Commutativity On σ Near-Rings and Its Structures

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ABSTRACT

If, in a ring $(N, +, \cdot)$ we ignore the commutativity of '+' and one of the distributive laws, (N, +, ·) becomes a Near-Ring. If we do not stipulate the left distributive law, (N, +, ·) is a right Near-Ring. This paper aims to introduce the concept of σ Near-Ring. A Near-Ring N is called σnear ring if N is a right near ring and $x^2y = xyx$ for all $x, y \in N$ .The element wise characterization for σ Near - Ring will be investigated and shall establish many theorems and properties.

Mathematics Subject Classification: 16Y30
Keywords : σNear -Ring, Commutativity, near-field.

DOI Number: 10.48047/NQ.2022.20.9.NQ44941 **Neuroquantology 2025; 23(1): 8101 -8106**

1 Introduction

A right near-ring is a non-empty set N together with two binary operations "+"and " · " such that (1) (N, +) is a group. (2) (N, ·) is a semi-group and (3) $(n_1 + n_2)n_3 = n_1 n_3 + n_2 n_3$ for all $n_1, n_2, n_3 \in N$.

Throughout this paper N stands for a right near-ring. $(N, +, .)$ With at least two elements 1 and '0' denotes the identity element of the group (*N*, +) obviously, 0n = 0 for all n in N. N is said to be zero-symmetric if n0 = 0 for all n in N. As in [2], a subgroup of $(M, +)$ of $(N, +)$ is called an *N*-subgroup of *N* if $NM \subset M$ and an invariant *N* subgroup of *N* if, in addition, $MN \subset M$. In [6], *N* is defined to be Pseudo commutative if $xyz = zyx$ for all

x, y, z in *N*. The concept of a mate function in N has been introduced in [4] with a view to handling the regularity structure with considerable case. A map '*f*' from *N* into N is called (i) a mate function for N if $x = xf(x)x$, (ii) a P_3 mate function, if, in addition, $xf(x) = f(x)x$ for all x in N. By identity 1 of N, we mean only the multiplicative identity of N.

eISSN: 1303-5150

Basic concepts and terms used but left undefined in this paper can be found in [2].

2 Notations

In this section, we furnish the notations which are used frequently throughout this paper.

(i) E denotes the set of all idempotent of *N.* (e in N is called an idempotent if $e^2 = e$)

(ii) L denotes the set of all nilpotent of *N*. (a in *N* is nilpotent if $a^k = 0$ for some positive integer k.) and N is said to be reduced if $L = \{0\}$

(iii) $N_0 = \{n \in N / n0 = 0\}$ - zero-symmetric part of N and N is called zero symmetric if $N = N_0$.

(iv) $N_d = \{n \in N / n(x + y) = nx + ny$ for all x, y in $N\}$ - set of all distributive element of N and N is called distributive if $N = N_d$.

(v) $C(N) = \{n \in N / nx = xn$ for all x in $N\}$ - centre of N.

(vi) If A is any non – empty subset of N, then

i) $A^* = A - \{0\}$

ii) $C(A) = \{n \in N / na = an$ for all $a \in A\}$

iii) When $A = N, C(N) = \{na = an$ for all $a \in N\} -$ called the centre of N.

(vii) When $E \subseteq C(N)$, we say that the idempotents are central.

3 Preliminary Results

We freely make use of the following results and designate them as **R(1)**, **R(2)**....etc.

R(1) N is sub directly irreducible if and only if the intersection of any family of non-zero ideals of N is again non-zero (Theorem 1.60, p.25 of [2])

R(2) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Problem 14, p.9 of [3]).

R(3) If f is a mate function for N, then for every x in $N, xf(x), f(x)x \in E$ and $Nx = Nf(x)x, xN = xf(x)N$ (Lemma 3.2 of [5]).

