WEAK RUPTURE DEGREE OF SOME CLASSES OF GRAPHS

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Abstract:

The weak rupture degree of a connected graph G is defined by $R_w(G) = max\{\omega(G-S) - |S| - m_e(G-S): S \subseteq V(G),\ \omega(G-S) > 1\}$ where $\omega(G-S)$ is the number of the components of $G-S$ and $m_e(G-S)$ is the number of edges of the largest component of $G-S$. In this paper we determine the exact values of the weak rupture degree of $K_{r(s)} \times K_{m(n)}$, $K_{m(n)} \times K_{a,b}$, $C_m \times K_{s,s}$ and $C_m \times K_{r(s)}$.

Keywords: Weak rupture degree, Vulnerability, Cartesian product, Tensor product.

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1.Introduction

A communication network consists of some centers and links which connect these centers. In a communication network the vulnerability measures the resistance of the network to disruptionof operation after the failure of certain stations or communication links. A communication network is modelled by a graph to measure the vulnerability as centers corresponding to the vertices of a graph and communication links corresponding to the edges of a graph. To measure vulnerability of a graph G, there are some parameters such as connectivity, toughness,scattering number, integrity, rupture degree [9] etc.

The known parameters do not deal with the number of the removing vertices, the number of the components and the number of the edges in the largest component of the remaining graph in a disrupted network simultaneously, In order to fill this void in the literature, Aslan introduced weak rupture degree of graphs in [4].

2.Preliminaries

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Then the following parameters are defined.

Definition 1.1: The *rupture degree* for a connected graph G is defined to be
$r(G)=max\{\omega(G-S) - |S| - m(G-S) : X \subset V(G), \omega(G-S) > 1\}$,
where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. The rupture degree for a complete graph $r(K_n) = 1-n$.

Definition 1.2: The *weak rupture degree* of a connected graph G is defined to be

$R_w(G) = \max\{\omega(G-S) - |S| - m_e(G-S) : S \subseteq V(G),\ \omega(G-S) > 1\}$, Where $\omega(G-S)$ is the number of the components of $G-S$ and $m_e(G-S)$ is the number of edges of the largest component of $G-S$. The weak rupture degree for a complete graph $R_w(K_n) = 2-n$.

Definition 1.3: The *tensor product* of the graphs G and H, denoted by $G \times H$, has the vertex set $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u,x)(v,y): uv \in E(G)\ and\ xy \in E(H)\}$.

Definition 1.4: A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y if |X|=m and |Y|=n such a graph is denoted by $K_{m,n}$.

Definition 1.5: A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set S is a maximum if G has no independent set S' with $S' > |S|$. The independent set S is a maximum number of G, $\alpha(G)$ is the number of vertices in a maximum independent set of G.

Definition 1.6: A subset S of V is called an *covering number* of G if every edge of G has at least one end in S. A covering S is a minimum

covering in G has no covering S' with $S' > |S|$. The covering number, β(G) is the number of vertices in a minimum covering of G.

The order n of a graph G is defined as $n = \alpha(G) + \beta(G)$.

Theorem 2.1. [4] Let G be a connected graph. Then

$$R_w\ (G) \leq r(G) + 1$$

Theorem 2.2. [4] If G is a graph of order n, independence number $\alpha(G)$ and covering number $\beta(G)$. Then

$$R_w\ (G) \geq \alpha(G) - \beta(G)$$

Theorem 2.3. [4] The weak rupture degree of the cycle C_n is

$$R_w(C_n) = \begin{cases} 0, & if\ n\ is\ even, \\ -1, & if\ n\ is\ odd\ . \end{cases}$$

Theorem 2.4. [14] If $m, n \geq 3$ then the weak rupture degree of $C_m \times C_n$ is

$$R_w(C_m \times C_n) = \begin{cases} max\{-m, -n\}, & if\ m\ and\ n\ are\ odd, \\ 0, & if\ m\ and\ n\ are\ diffrent\ parity. \end{cases}$$

For notation and definitions which are not given here can be found in [2]

3. Main Results

In this section, we determine the weak rupture degree of the graph $K_{r(s)} \times K_{m(n)}$, $K_{m(n)} \times K_{a,b}$, $C_m \times K_{r(s)}$ for $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$ where $K_{r(s)}$ denotes the complete r-partite graph in which each partite set has s vertices.

