DISJUNCTIVE DOMINATION POLYNOMIAL OF SOME GRAPHS

K.Balasubramanian[1] and R. Ganapathy Raman[2]

[1]Assistant Professor, Department of Mathematics
Government Arts College for Men, Nandanam, Chennai
E-mail: kbmanian123@gmail.com
[2]Assistant Professor, Department of Mathematics
Pachaiyappa's College, Chetpet, Chennai
E-mail: sirgana1@yahoo.co.in

Abstract

A subset *S* of vertices in a graph *G = (V, E)* is called a **Disjunctive dominating set** if every vertex *v* in *V* – *S* is adjacent to an element of *S* or at a distance of two from at least two vertices of *S* and minimum cardinality of *S* is **Disjunctive domination number** of G denoted by $\gamma_d(G)$.

Disjunctive Domination polynomial of a graph G is defined as $D_d(G,x) = \sum_{i=\gamma_d(G)}^{|V(G)|} d_d(G,i)x^i$ where $\gamma_d(G)$ is the disjunctive domination number of G, $d_d(G,i) = |\mathfrak{D}_d(G,i)|$ where $\mathfrak{D}_d(G,i)$ is the family of disjunctive dominating sets of G with cardinality *i*.

Our study is to find how many disjunctive domination sets can be formed for a given graph. In this paper we have found these for some standard graphs and some constructed graphs and it is represented as a polynomial

Key Words: Disjunctive domination number, Disjunctive domination polynomial
AMS Subject Classification: 05C 69
DOI Number: 10.48047/nq.2022.20.19.NQ99231 **NeuroQuantology2022;20(19): 2712-2722**

1.Introduction

Finding domination sets and domination number of various types of graphs are the active research area in the field of Graph theory using various domination parameters like domination number, disjunctive domination number, total domination number, disjunctive total domination number etc.

S. Alikhani and Y.H.Peng introduced dominating sets and domination polynomial of paths in March 2009[1]. In May 2009 published a paper" Dominating sets and domination polynomials of Cycles"[2].

Domination polynomial of a graph G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$ where γ(G) is the domination number of G, $d(G,i) = |\mathfrak{D}(G,i)|$ where $\mathfrak{D}(G,i)$ is the family of dominating sets of G with cardinality *i*.

In 2012, A.Vijayan and S.sanal kumar introduced the concept of total domination polynomial in graphs[3]. Later they have presented paper for Path, Cycle[4],[5] .

In this sequence we are defining disjunctive domination polynomial of graphs. Disjunctive Domination polynomial of a graph G is defined as $D_d(G,x) = \sum_{i=\gamma_d(G)}^{|V(G)|} d_d(G,i)x^i$ where $\gamma_d(G)$ is the disjunctive domination number of G, $d_d(G,i) = |\mathfrak{D}_d(G,i)|$ where $\mathfrak{D}_d(G,i)$ is the family of disjunctive dominating sets of G with cardinality *i*.

In this paper disjunctive domination polynomials of some graphs are found.

2.Our results

Theorem 2.1

Disjunctive domination polynomial of the Bull graph B is given by

$$\mathrm{D}_d\left(B,x\right) = (1+x)^5 - (1+5x+4x^2)$$

Proof

Let B be the Bull graph. Vertex set is $V(B) = \{u_i / \ \ 0 < i < 6\}$ and the edge set $E(B) = \{u_i\ u_{i+1} / \ \ 0 < i < 5\} \cup \{u_2\ u_4\}$.

Disjunctive domination number of B is $\gamma_d(B) = 2$

Number of disjunctive domination sets of cardinality 2 is given below

$$\{(u_1,u_4),(u_1,u_5),(u_2,u_3),(u_2,u_4),(u_2,u_5),(u_3,u_4)\}$$

Hence $d_d(W,2) = 6$

Disjunctive domination set of B with cardinality *i* = 3, 4, 5

$$d_d(W,i) = \binom{5}{i}$$

$$\mathrm{D}_d\left(B,x\right) = 6x^2 + \sum_{i=3}^{5}\binom{5}{i}x^i = (1+x)^5 - (1+5x+4x^2)$$

Hence disjunctive domination polynomial of bull graph is $\mathrm{D}_d\left(B,x\right) = (1+x)^5 - \left(1+5x+4x^2\right)$