R(4) If $L = \{0\}$ and $N = N_0$, then (i) $xy = 0 \Rightarrow yx = 0$ for all x, y in N. (ii) N has Insertion of factors property- IFP for short- i.e for x, y in N, $xy = 0 \Rightarrow xny = 0$. for all n in N . If N satisfies (i) and (ii) then N is said to have ($*$, IF P) (Lemma 2.3 of [5])

R(5) Any Pseudo commutative near-ring with a right identity is weak commutative ((i.e). $xyz = xyz$ for all x, y, z in N [2]) (Proposition 2.9 of [6])

R(6) A zero-symmetric near-ring N is a near- field if $N_d \neq \{0\}$ and for all $n \in N - \{0\}, N_n = N$ (Theorem 8.3, p.249 of [2]).

R(7) Let N be a right near-ring. If for every x, y in N, $xN\,y = N\,xy$ then we say N is a β_1 near-ring. (Definition 3.1.1 of [4])

Definition and Examples of σ Near – Rings

In this section we define σ Near – Rings and give certain examples of this new concept.

Definition 4.1:

Let N be a right near ring. Then N is said to be a σ Near -Ring if $x^2y = xyx$ for all $x, y \in N$.

Example (a):

The Near – Ring $(N, +, .)$ defined on Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$ where '.' is defined as per scheme 9, $P\,408$ of Pilz [3].

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	b
c	0	a	0	c

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is a σ Near – Ring

Example (b):

b) Let $(N, +)$ be the Klein's four group as in (a) above. If multiplication is defined as per scheme 8, P.408 of pilz[3]

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Then N is not a σ Near –Ring since $b^2a \neq bab\ ie)\ a \neq 0$.

5.Properties of σ Near rings

In this section we prove certain important properties of σ Near -Ring and give a complete characterization of such Near – Rings.

Proposition 5.1:

Let $N = N_d$ be a σ Near -Ring with identity. Then N is commutative.

Proof:

Let N be a σ Near – Ring

Then for all $x, y \in N, x^2y = xyx$ ……(1)

Replace the element x by $x + e$ in (1)

$(x + e)^2.y = (x + e).y(x + e).$

$(x + e)(x + e).y = (x + e(.y(x + e)$

$[x.x + x + x + e].y = (x + e).y(x + e)$

$x.(xy) + xy + xy + y = (xy + y)(x + e)$

$x(xy) + xy + xy + y = (xy)x + yx + xy + y$

$x(xy) + xy = (xy)x + yx$ (By right Cancellation law)

$(x.x)y + xy = x(yx) + yx$ (Associative law)

$= x.xy + yx$

$x^2y + xy = x^2y + yx.$

$xy = yx \quad \forall\, x, y \in N.$

$\therefore N$ is Commutative.

Consequently N becomes a σ Near – Ring

Lemma 5.2:

A σ Near –Ring is always zero symmetric.

Proof:

Suppose N is a σ Near –Ring. Then $x^2y = xyx$ for all $x, y \in N$.

When $x = 0$. then $0.y = 0.y.0 = 0$

It follows that N is Zero symmetric.

Theorem 5.3:

Let N be a σ Near –Ring. If N is Quasi Weak Commutative then $xy^2 = y^2x$ for all $x, y \in N$.

Proof:

Let N be a Quasi Weak Commutative Near – Ring.

Then $xyz = yxz$ …….(1)

Let N be a σ Near – Ring

Then $x^2y = xyx$ ……(2)

Now $xy^2 = xyy = yxy$ [By Equ (1)] $= y^2x$ [By Equ (2)]

Therefore $xy^2 = y^2x.$

Proposition 5.4:

Every weak Commutative Near – Ring is a σ Near Ring.

Proof:

Let N be a weak commutative near-ring, then $xyz = xzy$ for all $x, y, z \in N$ …… (1)

Let $x = ab.$ Then $xyx = (ab)y(ab) = a(bya)b = a(bay)b$ [By equation (1)] $= ab(ayb) = ab(aby) = ab(ab)y$ [By equation (1)]$= (ab)^2y = x^2y.$

Consequently, $xyx = x^2y$ for every x, y in N and the desired result follows

Proposition 5.5:

Homomorphic image of a σ Near -Ring is also a σ Near -Ring.

eISSN: 1303-5150

Proof:

The Proof is straight forward.

Proposition 5.6:

If I is an ideal of the σ Near -Ring N then N / I is also an σ Near - Ring

Proof:

The function $\emptyset: N \to /I$ defined by $\emptyset(x) = I + x$ is an epimorphism. The rest of the proof is taken care of by above Proposition 5.6.

