

Quantum Model for Psychological Measurements: From the Projection Postulate to Interference of Mental Observables Represented As Positive Operator Valued Measures

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ABSTRACT

Recently foundational issues of applicability of the formalism of quantum mechanics (QM) to cognitive psychology, decision making, and psychophysics attracted a lot of interest. In particular, in (Khrennikov *et al.*, 2014) the possibility to use of the projection postulate and representation of “mental observables” by Hermitian operators was discussed in very detail. The main conclusion of the recent discussions on the foundations of “quantum(-like) cognitive psychology” is that one has to be careful in determination of conditions of applicability of the projection postulate as a mathematical tool for description of measurements of observables represented by Hermitian operators. To represent some statistical experimental data (both physical and mental) in the quantum(-like) way, one has to use generalized quantum observables given by positive operator-valued measures (POVMs). This paper contains a brief review on POVMs which can be useful for newcomers to the field of quantum(-like) studies. Especially interesting for cognitive psychology is a variant of the formula of total probability (FTP) with the interference term derived for incompatible observables given by POVMs. We present an interpretation of the interference term from the psychological viewpoint. As was shown before, the appearance of such a term (perturbing classical FTP) plays the important role in cognitive psychology, e.g., recognition of ambiguous figures and the disjunction effect. The interference term for observables given by POVMs has much more complicated structure than the corresponding term for observables given by Hermitian operators. We elaborate cognitive interpretations of different components of the POVMs-interference term and apply our analysis to a quantum(-like) model of decision making.

Key Words: quantum model, psychological measurements, projection postulate, decision making

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1. Introduction

Recently the mathematical formalism of QM started to be widely used in various domains outside of physics, see, e.g., (Acacio de Barros

and Suppes, 2009; Accardi *et al.*, 2008, 2009; Asano *et al.*, 2010a, 2010b, 2011a, 2011b, 2011c, 2012a, 2012b, 2012c, 2012d; Basieva *et al.*, 2010; Busemeyer and Wang, 2007; Busemeyer and Bruza, 2012; Busemeyer *et al.*, 2006a, 2006b, 2008, 2009, 2011; Cheon and Tsutsui, 2006; Cheon and Takahashi, 2010; Conte *et al.*, 2006, 2008, 2009; Dzhafarov and Kujala, 2012, 2013, 2014; Haven and Khrennikov, 2009, 2012; Fichtner *et al.*, 2008; Khrennikov, 2003, 2004a, 2006, 2008, 2009a, 2010; Ohya and Volovich, 2011; Pothos and Busemeyer, 2009, 2013; Pothos *et al.*, in press; Wanke *et al.*,

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2009). We are interested in applications to cognitive psychology, especially decision theory (Accardi *et al.*, 2009; Asano *et al.*, 2010a, 2010b, 2011a; Busemeyer and Wang, 2007; Busemeyer *et al.*, 2006a, 2006b, 2008; Cheon and Tsutsui, 2006; Cheon and Takahashi, 2010; Khrennikov, 2009a, 2010; Pothos and Busemeyer, 2009, 2013; Pothos *et al.*, in press), and psychophysics (Dzhafarov and Kujala, 2012). There is a plenty of statistical data from experiments on recognition of ambiguous figures (Conte *et al.*, 2006, 2008, 2009), the disjunction effect (as well as other probability judgment fallacies) (Tversky and Kahnemann, 1981; Tversky and Shafir, 1992; Busemeyer and Wang, 2007; Busemeyer and Bruza, 2012; Busemeyer *et al.*, 2006a, 2006b, 2008, 2009, 2011), and psychophysics (Dzhafarov and Kujala, 2012), violating the classical formula of total probability (FTP). It was well known, see, e.g., Feynman *et al.* (Feynman and Hibbs, 1965), that statistical data obtained in quantum experiments of the interference type, e.g., the two slit experiment, violate FTP, see (Khrennikov, 2009b) for detailed analysis. The additional interference term in the quantum version of FTP can be naturally expressed through the wave function and eigenvectors of Hermitian operators representing observables with the aid of the projection postulate. The latter describes measurements of quantum observables (represented by Hermitian operators).

Since FTP is violated for data from both QM and cognitive psychology and the former developed the advanced mathematical apparatus for study of the interference effect, it is natural to apply this apparatus to describe statistical data from the latter, see above references and especially Khrennikov (Khrennikov, 2004a, 2006, 2010), Busemeyer *et al.* (Busemeyer *et al.*, 2006a), Dzhafarov *et al.* (Dzhafarov and Kujala, 2012). However, there is one pitfall; one problem has to be taken into account and carefully analyzed. The experimentally obtained matrices of transition probabilities² are not *doubly stochastic* (Khrennikov, 2009a, 2010; Haven and Khrennikov, 2012) - as they have to be for transitions from one orthonormal basis to another. Here we discuss the bases of the

eigenvectors of the Hermitian operators representing quantum observables. Therefore “cognitive observables” cannot be represented by Hermitian operators (at least with nondegenerate spectra). In quantum physics the same problem was resolved by invention of generalized quantum observables, *positive operator-valued measures* (POVMs). In general the matrices of transition probabilities for measurements represented by POVMs are not doubly stochastic. Therefore it is natural to apply POVMs to describe the aforementioned statistical data. In this paper we proceed in this direction.

