THE ROLE OF FIXED POINT THEOREM IN FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

A fractional differential equation of the Caputo type is investigated by first inverting it as an integral equation, then confirming that the kernel is completely monotone, and finally transforming it into another integral equation with a kernel that supports both contractions and compact mappings.

This kernel enables us to derive qualitative features of solutions using fixed point theory. A list of five transformations that turn difficult problems into straight forward fixed point problems. Using Krasnoselskii's fixed point theorem, we handle cases that are linear, superlinear, and sublinear.

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Introduction:

Many real-world phenomena have been found to be difficult to model using ordinary or partial differential equations or common difference equations formulated using the classical derivatives and integrals. In actuality, these issues arose after fractional calculus (fractional derivatives and integrals), which was developed to address issues where the classical calculus fell short. The theory and applications of ordinary and partial differential equations with fractional derivatives became one of the most researched areas in applied mathematics along with the advancement and development of fractional calculus. Numerous publications, books, and scientific events have highlighted the broad application potential of fractional differential equations in various scientific domains.

On the other hand, fixed point theory provides a highly powerful mathematical instrument to prove the existence and exclusivity of nearly all issues characterized by nonlinear relations. As a result, fixed point theory is used to study the existence and uniqueness issues of fractional differential equations. Fixed point theory has been quickly developing for about a century now. Fixed point theory continues to be researched and valued because of its applications. Additionally, this theory can be used in a variety of spatial contexts, including metric, abstract, and Sobolev spaces. Due to this characteristic, fixed point theory is extremely useful in the analysis of a wide range of practical science problems characterized by fractional ordinary, partial differential, and difference equations.

This special issue offers suggestions for theoretical developments in fixed point theory as well as applications to differential equations involving fractional ordinary, partial, and difference variables. The technological and physical processes that fractional differential equations are found to best explain have been simulated using this method. Fractional derivative models are utilised for accurate modeling of systems that require correct modeling of damping. The goal of this special issue on "Fractional Calculus and its Applications in Applied Mathematics and Other Sciences" is to review the most recent studies on fractional calculus undertaken by the world's leading researchers in the aforementioned fields.

The papers for this different edition were chosen following a careful and rigorous noble review procedure. Mathematical modeling of practical concerns sometimes leads to fractional differential equations and other problems including specific mathematical physics functions, as well as their expansions and generalizations in one or additional variables.

The majority of physical phenomena are also governed by fractional order PDEs in many other models, such as those of fluid dynamics, quantum physics, electricity, ecological systems, and many others.Understanding all currently used and recently discovered methods for solving fractional order PDEs, as well as the applications of these methods, is essential.

This special issue's goal is to get composed leading academics from many engineering fields, including applied mathematicians, and provide them with a stage on which to share their inventive research. Analytical and numerical techniques, state-of-the-art mathematical modeling, and novel developments in differential and integral equations of any order arising in physical systems are the main topics of this article.

Main aim of fractional Derivative

Now let us remember that a derivative of y w.r.t x indicates the rate of change of y wrt x Now If we are dealing with viscous liquids, then stress–strain relationship of fluids is given by Newton's law σ (t) =η D^1 ε (t) Hooke's law for modelling the stress–strain relationship elastic solids is given by σ (t) = E D^0 ε (t) . Now to model the relation between stress and strain for a viscoelastic Material one can consider an equation of the form

σ (t) = vD^k ε (t), 0<k <1

this gave rise to the fractional derivative in modelling Historically, the name fractional derivative was coined when Leibnitz introduced the notation d ny\dxn to denote the nth derivative of y wrt x L;Hopital asked what if n = ½.

Definition 1.1. Fixed point. Fixed point is a point in the field that maps to itself. That is, Function F of a set X into itself such that, a point x ∈ X such that f(x) = x.

Definition 1.2: Let (X,d) be a metric space and let P be a mapping on X . Then P is called a "Contraction" if there exists $r \in [0,1)$ such that $d(Px,Py) \le rd(x,y)$ for all $x,y \in X$.