Theorem 3.1. [13] The rupture degree of $K_{r(s)} \times K_{m(n)}$ is

$r\left(K_{r(s)} \times K_{m(n)}\right) = max\{snm(2 - r) - 1, snr(2 - m) - 1\}$, where $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$.

Theorem 3.2. For $r \geq 3, m \geq 3, s \geq 1$ and $n \geq 1$, the weak rupture degree of $K_{r(s)} \times K_{m(n)}$,

$$R_w\left(K_{r(s)} \times K_{m(n)}\right) = max\{snm(2 - r), snr(2 - m)\}.$$

Proof: Let G=$K_{r(s)} \times K_{m(n)}$.

Let $\{U_1, U_2, \dots, U_r\}$ and $\{V_1, V_2, \dots, V_m\}$ be the partite sets of $K_{r(s)}$ and $K_{m(n)}$ respectively.

Let $B_{ij} = U_i \times V_j$, $1 \leq i \leq r$ and $1 \leq j \leq m$.

Let $r \geq m$. By Theorem 3.1, we have $r(G) = snr(2 - m) - 1$; further B_{ij} has sn vertices.

By Theorem 2.1,

$$\begin{aligned} R_w(G) &\leq r(G) + 1 \\ &\leq snr(2 - m) - 1 + 1 \\ &\leq snr(2 - m). \end{aligned} \quad (1)$$

On the other hand, $S = \bigcup_{1 \leq i \leq r} B_{ij}$, for some j, $1 \leq j \leq m$, is the maximum independent set of G with $|S| = \alpha(G) = snr$ and $\beta(G) = snr(m - 1)$. Therefore by Theorem 2.2,

$$\begin{aligned} R_w(G) &\geq \alpha(G) - \beta(G) \\ &\geq snr(2 - m) - 1 + 1 \\ &\geq snr(2 - m). \end{aligned} \quad (2)$$

From (1) and (2)

$$R_w(G) = snr(2 - m). \quad (3)$$

Let $m \geq r$. Then proceeding as above we have,

$$R_w(G) = snm(2 - r). \quad (4)$$

By (3) and (4)

$$R_w\left(K_{r(s)} \times K_{m(n)}\right) = max\{snm(2 - r), snr(2 - m)\}.$$

Theorem 3.3. [13] If m, n, a, b are integers with $m \geq 3, n \geq 1$ and $a \geq b$, then the rupture degree of $K_{r(s)} \times K_{a,b}$,

$$r\left(K_{m(n)} \times K_{a,b}\right) = mn(a-b) - 1$$

Theorem 3.4. If m, n, a, b are integers with $m \geq 3, n \geq 1$ and $a \geq b$, then the weak rupture degree of $K_{r(s)} \times K_{a,b}$,

$$R_w\left(K_{m(n)} \times K_{a,b}\right) = mn(a-b)$$

Proof: : Let G=$K_{m(n)} \times K_{a,b}$.

Let $\{U_1, U_2, \dots, U_m\}$ and $\{V_1, V_2\}$ be the partite sets of $K_{m(n)}$ and $K_{a,b}$ respectively, where we assume $|V_1| = a$ and $|V_2| = b$.

Let $B_{ij} = U_i \times V_j$.

Let $X = \bigcup_{i=1}^{m} B_{i1}$ and $Y = \bigcup_{i=1}^{m} B_{i2}$; clearly G is a bipartite graph with bipartition (X, Y).

By Theorem 3.3, $r(G) = mn(a-b) - 1$.Hence by Theorem 2.1 ,

$$\begin{aligned} R_w(G) &\leq r(G) + 1 \\ &\leq mn(a-b) - 1 + 1 \\ &\leq mn(a-b). \end{aligned} \quad (5)$$

On the other hand, $S = \bigcup_{1 \leq i \leq m} B_{i1} = X$ is the maximum independent set of G with $|S| = \alpha(G) = mna$ and $\beta(G) =$ mnb. Therefore by Theorem 2.2,

$$\begin{aligned} R_w(G) &\geq \alpha(G) - \beta(G) \\ &\geq mna - mnb \\ &\geq mn(a-b). \end{aligned} \quad (6)$$

From (5) and (6)

$$R_w(G) = mn(a-b).$$

As the tensor product of $C_m \times K_{s,s}$ is disconnected when m is even, we consider the case m is odd in the next theorem.