We remark that in cognitive psychology, decision making, and psychophysics there are other important reasons to proceed with generalized quantum observables given by POVMs (and not with the Dirac-von Neumann observables given by Hermitian operators), see (Khrennikov *et al.*, 2014) for detailed analysis. However, in this paper we concentrate mainly on violation of double stochasticity for statistical data with nonclassical features, e.g., in the form of violation of FTP collected in experiments in cognitive psychology.

We start with analysis of possible sources of violation of double stochasticity and point to violation of the projection postulate as the key point, see section 2 on a general discussion on applicability of this postulate in physics, cognitive science, psychology, and psychophysics.

As an example illustrating possible applications of POVMs, we present a quantum-like model of decision making based on representation of the *decision and intention observables* by POVMs. Here (as well as in models based on the standard representation of observables by Hermitian operators) the violation of classical FTP is of the fundamental value.

Although the inter-relation between violation of the classical FTP and quantum interference of probabilities is well studied (Feynman and Hibbs, 1965; Khrennikov, 2009b), typically the situation was restricted to probabilities corresponding to the conventional quantum observables and the pure states (Hermitian operators and complex state vectors). The FTP with interference term for the POVM-observables and the mixed states (density operators) was derived in (Khrennikov and Loubenets, 2004b). In this case the

²The elements of such a matrix are probabilities of transition from one state to another as the result of measurement. Matrices of transition probabilities of QM can be considered as analogs of matrices of transition probabilities in theory of classical Markov systems (However, one has to be careful when exploring such an analogy).



interference term cannot be parametrized just by the “interference angle” (phase) θ . It contains a few parameters having interesting cognitive interpretations, see section 7. Here we present general theoretical analysis of the structure of the POVM-interference. However, the real cognitive meaning will become clear only in future through study of various concrete examples.

We also point out that the POVMs approach dismiss the exceptional role of *pure quantum states* (which are mathematically represented by normalized vectors in complex Hilbert space). This approach is based on usage of general quantum states given by *density operators*³ which represent in general classical statistical mixtures of superpositions.

The quantum-like model of decision making in the two party games based on usage of pure states (to represent mental states), Hermitian operators (to represent decision and intention observables), and the projection postulate (to represent measurements) (Busemeyer and Wang, 2007; Busemeyer *et al.*, 2006a, 2006b, 2008; Cheon and Tsutsui, 2006; Cheon and Takahashi, 2010; Khrennikov, 2009a, 2010; Pothos and Busemeyer, 2009, 2013; Pothos *et al.*, in press) can be generalized to the model with density operators, POVMs, and the corresponding formulas for measurement output probabilities and states.

One party, say Bob, has to make decision on some problem B . The payoff depends on the actions of another party, say Alice. Therefore Bob has to take into account intentions A of Alice. The mental state of Bob is described by the density operator ρ , his decision observable is given by the POVM M^b ; Bob also has to analyze possible actions of Alice, output of this analysis is treated as (self-) measurement and it is represented by the POVM M^a .

We emphasize that both the decision making observable M^b and the intention observable M^a represent *self-measurements*. This is the crucial interpretational difference from quantum measurement theory. In the latter, an observer is sharply separated from a system. The simplest resolution of the “self-measurement problem” is through consideration of the brain as a complex system unifying numerous subsystems having some

degree of independence. We might assume that some subsystems process the mental states and other subsystems perform measurements. However, we understood well that the problem is of extreme complexity. In fact, this is the problem of transition from unconsciousness to consciousness. We shall not try to go deeper into this problem and we shall proceed at the formal operational level, cf. (Busch *et al.*, 1995; De Muynck, 2002; D'Ariano, 2007).

In section 5 we start with a brief introduction to theory of generalized quantum observables, POVMs. Then we present the FTP with interference term corresponding to the most general quantum interference: a quantum state given by a density operator and incompatible generalized observables given by a pair of POVMs, see (Khrennikov and Loubenets, 2004b). Finally, we proceed to a quantum-like model of decision making based on POVMs. (We call our model quantum-like to distinguish it from really quantum physical models which were use by some authors to model information processing by the brain, e.g., (Penrose, 1989, 1994; Hameroff, 1994a, 1994b). Our model has no relation to quantum physical carriers of information.)

Finally, we remark that quantum measurement theory is far from being satisfactory (and this is well recognized by physicists). It might happen that the present experiments with extension of its domain of applications to cognitive psychology, decision making, and psychophysics would clarify some of its problems.

2 Projection postulate in physics and psychology

2.1 Projection postulate in quantum physics

For readers convenience, we recall this postulate of QM; for simplicity we consider the finite dimensional case.

Suppose that the state of a quantum system is represented by the (norm-one) vector ψ of complex Hilbert space and suppose that a quantum observable is represented by the Hermitian operator A . If measurement of this observable is resulted in the concrete eigenvalue a of A (and by the spectral postulate of QM only eigenvalues can be observed), then the state-vector ψ is projected onto the eigensubspace of H corresponding to a . (Since the projection can have the norm less

³These are Hermitian, positively defined and trace one operators.



than one, it has to be renormalized by dividing by its norm⁴).