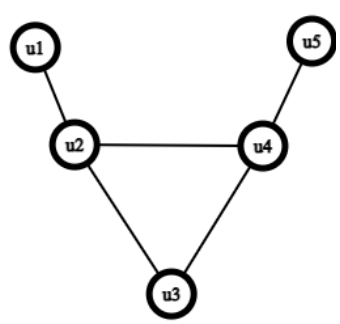

Figure 2.1 Bull graph

Theorem 2.2

Let G be a graph obtained by joining n_1 pendent vertices at v_1 and n_2 pendent vertices at v_5 of path P_5. Disjunctive domination polynomial of this graph G is

$$\mathrm{D}_d\left(G,x\right)=x^2(1+x)^{n_1+n_2+3}+(n_1+n_2)x^3(1+x)^{n_1+n_2+1}+\frac{1}{2}n_1n_2(n_1+n_2+2)x^4(1+x)^{n_1+n_2-1}$$

Proof:

Let G be the graph as defined in the theorem.

$V(G)=\{v_i:1\le i\le 5\}\cup \mathrm{A}\cup \mathrm{B}$, where $\mathrm{A}=\{a_i:1\le i\le n_1\}, B=\{b_i:1\le i\le n_2\}$

$$|V(N_k)|=n_1+\mathrm{n}_2+5 \quad \text{and} \quad |E(N_k)|=n_1+\mathrm{n}_2+4$$

Disjunctive domination number of this graph is 2 and Disjunctive domination set $\{v_1,v_3\}$ and $d_d(G,2)=1$. We have 3 cases of selecting disjunctive domination sets

Case 1. Any combination of sets containing v_1,v_5 disjunctively dominates the graph. Hence Disjunctive domination polynomial of this graph under this case is $\mathrm{D}_d\left(N_k,x\right)=x^2(1+x)^{n_1+n_2+3}$

Case 2. Any one vertex from set A, v_2 and any two vertices from set $B\cup v_4$ (**excluding** v_1 **,** v_5) disjunctively dominates this graph. Any set containing these combinations of vertices will also disjunctively dominates this graph**.** Similarly any one vertex from set B, v_4 and any two vertices from set $A\cup v_2$ (**excluding** v_1 , v_5) disjunctively dominates this graph. Any set containing these combinations of vertices will also disjunctively dominates this graph**.** Hence Disjunctive domination polynomial of this graph under this case is

$$\mathrm{D}_d\left(G,x\right)=\left(\binom{n_1}{1}\binom{n_2+1}{2}+\binom{n_2}{1}\binom{n_1+1}{2}\right)x^4(1+x)^{n_1+n_2-1}=\frac{1}{2}n_1n_2(n_1+n_2+2)x^4(1+x)^{n_1+n_2-1}$$

Case 3. v_1 , v_4 and any one vertex set B (**excluding** v_5)disjunctively dominates this graph. Similarly v_2 , v_5 and any one vertex set A (**excluding** v_1)disjunctively dominates this graph. Any set containing these combinations of these vertices will also disjunctively dominates the graph **.** Hence Disjunctive domination polynomial of this graph under this case is

$$\mathrm{D}_d\left(G,x\right)=(n_1+n_2)x^3(1+x)^{n_1+n_2+1}.$$

The above three cases exhausts all possibilities of obtaining disjunctive domination sets.Combining all these cases we get

$$\mathrm{D}_d\left(G,x\right)=x^2(1+x)^{n_1+n_2+3}+(n_1+n_2)x^3(1+x)^{n_1+n_2+1}+\frac{1}{2}n_1n_2(n_1+n_2+2)x^4(1+x)^{n_1+n_2-1}$$

Example :

Consider the following graph. Total number of vertices 12 and total number of edges 11,