Proposition 5.7:

Every σ Near - Ring N is isomorphic to a sub direct product of sub directly irreducible Near - Rings.

Proof:

By theorem 1.62, P 26 of pilz[3], N is isomorphic to a sub direct product of sub directly irreducible – Near -Rings.N_i'S and each N_i is a homorphic image of N under the multiplication π_i . Now the desired result follows from above proposition 5.7.

Proposition 5.8:

Let N be a σ Near -Ring with a mate function f.

Then we have

i) $L = \{0\}$
ii) N has (*IFP)
iii) $E \subseteq C(N)$

Proof:

i) Let N be a σ Near –Ring, then $x^2y = xyx \quad \forall\, x, y \in N$. ……(1)
Since f is a mate function for N then $x = x(f)x \in xNx$ for all $x \in N$ .Therefore $x = xnx$ for some n. …… (2).
Now, for $n, x \in N$, $x^2n = xnx$ [By equation (1)] $= x$[By equation (2)]
Suppose $x^2 = 0$. Clearly then $x = 0$. Then $R(2)$ guarantees that $L = \{0\}$

ii) By i) $L = \{0\}$, R(8) guarantees that N has (*IFP)

iii) Let $e \in E$. Since N is a σ Near -Ring $e^2n = ene \implies ne^2 = ene$ for all n in N (By theorem 5.3) $\implies ne = ene$ for all n in N … … … (3). Also we have $(ene - en)e = 0$
This implies $e(ene - en) = 0$ and $en(ene - en) = 0[by(ii)]$.
Also $ene(ene - en) = en.0 = 0$ [Since N is Zero symmetric].
Now $ene(ene - en) - en(ene - en) = 0$. Consequently, $(ene - en)^2 = 0$ and (i) guarantees $ene - en = 0$. Therefore $ene = en$ for all n in N …….(4)
From Equations (3) and (4), we get $en = ne$ for all n in N thus $E \subseteq C(N)$

Proposition 5.9:

Let N be a pseudo commutative near ring with right identity. Then if N is a σ Near -Ring then for any a, b in N, $ab = 0$ implies $ba = 0$

Proof:

Let N be a Pseudo commutative Near Ring. Then $xyz = zyx$ for all $x, y, z \in N$ …….(1)
Now R(5) guarantees that N is a weak Commutative, $xyz = xzy$ for all $x, y, z \in N$ ……..(2)
Since N is a σ near ring. Then $x^2a = xax, y^2b = yby$ for all $a, b, x, y, \in N$
Now ,$(xax)(yby) = (x^2a)(y^2b)$
$xa(xyb)y = (xxa)(yyb)$
$xa(xyb)y = x(xay)yb$
$xa(byx)y = x(yax)yb$ [By Equation (1)]
$(xab)yxy = (xya)xyb$
$baxyxy = (ayx)(xyb)$ [By Equation (1)]
$ba\,xy\,xy = a(yxb)yx$ [By Equation (1)]
$ba(xy)^2 = abx(yyx)$ [By Equation (2)]
$ba\,(xy)^2 = ab(xyxy)$ [By Equation (2)]
$ba(xy)^2 = ab(xy)^2$
$ba = ab$
Since $ab = 0$ it follows that $ba = 0$.

Proposition 5.10:

Any weak commutative Near ring with Left identify is a σ Near – Ring.

Proof:

Let N be a weak Commutative Near ring, $xyz = xzy$ for all $x, y, z \in N$

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Let $e \in N$ be a left identity. For any $x, y \in N$, $xyx = e(xyx) = e(xxy) = e(x^2y) = x^2y$

Proposition 5.11:

Any pseudo Commutative Near – Ring N with Identity is σ Near – Ring

Proof:

Let N be a pseudo Commutative Near ring, $xyz = zyx$ for all $x, y, z \in N$. Let $x, y \in N$ and $e \in N$ be an identity. Then $x^2y = (x^2y)e = xxye = x(xye) = x\,eyx$ [pseudo Commutative] $= xyx$. Hence $x^2y = xyx$ for all $x, y \in N$. Therefore N is a σ Near Ring.