In the case of nondegenerate spectrum all eigensubspaces are one-dimensional and the aforementioned projection (after renormalization) is just the eigenvector e_a of A corresponding to a . One typically says that the state ψ collapsed to the eigenstate e_a . In the case of continuous variables (with representation in the infinite-dimensional Hilbert space L_2 of square integrable functions) states are given by complex valued functions $\psi(x)$. Here one says about collapse of the wave function as the result of measurement.

This postulate plays a special role in axiomatics of QM. Although majority of quantum community uses it routinely and operates with the notion of collapse of the wave function, a part of this community handle this postulate with suspicion. This postulate remains controversial, see Auyang (Auyang, 1995), p.23, for the detailed discussion. In particular, E. Beltrametti and G. Cassinelli (who are among the world leading experts in quantum logic and foundations) remarked (Beltrametti and Cassinelli, 1979) that “it does not have the status of postulates of quantum theory, necessary for its internal coherence.”

L. E. Ballentine (who is also one of the world leading experts in quantum foundations, one of creators of the “statistical interpretation of QM”) pointed out (Ballentine, 1990) that this postulates leads to wrong conclusions. Even if a quantum system through interaction with a measurement device triggers it to produce one fixed eigenvalue, in general the state of this system does not collapse. As an example of inapplicability of the projection postulate, he considered the track left by a quantum (charged) particle in a cloud chamber. Typically the state of the incoming particle is given by a momentum amplitude. Particles ionizes the first atom in chamber and this process is considered as position measurement. By the projection postulate particles state should collapse to the corresponding eigenstate of the position operator, but the latter is a spherical wave which spreads out uniformly in all directions. Hence, it would be impossible for for this particle to ionize subsequent atoms to form a track which indicates the direction of original

momentum, which is allegedly destroyed by the first ionization, see (Ballentine, 1990).

One of coauthors of this paper (Khrennikov, 2009c, 2009d) analyzed the role of the projection postulate in reasoning on incompleteness/non-locality of QM which was presented in the famous argument of Einstein, Podolsky, and Rosen, known as the EPR paradox. In fact, this paradox can be reduced to the use of the projection postulate (Khrennikov, 2009c, 2009d).

Finally, we remark that the projection postulate in the presently common form of the projection of the state vector (collapse of the wave function) was proposed by J. von Neumann only for observables with *nondegenerate spectra* (Von Neuman, 1955). It was extended to all quantum observables by Lüders, see, e.g., (Busch *et al.*, 1995) for details. However, from the very beginning of QM majority of physicists applied this postulate without paying attention to the form of spectra of observables, e.g., in the aforementioned EPR argument. At the same time von Neumann pointed out that one must differ sharply the cases of nondegenerate and degenerate spectra. In the latter case he elaborated a more complicated mathematical scheme than simply vector reduction (Thus von Neumann did not accept “collapse on eigensubspace” of the dimension larger than one).

In physics understanding of the “state projection problem” in 1970th generated a constructive reply in the form of generalization of the class of quantum observables and, hence, the mathematical scheme of quantum measurement. The first step in this direction was consideration of quantum observables represented not by Hermitian operators, but by POVMs. The main idea beyond this generalization is very simple. Any Hermitian operator can be characterized by its spectral family. In the finite dimensional case this is just the family of projectors to its eigensubspaces. These projectors are mutually orthogonal. Any projector is by itself Hermitian (self-adjoint), *i.e.*, $P^* = P$, idempotent, *i.e.*, $P^2 = P$, positively defined, *i.e.*, $\langle P\psi, \psi \rangle \geq 0$, for any vector ψ , and the projectors from the spectral family sum up to the unit operator, in the case of discrete spectral family (P_n) we have

$$\sum_n P_n = I,$$

⁴Thus the operation of measurement is nonlinear - opposite to the Schrödinger dynamics of an isolated quantum system.



where I is the unit operator in the Hilbert state space. In the definition of POVMs operators need not be idempotent, *i.e.*, these are Hermitian positively defined operators.

2.2 Projection postulate in cognitive psychology and decision making

In applications of quantum theory to cognitive psychology, decision making, and psychophysics the role of the projection postulate (in the Lüders form) was analyzed in very detail in the paper (Khrennikov *et al.*, 2014 and see also Khrennikov, 2010; Haven and Khrennikov, 2012). The authors of (Khrennikov *et al.*, 2014) elaborated examples of mental measurements similar to the Ballentine example with a charged particle interacting with ions in a cloud chamber. In combination with aforementioned problem of violation of double stochasticity of the matrix of transition probabilities the examples from (Khrennikov *et al.*, 2014) motivate the use of POVMs, instead of Hermitian operators. However, the authors of the paper (Khrennikov *et al.*, 2014) also found that there exist problems of decision making which seems to be impossible to describe operationally even with the aid of POVMs, cf. (Basieva and Khrennikov, 2012).