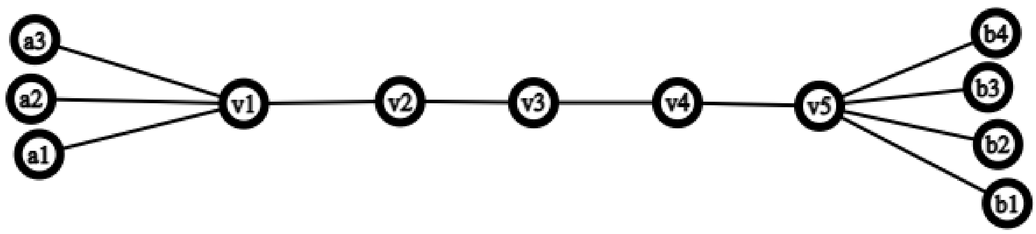

Figure 2.2

Disjunctive domination polynomial of this graph is

$$\mathrm{D}_d(G,x) = x^2(1+x)^{10} + 7x^3(1+x)^8 + 54x^4(1+x)^6$$

Theorem 2.3

Disjunctive domination polynomial of necklace graph N_k is given by

$\mathrm{D}_d(N_k,x) = x^2(1+x)^{3k+1} + \binom{k}{2}\binom{k-2}{2}x^k(1+x)^{2k+1} + k(k-1)x^{k+1}(1+x)^{2k+1}$, where N_k is a graph obtained by taking k copies of P_5 and fusing both end vertices of all P_5 paths and merging fused vertices with end points of a P_3 path.

Proof:

Let G be the graph N_k as defined in the theorem.

$V(N_k) = \{v_1,v_2,v_3\} \cup \mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$, where $\mathrm{A} = \{a_i : 1 \le i \le k\}, B = \{b_i : 1 \le i \le k\}, C = \{c_i : 1 \le i \le k\}$

$$|V(N_k)| = 3k+3 \quad \text{and} \quad |E(N_k)| = 4k+2$$

Disjunctive domination number of this graph is 2

Disjunctive domination set $\{v_1,v_3\}$ and $d_d(N_k,2) = 1$

We have 3 cases of selecting disjunctive domination sets

Case 1 Number of ways of selecting disjunctive domination sets with *i* vertices (*i > 2*)from the remaining 3*k*+1 vertices is given by $\binom{3k+1}{i-2}$. Each of these sets contains v_1, v_3.

$$\mathrm{D}_d(N_k,x) = x^2(1+x)^{3k+1}$$

Case 2 It is possible to construct disjunctive domination sets which **does not contain** v_1, v_3 by taking any of the two vertices from set A and choose suitably two vertices from set C and remaining k -4 vertices from set A,B, C suitably so that the set contains k vertices.

Number of ways of constructing disjunctive sets having k vertices is $\binom{k}{2}\binom{k-2}{2}3^{k-4}$.

Number of ways of selecting disjunctive domination sets with i vertices ($i > k$) *is* select k vertices as above and remaining i – k from the remaining 2k+1 vertices is given by $\binom{2k+1}{i-k}$.

$$\mathrm{D}_d\left(N_k,x\right)=3^{k-4}\binom{k}{2}\binom{k-2}{2}x^k(1+x)^{2k+1}$$

Case 3 Select v_1 and reject v_3 and any two vertices from set C and suitably select k -2 vertices from set A,B and C to get a disjunctive domination set containing k + 1 vertices, number of such ways is $3^{k-2}\binom{k}{2}$. Select v_3 and reject v_1 and any two vertices from set A and suitably select k -2 vertices from set A,B and C to get a disjunctive domination set containing k + 1 vertices, number of such ways is $3^{k-2}\binom{k}{2}$.

Number of ways of selecting disjunctive domination sets with i vertices ($i > k+1$) *is* select k+1 vertices as above and remaining i – k from the remaining 2k+1 vertices is given by $\binom{2k+1}{i-k-1}$.

Hence $\mathrm{D}_d\left(N_k,x\right)=3^{k-2}k(k-1)x^{k+1}(1+x)^{2k+1}$.

These cases exhaust all possible combinations of obtaining disjunctive domination sets. Hence Combining all these we get

$$\mathrm{D}_d\left(N_k,x\right)=x^2(1+x)^{3k+1}+3^{k-4}\binom{k}{2}\binom{k-2}{2}x^k(1+x)^{2k+1}+3^{k-2}k(k-1)x^{k+1}(1+x)^{2k+1}$$

Example :

