B_{y_{2n+1}})}\psi(t)dt +$

$$k_3 \int_0^{d_\alpha(Ty_{2n+1},Sy_{2n})+\frac{d_\alpha(By_{2n+1},Sy_{2n})+d_\alpha(Ty_{2n+1},Ay_{2n})}{2}} \psi(t)dt$$

$$\leq k_1 \int_0^{d_\alpha(y_{2n+1},y_{2n})+d_\alpha(y_{2n},y_{2n+1})} \psi(t)dt + k_2 \int_0^{d_\alpha(y_{2n+1},y_{2n})+d_\alpha(y_{2n+1},y_{2n+2})} \psi(t)dt +$$

$$k_3 \int_0^{d_\alpha(y_{2n+1},y_{2n})+\frac{d_\alpha(y_{2n+1},y_{2n})+d_\alpha(y_{2n+1},y_{2n})}{2}} \psi(t)dt$$

$$\leq 2k_1 + k_2 + \frac{3}{2}k_3 \int_0^{d_\alpha(y_{2n+1},y_{2n})} \psi(t)dt$$

$$\int_0^{d_\alpha(y_{2n+1},y_{2n})} \psi(t)dt \leq \frac{2k_1 + k_2 + \frac{3}{2}k_2}{1 - k_2 - \frac{1}{2}k_3} \int_0^{d_\alpha(y_{2n+1},y_{2n})} \psi(t)dt$$

Continuing in this way

$$\int_0^{d_\alpha(y_{2n},y_{2n+1})} \psi(t)dt \leq \frac{2k_1 + k_2 + \frac{3}{2}k_3}{1 - k_2 - \frac{1}{2}k_3} \int_0^{d_\alpha(y_{2n+1},y_{2n})} \psi(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Denote 2715

$$\rho_n = \int_0^{d_\alpha(y_{2n+1},y_{2n+2})} \psi(t)dt + \int_0^{d_\alpha(y_{2n+2},y_{2n+1})} \psi(t)dt$$

Adding equations (5) and (6), we get

$\rho_n \leq p\rho_{n-1}$ with

$$0 \leq p = \frac{2k_1 + k_2 + \frac{3}{2}k_3}{1 - k_2 - \frac{1}{2}k_3} < 1$$

Since $2(\kappa_1 + \kappa_2 + \kappa_3) < 1$. By induction we get

$\rho_n \leq p^n \rho_0$ and $\int_0^{d_\alpha(y_{2n+1},y_{2n+2})} \psi(t)dt \leq \rho_n \leq p^n \left[\int_0^{d_\alpha(y_0,y_1)} \psi(t)dt + \int_0^{d_\alpha(y_1,y_0)} \psi(t)dt \right]$

Suppose $m > n \geq 1$.Then it follows that

$$\int_0^{d_\alpha(y_n,y_m)} \psi(t)dt \leq \frac{p^n}{1-p}\left[\int_0^{d_\alpha(y_0,y_1)} \psi(t)dt + \int_0^{d_\alpha(y_1,y_0)} \psi(t)dt\right] = \rho_n$$

hencefromLemma2.3itisobviousthat$\{\rho_n\}$isaCauchysequenceinX_1.Since the given set X_1as complete, then $y_n \to y^\star asn \to \infty$.

Since A, B, S, T are continuous mappings then $y_{n+1}= Sy_{n+1}= Ay_n Ay^\star$and sincethelimitofasequenceintvs-conemetricspaceisunique,wegetAy*=y*i.ey*is the unique common fixed point of$A$

and S.Similarly we can prove that $y^\star$is unique common fixed point of B and T .Hence$y^\star$is the common fixed point of $A, B, S,$ &T .

Suppose that $Ax = x$,Then b i) implies that

$$\int_0^{d_\alpha(\rho,\rho)} \psi(t)dt = \int_0^{d_\alpha(A\rho,B\rho)} \psi(t)dt$$

$$\leq k_1 \int_0^{d_\alpha(S\rho,T\rho)+d_\alpha(A\rho,S\rho)} \psi(t)dt + k_2 \int_0^{d_\alpha(S\rho,T\rho)+d_\alpha(B\rho,T\rho)} \psi(t)dt + k_3 \int_0^{d_\alpha d_\alpha(S\rho,T\rho)+\frac{d_\alpha(S\rho,B\rho)+d_\alpha(A\rho,T\rho)}{2}} \psi(t)dt$$

$$\leq 2(k_1 + k_2 + k_3) \int_0^{d_\alpha(\rho,\rho)} \psi(t)\,dt < 1 \; for \; \rho \in X1$$

then by property (P₅) $\int_0^{d_\alpha(\rho,\rho)} \psi(t)dt = \theta$ it shows that

Finally let us assume that there is another fixed point$z^\star$of $A, B, S,$ &T then we have

$$\int_0^{d_\alpha(y^*,z^*)} \psi(t)dt = \int_0^{d_\alpha(Ay^*,z^*)} \psi(t)dt$$

$$\leq k_1 \int_0^{d_\alpha(Sy^*,T\rho)+d_\alpha(Ay^*,Sz^*)} \psi(t)dt$$
$$+ k_2 \int_0^{d_\alpha(Sy^*,Tz^*)+d_\alpha(Bz^*,Tz^*)} \psi(t)dt$$
$$+$$

$$k_3 \int_0^{d_\alpha(Sy^*,Tz^*)+\frac{d_\alpha(Sy^*,Bz^*)+d_\alpha(Ay^*,Tz^*)}{2}} \psi(t)dt.$$

$$\leq 2(k_1 + k_2 + k_3)\int_0^{d_\alpha(y^*,z^*)} \psi(t)dt \quad for\, \rho \in X_1$$

$2(k_1 + k_2 + k_3) < 1$ and by property (P_5) shows that $\int_0^{d_\alpha(y^*,z^*)} \psi(t)dt = \theta$ and also we have $\int_0^{d_\alpha(y^*,z^*)} \psi(t)dt = \theta$ Hence by lemma 1(1) $y^\star = z^\star$. The fixed point is unique.

5.Conclusion:

In this attempt, unique common fixed point results for four mappings in tvs- cone metric space is proved with conclusions. The outcomes generalizes and extends the ongoing aftereffects of Dubey A.K et.al., 5] with new contractive conditions.

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