3. Neumark dilation theorem and theory of open quantum systems in psychology

Another possibility to solve the problem of *non-double stochasticity of matrices of transition probabilities for data on recognition of ambiguous figures and the disjunction effect* is to consider Hermitian operators with degenerate spectra acting in Hilbert space having the dimension which is higher than the “natural dimension” of the decision problem. In the simplest case, the decision and intention observables are dichotomous and the “natural Hilbert space of the game” is two dimensional, *i.e.*, observables are represented by 2x2 matrices. We state again that it is impossible to represent some psychological statistical data by Hermitian 2x2 matrices. If, instead of using POVMs, one considers Hermitian operators in Hilbert spaces of higher dimensions, the problem of the interpretation of these extra dimensions arises. At the present time neurophysiology cannot present the complete set of variables involved in the decision making. Therefore we do not know what and how many mental variables have to be added. The “neurophysiological dimension” of the Hilbert

space of mental states may be millions or even billions. Therefore a possibility to solve the problem of non-double stochasticity by using 2x2 POVMs, instead of billionxbillion Hermitian matrices, is more attractive.

We recall that by *Neumark's theorem* (Gelfand and Neumark, 1943) any generalized quantum observable given by a POVM M can be represented as a conventional quantum observable given by a Hermitian operator A acting in the Hilbert space of higher dimension. This extended Hilbert space can be represented as the tensor product of the original Hilbert space H and another Hilbert state space K . Here A acts in $H \otimes K$. It can be considered as a measurement on a compound system, the original system S and another system S' . An adequate mental interpretation of such lifting of (self-)measurement on S to (self-)measurement on a compound mental system (S, S') has not yet been elaborated. Representations of measurement by POVM M acting in H and Hermitian operator A acting in $H \otimes K$ are coupled by an isometric operator, say B . Its cognitive meaning is neither clear.

We point out that in physics appearance of POVMs and the Neumark theorem are typically treated in the framework of theory of open quantum systems. Additional degrees of freedom represented by the Hilbert state space K in the above consideration are interpreted as representing an environment, in our case, so to say, the *mental environment*. In physics the situation is essentially simpler (from the interpretational viewpoint): one can identify a quantum system as a physical entity sharply separated from the environment; for example, an electron under influence of the electromagnetic fluctuations. In cognitive science, as was already emphasized in introduction, it is not easy to extract physically a cognitive system, as a decision maker, from its mental environment. The brain is at the same time a system and observer, self-observer, and more generally it is at the same a system and its mental environment, *e.g.*, in the form of memory.

Therefore it is natural to distance from the physical structure of the aforementioned entities and proceed at the purely information level. Both a decision maker and its environment are considered as information systems exchanging information. The decision is coming as the result of such an information



exchange which resolves the information uncertainty in a decision maker. Since the mental (as well as physical) environment has huge complexity, one does not try to describe explicitly the compound system, the decision maker plus the environment. The operational description based on POVMs is explored. In principle, one can proceed at a deeper level of theory of open quantum systems and describe the dynamics of the state of a decision maker interacting with the mental environment, see Asano *et al.* (Asano *et al.*, 2010a, 2010b, 2011a, 2011b, 2011c, 2012a, 2012b, 2012c, 2012d; Basieva *et al.*, 2010).

4. Mathematical structure of projection postulate

4.1 States and observables

Starting from Heisenberg, observables are represented by Hermitian matrices or in the abstract framework by Hermitian operators. These operators act in complex Hilbert space H , *i.e.*, a complex linear space endowed with a scalar product denoted as $\langle \psi_1 | \psi_2 \rangle$.⁵

The norm (the abstract analog of the Euclidean length) of a vector is defined as $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$. In the fixed system of coordinates $\|\psi\| = \sqrt{\sum_j |z_j|^2}$. Normalized vectors of H , *i.e.*, ψ such that $\|\psi\| = 1$, represent a special (and the most important) class of states of quantum systems, *pure states*. Each pure state ψ can be represented as an

⁵The latter is a function from the Cartesian product $H \times H$ to the field of complex numbers \mathbb{C} , $\psi_1, \psi_2 \rightarrow \langle \psi_1 | \psi_2 \rangle$, having the following properties:

1. Positive definiteness: $\langle \psi | \psi \rangle \geq 0$ with $\langle \psi | \psi \rangle = 0$ if and only if $\psi = 0$.
2. Conjugate symmetry: $\langle \psi_1 | \psi_2 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$
3. Linearity with respect to the first argument: $\langle k_1 \psi_1 + k_2 \psi_2 | \phi \rangle = k_1 \langle \psi_1 | \phi \rangle + k_2 \langle \psi_2 | \phi \rangle$, where k_1, k_2 are complex numbers.

Here bar denotes complex conjugation; for $z = x + iy$, $\bar{z} = x - iy$. From the second and third properties it is easy to obtain that, for the second argument, $\langle \phi | k_1 \psi_1 + k_2 \psi_2 \rangle = \bar{k}_1 \langle \phi | \psi_1 \rangle + \bar{k}_2 \langle \phi | \psi_2 \rangle$. By fixing in H an orthonormal basis (e_j) , *i.e.*, $\langle e_i | e_j \rangle = \delta_{ij}$, we represent vectors by their coordinates $\psi_1 = (z_1, \dots, z_n, \dots)$, $\psi_2 = (w_1, \dots, w_n, \dots)$. In the coordinate representation the scalar product has the form $\langle \psi_1 | \psi_2 \rangle = \sum_j z_j \bar{w}_j$. By using this representation the reader can easily verify the aforementioned properties of the scalar product.