Definition 1.3: Caputo fractional Dini derivative

The Caputo fractional Dini derivative of a function x(t) is defined as

$$ {}^cD^q x(t) = \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} x(s)]ds $$

Definition 1.4: L^p space- The *p*-norm can be prolonged to vectors that have an infinite number of sequences, which yields the space ℓ^p. This contains as special cases:

ℓ^1, the space of sequences whose series is absolutely convergent,

ℓ^2, the space of square-summable sequences, which is a Hilbert space, and

ℓ^∞, the space of bounded sequences.

Theorem1.1: Krasnoselskii's theorem can be integrated with the fixed point theorems of Banach and Schauder. Krasnoselskii demonstrated that the sum of A+B has a fixed point in M.

i) A is continuous and compact

ii) Ax + By ∈ M for every x, y ∈ M and (iii) B is a strict contraction.

Theorem 1.2.[9] Banach fixed point theorem. Let (X, d) be a complete metric space and let F be a contraction on X. Then F has a unique fixed point.

Main Result:

Statement:2.1

We consider a fractional differential equation of Caputo type

$$cD^q = f(t) - [a(t) + b(t)]x(t), 0 < q < 1; \;\; x(0) = x_0 \qquad ......(2.1)$$

inverted as

$$x(t) = x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}[a(s) + b(s)]x(s) - f(s)]ds \qquad ....(2,2)$$

..

where a, f: $[0, \infty] \to R$ are continuous and there are positive numbers ∈ and M with

$$0 < \in \leq a(t) + b(t) \leq M \qquad \text{-------}(2.3)$$

We draw attention to the fact that due to the huge kernel and the fact that a(t),b(t) are allowed to be arbitrarily big, (2.2) would not be able to define a contraction on the space of bounded continuous functions. Prior to exchanging R(t-s) for the kernel, we must reduce [a(t)+b(t)] to a function with boundary conditions of $\alpha + \beta < 1$.

Here, BC denotes the Banach space of bounded continuous functions ψ: $[0, \infty] \to R$ with the supremum norm ||. ||

Theorem 2.1: Let (2.3) hold. If f ∈ BC then for every x(0) ∈ R there is a unique solution x(t) of (2.2) and it is also in BC. If $f \in L^2[0, \infty)$ with f bounded or if $f(t) \to 0$as $t \to \infty$ then $x(t) \to 0$

Proof: Define $J = \in + (1/2)(M - \in)$.Then there is an $\alpha + \beta$ with J > 0, $0 < \alpha + \beta < 1$, |a(t)+b(t) − J| < $(\alpha + \beta)$J. ---------(2.4)

We have

$$x(t) = x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\,[Jx(s) + (a(s) + b(s) - J)x(s) - f(s)]ds$$

$$= x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\,[Jx(s) + \frac{(a(s) + b(s) - J)}{J}x(s) - \frac{f(s)}{J}]ds$$

Which we decompose into

$$z(t) = x(0) - \frac{J}{\Gamma q}\int_0^t (t-s)^{q-1}\, z(s)ds$$

With solution

$$z(t) = x(0) - \int_0^t R(t-s)x(0)ds$$

Theorem 1.3. Schauder's fixed point Theorem. Let X be a Banach space M is subset of X be nonempty, convex, bounded, closed and P: X $\to M$ be a compact operator. Then T has unique fixed point and

$$x(t) = z(t) + \frac{1}{J}\int_0^t R(t-s)f(s)ds - \int_0^t R(t-s)\frac{(a(s)+b(s)-J}{J}x(s)ds$$

This will define our contraction mapping on BC.For $\psi \in BC$ we define $P: BC \to BC$ by

$$(P\psi)t = z(t) + \frac{1}{J}\int_0^t R(t-s)f(s)ds - \int_0^t R(t-s)\frac{(a(s)+b(s)-J)}{J}\psi(s)ds$$