Consider the following N_4 graph. Total number of vertices 15 and total number of edges 18, Disjunctive domination polynomial of this graph is

$$\mathrm{D}_d\left(N_4,x\right)=x^2(1+x)^{13}+6x^4(1+x)^9+108x^5(1+x)^9$$

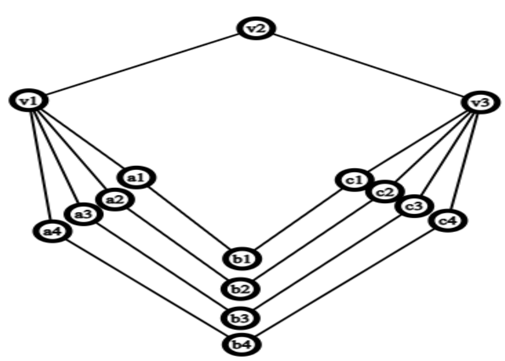

Figure 2.3 N_4 graph

Theorem 2.4

Let G be any graph of order n, then disjunctive domination polynomial of $G\square\ K_1$ is

$\mathrm{D}_d\left(G\square\ K_1, x\right) = x^n(2+x)^n$

Proof

Let G be any graph of order n, then disjunctive domination number of $G\square\ K_1$ is *n*.

Number of ways of selecting a Disjunctive domination set having *n* vertices is given as follows

Select any k vertices from n vertices namely U = $\{u_1, u_2, u_3, \ldots, u_n\}$ and remaining n-k vertices from V = $\{v_1, v_2, v_3, \ldots, v_n\}$ such that whatever u_i selected from U corresponding v_i from V should be rejected. Hence number of ways is $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$

Hence $d_d\left(G\square\ K_1\ , n\right) = 2^n$.

Number of ways of selecting *n+k* vertices from 2n vertices where $1 \le k \le n$ is given below.

If k vertices are selected from U we have to select n vertices suitably from V, if k+1 vertices are selected from U we have to select n-1 vertices suitably from V. Like this we have to find all possible cases to get a disjunctive domination sets containing *n+k vertices* and this will be $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}$. Hence Coefficient of x^{n+k} is $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k} = 2^{n-k}\binom{n}{k}$ where $0 \le k \le n$

.Hence $\mathrm{D}_d\left(G\square\ K_1, x\right) = x^n(2+x)^n$

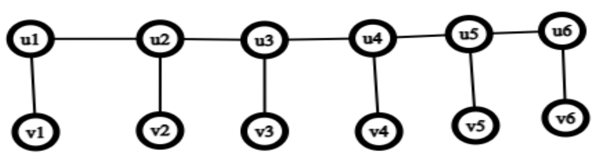

Figure 2.4 $P_6\square\ K_1$

Example : Consider the graph G as a path having 5 vertices. Hence the disjunctive domination polynomial of $\mathrm{D}_d\left(P_5\square\ K_1, x\right) = 32x^5+80x^6+80x^7+40x^8+10x^9+x^{10}$

Theorem 2.5

Disjunctive domination polynomial K_n is $D_d(K_n, x) = (1+x)^n - 1$

Proof

Cardinality of vertex set and edge set of K_n is $|V(G)| = n$ and $|E(G)| = \frac{n(n-1)}{2}$

Being a complete graph, every vertex of G is connected to every other vertex of G and hence domination number of G = disjunctive domination of G = 1. Hence $d_d(G,1) = n$.

Number of ways of selecting *i* vertices from n vertices are $\binom{n}{i}, 1 \le i \le n$.

Hence disjunctive domination polynomial of K_n is $D_d(K_n, x) = (1+x)^n - 1$

Example : Disjunctive domination polynomial of K_6 is $D_d(K_6, x) = 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Theorem 2.6

Disjunctive domination polynomial of $K_{p,q}$ is $\mathrm{D}_d\left(K_{p+q}, x\right) = (1+x)^{p+q} - (1+(p+q)x)$ where $p \ge 2, q \ge 2$.

Proof

Let G be a complete bipartite graph $K_{p,q}$

Domination set can be obtained choosing one element from u_i and one element from v_j.

Hence domination number of $K_{p,q}$ is two. Since the distance between any two vertices of *u*' s and any two vertices of *v*'s is two, we can select any two vertices of this graph to get a disjunctive domination set..