We remark that usage of *complex numbers* plays the crucial role. One cannot proceed with real Hilbert spaces. There are experimental statistical data which cannot be embedded in the real model.



operator acting in H , namely, the orthogonal projector on this vector. Thus, by fixing the orthonormal basis in H , we represent any pure state by a matrix $\rho = (\rho_{ij})$:

- a) it is positively defined⁶,
- b) Hermitian⁷,
- c) its trace equals 1.⁸

In the two dimensional case, one quantum bit (qubit) case, we have

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \quad (1)$$

In the two dimensional case positive definiteness means that $\rho_{ii} > 0, i = 1, 2$ and the determinate is also positive; finally, $\text{tr} \rho = \rho_{11} + \rho_{22} = 1$.

The condition guaranteeing that ρ is a projector can be written as the equality $\rho^2 = \rho$. The next step in development of the quantum formalism (due to Landau and von Neumann) was proceeding without the latter constraint, *i.e.*, considering all possible matrices satisfying conditions a)-c). They are called *density matrices* and they represent the most general states of quantum systems. In the abstract framework one considers operators satisfying conditions a)-c), density operators. Each density operator can be written as a weighted sum of projection operators corresponding to pure states. If such a sum contains more than one element, then the state represented by this density operator is called a *mixed state*: the mixture of pure states with some weights. Although this terminology is widely used, it is ambiguous. Representation of a density operator as a weighted sum of projectors corresponding to pure states is not unique. Thus by using the terminology mixed state one has to take into account this non-uniqueness.

For simplicity let us restrict consideration to the case of finite dimensional Hilbert space. Take an arbitrary quantum observable, represented by a Hermitian operator A . For any Hermitian operator, there exists an orthogonal basis of H consisting of its eigenvectors. The values of the observable

⁶Positive definiteness means that, for any vector ϕ , $\langle \rho \phi | \phi \rangle \geq 0$.

⁷Hermiticity means that $\rho_{ij} = \bar{\rho}_{ji}$, in particular, the diagonal elements are real.

⁸Thus $\text{Tr} \rho = \sum_j \rho_{jj} = 1$.

which is operationally represented by A are encoded in the eigenvalues of this operator, a_1, \dots, a_n . (We shall use the same symbol for an observable and its operator representation.)

Consider now the case of nondegenerate spectrum. Here all eigenvalues are nondegenerate, *i.e.*, all eigensubspaces are one dimensional. For an ensemble of quantum systems prepared in the same state which is represented by the density operator ρ , the probability $p(a_j)$ to obtain the fixed (eigen)value a_j is encoded in the corresponding matrix element of the operator ρ (in the basis of eigenvectors): $p(a_j) = \rho_{jj}$. This rule can be also written as

$$p(a_j) = \text{Tr} \rho P_j, \quad (2)$$

where P_j is the projector to the eigenvector e_j corresponding the eigenvalue a_j . This is one of the basic postulate of quantum mechanics. It connects experimental probabilities with operator representation of observables. (In the simplest form, for the pure state, this postulate was proposed by M. Born.

If the spectrum of A is degenerate, *i.e.*, eigensubspaces are in general multi-dimensional, then the same rule (2) for calculation of probabilities is applicable with P_j as the projector to the eigen-subspace L_j corresponding the eigenvalue a_j .

4.2 Superposition

Let the state space of some system (physical or cognitive) be represented by finite-dimensional Hilbert space H . Consider the pure state ψ and the observable A , denote its eigenvalues by a_1, \dots, a_m and the corresponding eigenvectors by e_1, \dots, e_m . This is an orthonormal basis in H . (We again proceed under the assumption that spectrum is nondegenerate.) We expand the vector ψ with respect to this basis:

$$\psi = c_1 e_1 + \dots + c_m e_m, \quad (3)$$

where (c_j) are complex numbers such that the sum of their squared absolute values equals to one (this is the coordinate expression of the normalization by one of a pure state-vector): $|c_1|^2 + \dots + |c_m|^2 = 1$. By using the terminology of

linear algebra we say that the pure state ψ is *superposition* of pure states e_j . The density matrix corresponding to ψ has the elements $\rho_{ij} = c_i \bar{c}_j$. Hence, for the pure state ψ , the basic probabilistic postulate of quantum mechanics, (2), has the form: $p(a_j) = \rho_{jj} = c_j \bar{c}_j = |c_j|^2$. This postulate can be written without using the coordinates of the state vector ψ with respect to the basis of eigenvectors of a quantum observable. We remark that, since the basis of eigenvectors of a Hermitian operator can always be selected as orthonormal, the coordinates c_j can be expressed in the form: $c_j = \langle \psi | e_j \rangle$. Hence, the Born's rule takes the form:

$$p(a_j) = |\langle \psi | e_j \rangle|^2. \quad (4)$$

4.2.1 Projection postulate; resolution of uncertainty

Consider the case of nondegenerate spectrum. The next natural question is about the post-measurement state. What will happen with the state ψ after measurement? By the *projection postulate* the superposition (3) is reduced to just one term, the state e_j corresponding to the eigenvalue a_j which was obtained in the measurement. This procedure can be interpreted in the following way:

This superposition encodes uncertainty in results of measurements for the observable A . Roughly speaking before measurement, a quantum system “does not know how it will answer to the question A” The mathematical expression (3) encodes potentialities for different answers. Thus a quantum system in the superposition state ψ does not have any value of A as its objective property. After measurement superposition is reduced to just one term in the expansion (3) corresponding the value of A obtained in the process of measurement.