Theorem 3.5. [13] Let C_m be an odd cycle. Then the rupture degree of $C_m \times K_{s,s}$ is

$$r\left(C_m \times K_{s,s}\right) = \begin{cases} -1, & if\ s = 1, \\ -1, & if\ m = 3\ and\ s = 2, \\ -5, & if\ m \geq 5\ and\ s = 2, \\ -1, & if\ s \geq 3. \end{cases}$$

Theorem 3.6. Let C_m be an odd cycle. Then the weak rupture degree of $C_m \times K_{s,s}$ is

$$R_w\left(C_m \times K_{s,s}\right) = \begin{cases} 0, & if\ s = 1, \\ 0, & if\ m = 3\ and\ s = 2, \\ -4, & if\ m \geq 5\ and\ s = 2, \\ 0, & if\ s \geq 3. \end{cases}$$

Proof: Let $G = C_m \times K_{s,s}$, where $m \geq 3$ and $s \geq 1$.

Case 1: $s = 1$.

As m is odd and $s = 1$, $G = C_m \times K_{1,1} \cong C_{2m}$. By Theorem 2.3,

$$R_w(G) = R_w(C_{2m}) = 0.$$

Case 2: $s = 2$.

In this case $G = C_m \times K_{2,2} \cong C_m \times K_4$. By Theorem 2.4,

$$R_w(C_m \times C_4) = \begin{cases} 0, & if\ m = 3, \\ -4 & if\ m \geq 5. \end{cases}$$

Case 3: $s \geq 3$.

As m is odd and $s \geq 3$, $G = C_m \times K_{s,s}$ is bipartite graph with bipartition say (X, Y). It is clear that $\alpha(G) = sm$ and $\beta(G) =$ sm. Therefore

$$\begin{aligned} R_w(G) &\geq \alpha(G) - \beta(G) \\ &\geq sm - sm = 0 \end{aligned} \quad (7)$$

By Theorem 3.5, the rupture degree of G is $r(G) = -1$. Hence by Theorem 2.1,

$$\begin{aligned} R_w(G) &\leq -1 + 1 \\ &= 0 \end{aligned} \quad (8)$$

From (7) and (8)

$$R_w(G) = 0.$$

Hence the proof.

Theorem 3.7. [13] For $m, r \geq 3$ and $s \geq 1$, the rupture degree is

$$r\big(C_m \times K_{r(s)}\big) = rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil - 1.$$

Theorem 3.8. For $m, r \geq 3$ and $s \geq 1$, the weak rupture degree is

$$R_w\big(C_m \times K_{r(s)}\big) = rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil.$$

Proof: Let G=$C_m \times K_{r(s)}$, where $r \geq 3, m \geq 3$ and $s \geq 1$ and $K_{r(s)}$ denotes the complete r-partite graph in which each partite set has s vertices.

Let $V(C_m) = \{u_0, u_1, \dots, U_{m-1}\}$.

Let $\{V_0, V_1, \dots, V_{r-1}\}$ be the partite sets of $K_{r(s)}$ and let $W_{ij} = u_i \times V_j$.

Then $S_i = \bigcup_{j=0}^{r-1} W_{ij}$, $0 \leq i \leq m-1$, of G is an independent set of $C_m \times K_{r(s)}$ and $V\big(C_m \times K_{r(s)}\big) = \bigcup_{i=0}^{r-1} S_i = \bigcup_{\substack{0\leq i\leq m-1\\ 0\leq j\leq r-1}} W_{ij}$.

By Theorem 3.7 , the rupture degree of G is $r(G) = rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil - 1$. Hence by Theorem 2.1,

$$R_w(G) \leq rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil - 1 + 1 \leq rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil \quad (9)$$

Now $S = \bigcup_{i=0}^{\left\lceil\frac{m}{2}\right\rceil - 1} S_{2i}$ is a maximum independent set in G with $\alpha(G) = rs\left\lfloor\frac{m}{2}\right\rfloor$ and $\beta(G) = rs\left\lceil\frac{m}{2}\right\rceil$. Therefore by Theorem 2.2,

$$R_w(G) \geq \alpha(G) - \beta(G) \geq rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil \quad (10)$$

Hence from (9) and (10)

$$R_w(G) = rs\left\lfloor\frac{m}{2}\right\rfloor - rs\left\lceil\frac{m}{2}\right\rceil.$$

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