Proposition 5.12:

A σ near ring N with identity is always Commutative.

Proof:

Let N be a σ Near – Ring, Then $x^2y = xyx \ \forall x, y \in N$ .....(1)

Replace $y\ by\ y + 1$ in (1), $x^2(y + 1) = x(y + 1)x$

$x^2y + x^2 = (xy + x)x = xyx + x^2$

$xxy + x^2 = xyx + x^2$

$xxy = xyx$

$xy = yx$

Therefore N is Commutative.

Proposition 5.13:

Any σ Near – Ring N with right identity is pseudo Commutative.

Proof:

Let $x, y \in N$ and $e \in N$ be a right identity.

Let N be a σ Near – Ring, Then $x^2y = xyx$ .....(1)

$\therefore\ x^2y = xyx = (xyx)e = (xyx)e = x(yxe) = x(xye) = (xxy)e$

$xxy = (yxx)e$

$xxy = yxx$

∴ N is Pseudo Commutative

6 Structure Theorem of σ Near-Ring

In this section we establish a structure theorem for σ Near-Ring.

Theorem 6.1

N is an σ Near Ring if and only if every y in N can be written as $x^2y = u + v$ where $u \in N_o$ and $V \in N_c$ and $u = x_0(nx + m) - x_0\ y\ x_c, V = x_0y\ x_c + x_c, x = x_0 + x_c \in N_0 \oplus N_c$ where $x_0, n \in N_0\ x_c, m \in N_c$ further more $u \in N_0, V \in N_c$.

Proof:

For the 'only if' part

Let $y \in N$. Since N is an σ near-ring, there exists x in B such that $x^2y = xyx$

By using pierce decomposition we can write $y = n + m$ and $x = x_0 + x_c$. Where $x \in N, n, x_0 \in N_0$ and $m, x_c \in N_c$

Now $x^2y = (x_0 + x_c)(n + m)x$

$= (x_0 + x_c)(nx + mx)$

$= (x_0 + x_c)(nx + m)$ (Since $m \in N_c$)

$= (x_0 + x_c)(n + m) + x_c(nx + m)$

$= x_0(nx + m) + x_c(\because x_c \in N_c)$

$= x_0(nx + m) - x_0\ y\ x_c + x_0\ y\ x_c + x_c$

$u + v$ Where $u = x_0(nx + m) - x_0\ y\ x_c$

Now,

$u.o = [x_0(nx + m) - x_0\ y\ x_c].0$

$= x_0(nx + m)0 - x_0\ y\ x_c\ .0$

$= x_0(n\ x\ \ 0 + m\ 0) - x_0\ y\ x_c.0$

$= x_0(nx0 + m) - x_0\ y\ x_c$

(Since $m, x_c \in N_c$

$= x_0(nx_c + mx_c) - x_0\ y\ x_c$

($\because x_0 = x_c\ m \in N_c$

$= 0.$

$\because u \in N_c$

$^{rly}\quad v.0 = [x_0\ y\ x_c + x_c].0$

$= x_0\ y\ x_c 0 + x_c\ 0$

$= x_0\ y\ x_c + x_c \quad (\because x_c \in N_c)$

$= v$

Thus $x^2y = u + v$ where $u \in N_0$ and $v \in N_c$ for the "if part"

As some for every Y in N with $x^2y = u + v$ where $u \in N_0$ and $v \in N_c$ with $u = x_0(nx + m) - x_0y\ x_c \quad v = x_0\ y\ x_c +$

eISSN: 1303-5150

x_c where $x = x_0 + x_c, x_0$, $n \in N_c$ and $x_c, m \in N_c$

We shall show that N is an σ Near – Ring.

Now, $x^2y = u + v$

$= x_0 (nx + m) - x_0 y\, x_c +$

$x_0\, y\, x_c + x_c$

$= x_0(nx + mx) + x_c$ ($\because$ $m \in N_c$)

$= x_0(n + m)x + x_c$

$= x_0 y\, x + x_c\, y\, x (\because x \in N_c)$

$= (x_0 + x_c)yx$

$= xyx$

Thus for every $y \in N$, $x^2y = xyx$ for some x in N.

∴ N is an σ Near – Ring.

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