We remark that the state reduction is often called state's *collapse*. Some experts in quantum foundations treat superposition physically and not simply operationally; for them, the collapse is also the physical event. We proceed with the operational interpretation of the quantum formalism. By this interpretation superposition (3) express uncertainty in expected results of the A -measurement. When



the result a_j is detected, this uncertainty is resolved. Hence, “collapse” takes place not in physical space, but in information space.

Encoding of uncertainty in cognitive systems by superpositions is one of the cornerstones of quantum(-like) approach to cognition. A cognitive system as well as a quantum physical system can be in a state of uncertainty on possible reactions to measurements. Such states are *mathematically encoded* as linear superpositions. Measurement resolves such superpositions.

This picture of resolution of information uncertainty in the state of a cognitive system is very clear and simple and therefore it is very attractive. It can be applied to some measurements performed in cognitive science and psychology. It is also useful for presentation of the fundamentals of quantum measurement theory for researchers working in cognitive science and psychology (Khrennikov, 2004a, 2010; Busch *et al.*, 2012). In fact, the same happens in quantum physics: the majority of textbooks on quantum mechanics are based on this picture of measurement, only more advanced books, especially with applications to quantum information theory, present a more complex picture of resolution of quantum uncertainty.

In real experiments the situation is more complicated. It is not so often possible to resolve uncertainty completely as the result of measurement. Typically uncertainty is reduced only partially and the output of measurement cannot be presented by the projection postulate. As was emphasized in introduction, more general observables, POVMs, and corresponding state updating scheme, see section 5.1, have to be explored. The heuristic picture of POVM-measurement is that this is a kind of fuzzy measurement, see especially (Busch *et al.*, 1995; De Muynck, 2002). For a fixed input pure state, the output states corresponding to different eigenvalues are not mutually orthogonal, as it is in the case of observables represented by Hermitian operators. For dichotomous decision observable $A = 0, 1$, its values cannot be interpreted as sharp “no”, “yes”. The corresponding output subspaces are in general overlapping; Roughly speaking such “no”-decision is partially “yes” and vice versa. A proper cognitive interpretation has to be elaborated.

5 Generalized quantum observables

5.1 POVMs

Definition. A positive operator valued measure (POVM) is a family of positive operators $\{M_j\}$ such that $\sum_{j=1}^m M_j = I$, where I is the unit operator.

(We considered the simplest case of a discrete operator valued measure on the set of indexes $J = \{1, 2, \dots, m\}$, see, e.g., (Busch *et al.*, 1995) for POVMs on continuous sets.) It is convenient to use the following representation of POVMs:

$$M_j = V_j^* V_j, \quad (5)$$

where $V_j : H \rightarrow H$ are linear operators.

A POVM can be considered as a random observable. Take any set of labels $\alpha_1, \dots, \alpha_m$, e.g., for $m = 2, \alpha_1 = \text{yes}, \alpha_2 = \text{no}$. Then the corresponding observable takes these values (for systems in the state ρ) with the probabilities

$$p(\alpha_j) \equiv p_\rho(\alpha_j) = \text{Tr} \rho M_j = \text{Tr} V_j \rho V_j^*. \quad (6)$$

We are also interested in the *post-measurement states*. Let the state ρ was given, a generalized observable was measured and the value α_j was obtained. Then the output state after this measurement has the form

$$\rho_j = \frac{V_j \rho V_j^*}{\text{Tr} V_j \rho V_j^*}. \quad (7)$$

5.2 Interference of probabilities for generalized observables

Consider two generalized observables a and b corresponding to POVMs $M^a = \{V_j^* V_j\}$ and $M^b = \{W_j^* W_j\}$, where $V_j \equiv V(\alpha_j)$ and $W_j \equiv W(\beta_j)$ correspond to the values α_j and β_j .

If there is given the state ρ the probabilities of observations of values α_j and β_j have the form

$$p^a(\alpha) = \text{Tr} \rho M^a(\alpha) = \text{Tr} V(\alpha) \rho V^*(\alpha), \quad p(\beta) = \text{Tr} \rho M^b(\beta) = \text{Tr} W(\beta) \rho W^*(\beta). \quad (8)$$

Now we consider two consecutive measurements: first the a -measurement and



then the b -measurement. If in the first measurement the value $a = \alpha$ was obtained, then the initial state ρ was transformed into the state

$$\rho_\alpha^a = \frac{V(\alpha)\rho V^*(\alpha)}{\text{Tr}V(\alpha)\rho V^*(\alpha)}. \quad (9)$$

For the consecutive b -measurement, the probability to obtain the value $b = \beta$ is given by

$$\begin{aligned} p(\beta|\alpha) &= \text{Tr}\rho_\alpha^a(\alpha)M^b(\beta) \\ &= \frac{\text{Tr}W(\beta)V(\alpha)\rho V^*(\alpha)W^*(\beta)}{\text{Tr}V(\alpha)\rho V^*(\alpha)}. \end{aligned} \quad (10)$$

This is the conditional probability to obtain the result $b = \beta$ under the condition of the result $a = \alpha$.