That the mapping is into BC is obvious. Additionally, it is a contraction as a result of (2.4).The first conclusion is thus true. In order to reach the second conclusion, BC is supplemented with the requirement that the $\psi \to 0$ [2]

Theorem 2.2

If $|x(0)| < \sqrt{\frac{3}{9}}$ then the solution of $cD^q\ x = -x^3$ (2.5)

is bounded [1]

Proof: Invert the equation as

$$x(t) = x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\,[x(s) - (x(s) - x^3\ (s))]ds$$

And separate into

$$z(t) = x(0) - \frac{1}{\Gamma q}\int_0^t R(t-s)z(s)ds$$

$z(t) \le |x(0)| < \sqrt{\frac{3}{9}}$ and

$$x(t) = z(t) - \int_0^t R(t-s)^{q-1}\,[x(s) - (x(s) - x^3\ (s)]ds$$

A maximum of y = x - x^3 of $\frac{2\sqrt{3}}{9}$ occurs at x= $\frac{1}{\sqrt{3}}$ if we define

S ={$\emptyset$: $[0,\infty) \to R \| \emptyset \| \le \frac{\sqrt{3}}{3}$, $\emptyset$ *is continuous* (2.6)

and P by $\emptyset \in$ S implies

$$P(\emptyset)(t) = z(t) - \int_0^t R(t-s)^{q-1}\,[\emptyset(s) - \emptyset^3\ (s)]ds$$

Then $|P(\emptyset)(t)| \le \left|x(0) + \frac{2\sqrt{3}}{9}\right| \le \frac{\sqrt{3}}{3}$ so P $\emptyset \in$ S

Consider the equation y =x - x^3 with derivative y' = 1 – $3x^2$, which will give us the contraction constant at any value of x. It is obvious that this is a big contraction and that the mapping has a singular fixed point. This is how we can establish that P specifies large contraction in S.

Theorem 2.3: The zero solution of (2.5) is asymptotically stable.

Proof:[1] in this paper zero solution is stable is proved. Thus, we want to show that

x(t) → 0 as t → ∞ for any solution x(t) of (5) with $\left|x(0) + \frac{2\sqrt{3}}{9}\right| \le \frac{\sqrt{3}}{3}$ now for S is given above we define

S_0 = { $\emptyset \in$ S /x(t) → 0 as t → ∞}

Then S_0 is complete metric space with supremum metric $\rho(x,y) = \|x-y\|$.Let the mapping p defined abve. Then for ϕ ∈ S0, we have P ϕ ∈ S. We observe that R ∞ 0 R(s)ds = 1 implies that

z(t) = x(0)[1 − $\int_0^t R(s)ds$*] = x(0)* $\int_t^\infty R(s)ds$*→ 0 as t → ∞*

Also, since R ∈ L^1 (0,∞) and ϕ(t) → 0 as t → ∞, we have

$\int_0^t R(t-s)^{q-1}\,[\emptyset(s)-\emptyset^3\,(s)]ds$*→ 0 as t → ∞*

Hence $P(\emptyset)(t)$ → 0 as t → ∞ these yields to $P(\emptyset\,)$ ∈ S_0 we also see that P has large contraction on S_0 $P(\emptyset\,)$≤ √ 3/3 for all ϕ ∈ S0 and n ≥ 1. By Theorem 2.2, P has a unique fixed point ϕ ∈ S0 which is a solution of (5) with ϕ(0) = x(0) and ϕ(t) → 0 as t → ∞. Thus, the zero solution of (5) is asymptotically stable.

Theorem 2.4:

$^cD^q x = -x^{2n+1}$, *0 < q < 1.*

In reality, we can be taken into consideration $^cD^qx = -g(x)$, *0 < q < 1*, where g(0) = 0. We define G(x) = x − g(x) and ask that dG(x)/dx is continuous and positive except, possibly, at x = 0 on an interval [−b, b], and |dG(x)/dx| ≤ α < 1.