Hence number of ways of selecting *i* vertices from $p+q$ vertices is

$$d_d(K_{p,q}, i) = \binom{p+q}{i}, 2 \le i \le p+q$$

Hence disjunctive domination polynomial of this $K_{p,q}$ is $\mathrm{D}_d\left(K_{p,q}, x\right) = (1+x)^{p+q} - (1+(p+q)x)$

Example :

$$\mathrm{D}_d\left(K_{3,4}, x\right) = 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

Theorem 2.7

Disjunctive domination polynomial $K_{1,n}$ is $D_d(K_{1,n}, x) = (1+x)^{n+1} - 1 - nx$

Proof

We have two cases of finding disjunctive domination sets

Case 1: Apex vertex dominates all other vertices of $K_{1,n}$. Hence any set which contains this vertex is a domination set. Hence Disjunctive domination polynomial for this case is $x(1+x)^n$.

Case 2 Any two vertices of $K_{1,n}$ excluding the apex vertex disjunctively dominates all other vertices of $K_{1,n}$ and we can select such sets in $\binom{n}{2}$ ways. Hence any set which contains at least 2 vertices excluding apex vertex also disjunctively dominates $K_{1,n}$.

Hence Disjunctive domination polynomial for this case is $(1+x)^n-1-nx$.

These cases exhaust all possible combinations of obtaining disjunctive domination sets.

Hence combining all these we get $D_d(K_{1,n},x)=(1+x)^{n+1}-1-nx$

Theorem 2.8

Disjunctive domination polynomial W_n is $D_d(W_n,x)=(1+x)^n-1-(n-1)x$

Proof

Proceeding the same way as given in theorem 2.6 we get the result

$D_d(W_n,x)=(1+x)^n-1-(n-1)x$

Theorem 2.9

Disjunctive domination polynomial of Cycles C_6 with parallel P_2 chords is given by

$$\mathcal{D}_d(C_6(P_2),x)=(1+x)^6-1-6x-4x^2$$

Proof :

Let $C_6(P_2)$ represents cycles with 6 vertices and two parallel chords.

Vertex set is $V(B)=\{u_i/\ \ 0<i<7\}$ and the edge set

$E(W)=\{u_i\ u_{i+1}/\ \ 0<i<7\}\cup\{u_6\ u_1\}\cup\{u_2\ u_6\}\cup\{u_3\ u_5\}$.

Disjunctive domination number of B is $\gamma_d(C_6(P_2))=2$

Number of disjunctive domination sets of cardinality 2 is given below

$\{(u_1,u_3),(u_1,u_4),(u_1,u_5),(u_2,u_3),(u_2,u_4),(u_2,u_5),(u_3,u_6)(u_4,u_6),(u_5,u_6),(u_2,u_6),(u_3,u_5)\}$

Hence $d_d(C_6(P_2),i)=11$

Disjunctive domination set of B with cardinality *i* = 3, 4, 5 , 6 is $d_d(C_6(P_2),i)=\binom{6}{i}$

Hence $\mathrm{D}_d\left(C_6(P_2),x\right)=11x^2+\sum_{i=0}^{6}\binom{6}{i}x^i-1-6x-15x^2=(1+x)^6-1-6x-4x^2$

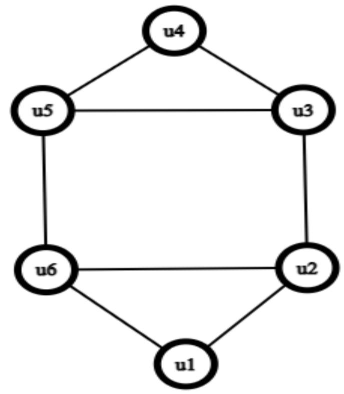

Figure 2.5 Cycles C_6 with parallel P_2 chords

Theorem 2.10

Disjunctive domination polynomial of the Wagner graph W is given by

$$\mathcal{D}_d(W,x) = (1+x)^8 - (1+8x)$$

Proof

Wagner graph is a 3 regular graph with 8 vertices and 12 edges. Let W be the Wagner graph. Vertex set is $V(W) = \{v_i / \ \ 0 < i < 9\}$ and the edge set

$E(W) = \{u_i\ u_{i+1} / \ \ 0 < i < 8\} \cup \{u_i\ u_{i+4} / \ \ 0 < i < 5\} \cup \{u_1\ u_8\}$.

Disjunctive domination number of W is $\gamma_d(W) = 2$

Every pair of vertices of W with $u_i \neq u_j\ with\ i \neq j$ disjunctively dominates this graph and number of such ways is $\binom{8}{2}$.