We set

$$p(\alpha, \beta) = p^a(\alpha)p(\beta|\alpha). \quad (11)$$

This is the probability to obtain the result (α, β) in the consecutive measurement of a and then b . To find $p(\alpha, \beta)$ with the aid of the probability $p(\alpha)$ and the conditional probability $p(\beta|\alpha)$, we apply the classical Bayes formula. Thus “quantumness” is in the probabilities $p(\alpha)$ and $p(\beta|\alpha)$; the probability $p(\alpha, \beta)$ is the result of their classical combination.

We remark that the probability $p(\alpha, \beta)$ can be expressed directly in terms of POVMs:

$$p(\alpha, \beta) = \text{Tr}\rho V^*(\alpha)W^*(\beta)W(\beta)V(\alpha). \quad (12)$$

In general, the family of probabilities $p(\alpha, \beta)$ cannot be considered as the joint probability distribution of two classical random variables, the pair (a, b) . In the same way we introduce the probability

$$p(\beta, \alpha) = p^b(\beta)p(\alpha|\beta). \quad (13)$$

This is the probability to obtain the result $b = \beta$ in the first measurement and then the result $a = \alpha$. One can easily find examples of POVMs (and even Hermitian operators), see, e.g., (Khrennikov, 2010), such that

$$p(\alpha, \beta) \neq p(\beta, \alpha). \quad (14)$$

We call $p(\alpha, \beta)$ and $p(\beta, \alpha)$ *ordered joint probabilities*.

6. Formula of total probability with the interference term

We recall that, for two classical random variables a and b which can be represented in the Kolmogorov measure-theoretic approach, the formula of total probability (FTP) has the form

$$p^b(\beta) = \sum_\alpha p^a(\alpha)p(\beta|\alpha). \quad (15)$$

Further we restrict our consideration to the case of dichotomous variables, $\alpha = \alpha_1, \alpha_2$ and $\beta = \beta_1, \beta_2$.

This formula which is one of the most fundamental and successful in applications (the law of total probability) is violated in quantum physics. For example, in the famous two-slit experiment, the probability that a photon is detected at position x on the photo-sensitive plate (used for measurement) is represented as

$$\begin{aligned} p(x) &= \left| \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x) \right|^2 \\ &= \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + |\psi_1(x)||\psi_2(x)|\cos\theta, \end{aligned}$$

where ψ_1 and ψ_2 are two wave functions, whose absolute values $|\psi_k(x)|^2$ give the distribution of photons which pass through the slit- k ($k = 0, 1$). The term of $|\psi_1(x)||\psi_2(x)|\cos\theta$ implies the interference effect of two wave functions. Let us denote $|\psi_k(x)|^2$ by $p(x|k)$, and then the above equation is represented as

$$\begin{aligned} p(x) &= p(1)p(x|1) + p(2)p(x|2) \\ &+ 2\sqrt{p(1)p(x|1)p(2)p(x|2)}\cos\theta, \end{aligned} \quad (16)$$

where $p(1) = p(2) = 1/2$. In the above form, it seems that a classical probability law (15) is violated, and the term of interference $2\sqrt{p(1)p(x|1)p(2)p(x|2)}\cos\theta$ specifies the violation.

We are now interested in the version of FTP with the interference term for in general nonpure states given by density operators and generalized quantum observables given by two (dichotomous) PVOMs:

$$\begin{aligned} p^b(\beta) &= p^a(\alpha_1)p(\beta|\alpha_1) + p^a(\alpha_2)p(\beta|\alpha_2) \\ &+ 2\lambda_\beta\sqrt{p^a(\alpha_1)p(\beta|\alpha_1)p^a(\alpha_2)p(\beta|\alpha_2)}, \end{aligned} \quad (17)$$



or by using ordered joint probabilities

$$p^b(\beta) = p(\alpha_1, \beta) + p(\alpha_2, \beta) + 2\lambda_\beta \sqrt{p(\alpha_1, \beta)p(\alpha_2, \beta)}. \quad (18)$$

Here the coefficient of interference λ_β has the form:

$$\lambda_\beta = \frac{\sum_{i=1,2} \text{Tr} \rho \{W^*(\beta)V^*(\alpha_i)V(\alpha_i)W(\beta) - V^*(\alpha_i)W^*(\beta)W(\beta)V(\alpha_i)\}}{2\sqrt{p^a(\alpha_1)p(\beta|\alpha_1)p^a(\alpha_2)p(\beta|\alpha_2)}} \quad (19)$$

Introduce the parameters

$$\gamma_{\alpha\beta} = \frac{\text{Tr} \rho W^*(\beta)V^*(\alpha)V(\alpha)W(\beta)}{\text{Tr} \rho V^*(\alpha)W^*(\beta)W(\beta)V(\alpha)} = \frac{p(\beta, \alpha)}{p(\alpha, \beta)}. \quad (20)$$

This parameter is equal to the ratio of the ordered joint probabilities of the same outcome, but in the different order, namely, “b then a” or “a then b”. Then (Khrennikov and Loubenets, 2004b)

$$\lambda_\beta = \frac{1}{2} \left[\sqrt{\frac{p(\alpha_1, \beta)}{p(\alpha_2, \beta)}} (\gamma_{\alpha_1\beta} - 1) + \sqrt{\frac{p(\alpha_2, \beta)}{p(\alpha_1, \beta)}} (\gamma_{\alpha_2\beta} - 1) \right]. \quad (21)$$

In principle, this coefficient can be larger than one (Khrennikov and Loubenets, 2004b). Hence, it cannot be represented as

$$\lambda_\beta = \cos \theta_\beta \quad (22)$$

for some angle (“phase”) θ_β , cf. (16). However, if POVMs M^a and M^b are, in fact, spectral decompositions of Hermitian operators, then the coefficients of interference are always less than one, i.e., one can find phases θ_β .