The work with the linear equation $^cD^q$ = f(t) – [a(t)+b(t)]x(t) with a(t) and b(t) large was not restricted to linear equations. It works perfectly on

$^cD^qx = -[a(t)+b(t)]x^3$, 0 < q < 1,

with 0 < ε ≤ [a(t)+b(t)] < M. In detail, a very exciting addition occurs. We obtain a huge contraction plus an ordinary contraction with reduction constant α +β< 1 and it is precisely this property which allows us to add the two contractions together and obtain a large contraction. Here are the details

Proof:

Invert the equations as

$$x(t) = x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\,[(a(s)+b(s)] - x^3\,(s)]ds$$

Exactly as in Section 2, we find J > 0, α+ β < 1 with |J – [a(t)+b(t)]| ≤ αJ. Then

$[a(s) + (s)]x^3(s) = Jx^3(s) - (Jx^3(s) - [a(s)+b(s)]x^3(s))$

$$=J[x^3\,(s) - \frac{J-[a(s)+b(s)]}{J}x^3\,(s)$$

$$= J[x(s) - x(s) - x^3\,(s) - \frac{J-[a(s)+b(s)]}{J}x^3\,(s)$$

So

$$x(t) = x(0) - \frac{J}{\Gamma q}\int_0^t (t-s)^{q-1}\,[x(s) - x(s) - x^3\,(s) - \frac{J-[a(s)+b(s)]}{J}x^3\,(s)]ds$$

Which we decompose as

$$z(t) = x(0) - \frac{J}{\Gamma q}\int_0^t (t-s)^{q-1} z(s)ds$$

$$x(t) = z(t) + \int_0^t R(t-s)\,[(x(s) - x^3\,(s) - \frac{J-[a(s)+b(s)]}{J}x^3\,(s)]ds$$

That integrand resolve a huge contraction. The contraction constant for the main term is acquired from $y = x - x^3$ with $y' = 1 - 3x^2$, while the x derivative of $\frac{J-[a(s)+b(s)]}{J}x^3$ is bounded by $3(\alpha+\beta)\,x^2$, yielding the sum $1-3x^2 + 3(\alpha+\beta)\,x^2$. On a specific space of bounded continuous functions, this will result in a significant contraction.

Summary

There are a sum of trades happening and it is worth listing them here. Assume, for example, that we initiate with ${}^{c}D^{q}x = -[a(t)+b(t)]x^3$ with a(t) as above, and invert it as
with a(t) as above, and invert it as

$$x(t) = x(0) - \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1}\,[(a(s)+b(s)) - x^3\,(s)]ds$$

1. A factor [a(t)+b(t)] destroys x^3 local contraction features, therefore we swap out [a(t)+b(t)] for a function with a constant lower than 1.
2. The big kernel continues to disrupt the contraction; therefore, we must add and remove x(s) in order to obtain a linear portion, which we can then interchange for R(t-s).
3. However, throughout the last item's procedure, the contraction function x^3 is changed to x- x^3, which changes the concept of a contraction into that of a huge contraction.
4. At this point, the huge kernel $(t - s)^{q-1}$ is switched out for the little kernel R(t- s).
5. x(0) can be a challenge when using fixed point approaches to demonstrate that solutions gravitate to zero. However, as t increases, we trade x(0) for z(t), which goes to zero as $t \to \infty$. With these exchanges, we have a straightforward fixed point issue.

Conclusion:

Krasnoselskii's fixed point theorem can be utilized to resolve fractional differential equations. Their interest in the sum of a contraction and a compact map is thus well-motivated. The inversion of a fractional differential equation of Caputo type with continuous functions is actually nothing more than a well-known integral equation with a kernel of the form $(t-s)^{q-1}$.

Both a contraction and a compact map are completely insufficient to handle the original integral equation. All would fail if the kernel weren't totally monotone, which allows for the trading of that large kernel for R(t−s), which has the condition that 0 < R(t) and $\int_0^1 R(s)ds =$ 1.Equi-continuous sets are generated using the new integral equations. We added a kernel from that good fortune that strongly supports both.

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