$N(u_i)$ represents neighborhood of u_i which contains vertices adjacent to u_i,

Consider the set $\{u_1, u_2\}$. $N(u_1) = \{u_2, u_5, u_8\}$ and $N(u_2) = \{u_1, u_3, u_6\}$. Remaining 2 vertices $u_4 \& u_7$ are at a distance of two units from $u_1 \& u_2$. Hence $\{u_1, u_2\}$ is a disjunctive domination set of W.

Like these we have totally $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ ways of selecting 2 vertices which disjunctively dominates the Wagner Graph W. Hence $d_d(W,i) = |\mathcal{D}_d(W,i)| = \binom{8}{i}, 2 \leq i \leq 8$.

Hence $\mathcal{D}_d(W,x) = (1+x)^8 - (1+8x)$

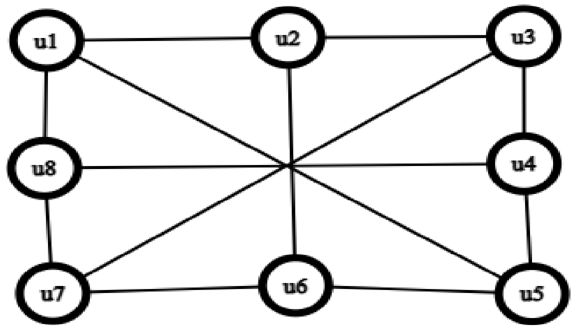

Figure 2.6 Wagner graph

3.Disjunctive domination polynomial of path and cycle for some particular cases.

1. $D_d(C_4, x) = 6x^2 + 4x^3 + x^4$
2. $D_d(C_5, x) = 10x^2 + 10x^3 + 5x^4 + x^5$
3. $D_d(C_6, x) = 9x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
4. $D_d(C_7, x) = 7x^2 + 28x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$
5. $D_d(C_8, x) = 4x^2 + 32x^3 + 62x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$
6. $D_d(P_4, x) = 4x^2 + 4x^3 + x^4$
7. $D_d(P_5, x) = 4x^2 + 8x^3 + 5x^4 + x^5$
8. $D_d(P_6, x) = 3x^2 + 12x^3 + 13x^4 + 6x^5 + x^6$
9. $D_d(P_7, x) = 1x^2 + 14x^3 + 25x^4 + 19x^5 + 7x^6 + x^7$
10. $D_d(P_8, x) = 12x^3 + 38x^4 + 44x^5 + 26x^6 + 8x^7 + x^8$

Observations on coefficients of disjunctive domination polynomial of P_n ,C_n

1 $d_d(P_n, n) = 1$

2 $d_d(P_n, n-1) = n$

3 $d_d(P_n, n-2) = \binom{n}{2} - 2$

4 $d_d(P_n, n-3) = \binom{n}{3} - (2n-4)$

5 $d_d(P_{4k}, k+1) = 4$

6 $d_d(P_{4k+1}, k+1) = 1$

7 $d_d(C_n, n) = 1$

8 $d_d(C_n, n-1) = n$

9 $d_d(C_n, n-2) = \binom{n}{2}$

10 $d_d(C_n, n-3) = \binom{n}{3}$

11 $d_d(C_{4k}, k) = 4$

12 $d_d(C_n, n-4) = \binom{n}{4} - (n-3)$

Conclusion

In this paper we have found disjunctive domination polynomial of some graphs. Some observations were made on the coefficients of disjunctive domination polynomial of paths and cycles. Our further work involves in establishing recurrence relation fordisjunctive domination polynomial of path P_n and cycle C_{n}..

References

1. Alikhani. S and Peng. Y. H , March 2009, Domination sets and Domination polynomial of paths, International Journal of Mathematical Sciences
2. Alikhani. S and Peng. Y. H , May 2009, Dominating sets and Domination polynomial of cycles, Global Journal of Pure and Applied Mathematics, 4 (2008), 202-210.
3. Vijayan. A, Sanal Kumar.S, on Total domination sets and polynomial of paths, International Journal of Mathematics Research, Volume 4(2012), 339-348
4. Vijayan. A, Sanal Kumar.S, On Total Domination sets and Polynomial of cycles, International Journal of Mathematical Archive, 3 (2012), 1379-1385.
5. Vijayan. A, Sanal Kumar.S, on Total domination polynomial of graphs, Global Journal of Theoretical and Applied Mathematical Sciences, 2012 Volume 2(2012), 91-97