The transition from the classical FTP to the FTP with the interference term (17) can be considered as an extension of the parameter space. Besides the probabilities for the results $a = \alpha$ and $b = \beta$ and the conditional probabilities $p(\beta|\alpha)$, new parameters λ_β , the coefficients of interference between observables a and b (for the state ρ), are invented in consideration.

We present an interpretation of the structure of the coefficients of interference. We

start with terms $\gamma_{\alpha\beta}$. They express *noncommutativity of measurements* or nonclassicality of the joint probability distribution. For classical probability all coefficients $\gamma_{\alpha\beta} = 1$ (and hence all interference coefficients $\lambda_\beta = 0$).

We also introduce the coefficients

$$\mu_\beta = \frac{p(\alpha_1, \beta)}{p(\alpha_2, \beta)}. \quad (23)$$

They express the relative magnitude of probabilistic influences of the results $a = \alpha_1$ and $a = \alpha_2$, respectively, on the result $b = \beta$.

Thus the coefficients of interference are composed of the coefficients of noncommutativity which are wighted with the the relative magnitudes of influences:

$$\lambda_\beta = \frac{1}{2} [(\gamma_{\alpha_1\beta} - 1)\sqrt{\mu_\beta} + (\gamma_{\alpha_2\beta} - 1) / \sqrt{\mu_\beta}]. \quad (24)$$

7 The scheme of decision making based on classical and quantum-like FTP

7.1 Classical scheme

There are two parties, Alice and Bob. They operate under some complex of conditions (physical, social, financial) C , context. Each can act only in two ways: $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$. Decisions of Bob depend crucially on actions of Alice. However, in general Bob does not know precisely which action $a = \alpha_1$ or $a = \alpha_2$ will be chosen by Alice, Therefore Bob can only guess. He estimates subjectively probabilities $p^a(\alpha_1)$ and $p^a(\alpha_2)$ of possible actions of Alice. He can also estimate probabilities of his own actions conditioned on actions of Alice, $p(\beta|\alpha)$. Finally, Bob estimates the probabilities $p^b(\beta_1)$ and $p^b(\beta_2)$ by using FTP (15). If, e.g., $p^b(\beta_1) > p^b(\beta_2)$, then Bob takes the decision $b = \beta_1$.

This scheme is realized in the brain on the unconscious level. Subjective estimates of probabilities for actions of Alice and conditional probabilities for his own actions conditioned by the actions of Alice typically not present on the conscious level.



7.2 Quantum-like scheme

This is a generalization of classical scheme. Besides the aforementioned probabilities, Bob estimates the measure of incompatibility of his possible actions and possible actions of Alice, the coefficients of interference. The later estimate is combined of the estimate of the effect of noncommutativity of Bob's and Alice's actions and relative magnitudes of probabilistic influences of Alice's actions to Bob's actions. To estimate $\gamma_{\alpha\beta}$, Bob has to estimate not only the ordered probability $p(\alpha, \beta)$ of his own action $b = \beta$ under the condition that Alice would act as $a = \alpha$, but also the ordered probability $p(\beta, \alpha)$ of possible Alice's action $a = \alpha$ under the condition that she assumes Bob's action $b = \beta$. However, the absolute magnitudes of these probabilities are not important, Bob is interested only in their ratio.

Finally, Bob estimates probabilities of his actions by using the FTP with the interference term (17).

This scheme is also realized on the unconscious level. In particular, all aforementioned measures of quantumness are estimated subjectively (on the basis of collected experience).

We can say that in the quantum-like scheme the counterfactual arguments play a crucial role. To estimate the coefficients of noncommutativity $\gamma_{\alpha\beta}$, Bob has to imagine how Alice would act if he acts as $b = \beta$. These

probabilities are compared with probabilities of Bob's own actions conditioned by possible actions of Alice. If Bob guess that Alice would make the same estimation of the probabilities of his actions conditioned by her actions as for her own actions conditioned the corresponding actions of Bob, then $p(\alpha, \beta) = p(\beta, \alpha)$ and $\gamma_{\alpha\beta} = 1$. In this case the quantum-like and classical FTP and, hence, decision making schemes coincide. Thus *in this decision making scheme quantumness is related assymetry in the estimation of the action-reaction probabilities*. Bob can imagine that Alice is friendlier than him or that Alice is more defect inclined (we consider the Prisoner's Dilemma game as the basic example of the situation under consideration).

In the operational quantum-like formalism all probabilities and components of the coefficients of interference are encoded in the Bob's mental state, the density operator ρ , his decision making observable, POVM M^b , and his observable for intentions of Alice, POVM $M^a_{i=1/2}$. (We remark these are self-observables.)

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