Numerical Simulation of Asano-Khrennikov-Ohya Quantum-like Decision Making Model

Xingdong Zuo

ABSTRACT

In classical game theory, the rational behavior for the player is to make the decision which is approaching Nash equilibrium. The Prisoner's Dilemma, which is a canonical game, is often used to present the rationality. In real experiment in cognitive psychology which were performed by Shafir and Tversky (Shafir and Tversky, 1992a, 1992b), the statistical data show the existence of the irrational behaviors in reality. The phenomenon is called disjunction effect. To explain why it probably happens, we review the Asano-Khrennikov-Ohya model (Asano *et al.*, 2011c; Khrennikov, 2011b) which is the mathematical modeling of the process of decision making in the game of Prisoner's Dilemma. It applies only the mathematical apparatus of quantum mechanics to the decision making process rather than the quantum physical model. In this paper, we present several numerical simulations for the Asano-Khrennikov-Ohya model together with the graphs of the von Neumann entropy for the solutions. By analyzing the simulation results, we explicitly and numerically present the existence of the irrational behavior for the player which is generated by the Asano-Khrennikov-Ohya model.

Key Words: Quantum-like model, Decision making, Lindblad equation, Prisoner's Dilemma, Disjunction effect

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1. Introduction

Traditionally, the development of quantum mechanics (QM) focuses on the world of the microscopic particles. However, recently the mathematical formalism of QM started to be applied in cognitive science (Acacio de Barros and Suppes, 2009; Accardi *et al.*, 2008, 2009; Asano *et al.*, 2011a, 2011b, 2012a, 2012b; Basieva *et al.*, 2010; Busemeyer *et al.*, 2006a, 2006b, 2008, 2009, 2012; Cheon and Takahashi, 2006, 2010; Conte *et al.*, 2006, 2008, 2009; Dzhafarov and Kujala, 2012; Haven and Khrennikov, 2009, 2013; Khrennikov *et al.*, 2003, 2004a, 2004b, 2006, 2009a, 2009b, 2011a, 2011b, 2014; Ohya and Volovich, 2011; Pothos and Busemeyer, 2009).

In classical game theory, the prisoner's dilemma is a canonical example (Axelrod, 2006; Aumann, 1959; Bicchieri, 1993; Chess, 1988; Dresher, 1961; Rapoport and Albert, 1965), and the rational behavior for the player is to make the decision which approaches Nash equilibrium. But in reality, there are numerous experimental evidences of the irrational behaviors. In the experiments in cognitive

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psychology which were performed by Tversky and Shafir (Shafir and Tversky, 1992a, 1992b), the statistical data show the existence of the irrational behaviors. The irrational behavior for the player is to choose the strategies different from the Nash equilibrium. The classical game theory cannot explain it. In order to explain the observed deviations from classical game theory (irrational behavoir), we can apply the mathematical apparatus of QM.[2] In this paper, we analyze Asano-Khrennikov-Ohya model (Asano *et al.*, 2011c; Khrennikov, 2011b) of the quantum-like decision making, in which the mental dynamics for the player is described by the Lindblad equation (Lindblad, 1976; Gorini *et al.*, 1976). As a result of the stabilization of its solution, the player makes a decision probabilistically. This combination of the Lindblad dynamics with the asymptotic stabilization can be treated as a model of the quantum measurement, as it was proposed by W. Zurek (Wheeler and Zurek, 2014).

The main aim of this paper is to perform the numerical simulations of the Asano-Khrennikov-Ohya decision making model where we show the relation between the stabilization of the solution of the Lindblad equation and its cognitive meaning in the process of decision making.

We present several examples of the Lindblad equation with their numerical and/or analytic solutions. The corresponding von Neumann entropy for the density operator at different moments of time is presented as a graph. This mathematical modeling and numerical simulation for the Lindblad equation are important for the further mathematical modeling of cognitive behavoir of the players in the process of decision making. Our examples illustrate in the simple graphical form (the output of the numerical simulation) different types of behavoirs for the solutions of the Lindblad equation. The stabilized solution will lead the density operator to an equilibrium state, which is regarded as the action of the termination of the comparison process for a decision maker. The resulting pure stategy is approached probabilistically by performing the quantum measurement. Moreover, to illustrate the difference between the dynamics of the isolated and open systems, we study the stabilization of the solutions of the von Neumann equation by proving two main theorems.

We remark that the Asano-Khrennikov-Ohya model has not any direct relation to quantum physical models of brain's functioning, see papers by Penrose, Hameroff and Tegmark (Penrose, 1989, 1994; Hameroff, 1994a, 1994b, 1996; Tegmark, 2000)

2. Asano-Khrennikov-Ohya quantum-like decision making model

The study of the analysis of the strategies to make decisions is called game theory. It started from the works of John von Neumann *Zur Theorie der Gesellschaftsspiele* (von Neumann, 1928).

In this paper, we deal with a famous game so-called Prisoner's Dilemma. In the classical game theory, a behaviour rationality is to make the decision which maximizes the player's own payoff. But in reality, there exists irrational behaviors, for example in an statistical experiment by Shafir and Tversky (Shafir and Tversky, 1992a, 1992b) which presented that there are only 63% of the rational choice. In the case of Prisoner's Dilemma, it implies that there are many people decided to coorperate which is classically considered as the irrational behavoir. In order to explain why it probably happens, we introduce a quantum-like decision making model so-called Asano-Khrennikov-Ohya model (Asano *et al.*, 2011c; Khrennikov, 2011b).

2.1. Classical 2-player Prisoner's Dilemma

We describe the game of Prisoner's Dillemma as follows. Suppose there are two criminals, Alice and Bob, they are arrested by the police due to the robbery. Then they are separated without any communication in between. When they are individually investigated by the policeman, they are notified with the following possible consequences: If both of them confess the crime, they will both be served in prison for 5 years. If both of them deny the crime, it is 2 years for each. And if one confess and the other deny, the one who confesses will only serve for 1 year and 10 years for who denies.

Each player has two strategies, either cooperate denoted by C or defect denoted by D. If they trust each other, they might cooperate, i.e. both deny the crime, in order to

[2] We do not try to use quantum mechanics to model brain's functioning. We formally use the mathematical apparatus of QM to model information processing by cognitive systems.

obtain the payoffs $(C,C)=(2,2)$ which is good for the group. But they are separated without any communication, they are both scared to be betrayed, because if the player denies the crime and the other confesses it, there will be 10 years in prison, and therefore they both want to betray each other to have the shortest years in prison, i.e. $(D,C)=(1,10)$ and $(C,D)=(10,1)$. It is the rational behavior in classical game theory.

We present the following payoff table

	C_B	D_B
C_A	(2,2)	(10,1)
D_A	(1,10)	(5,5)

Then for a more general case: Let $a<b<c<d$ be the payoff, we have the table

	$\vert C_B\rangle$	$\vert D_B\rangle$
$\vert C_A\rangle$	(b,b)	(d,a)
$\vert D_A\rangle$	(a,d)	(c,c)

2.1.1 Nash equilibrium

In addition to John von Neumann's works in classical game theory, the papers of John Nash must be mentioned, especially his papers about the equilibrium points for non-cooperative game, which is called Nash equilibrium (Nash, 1950, 1951).

Let us introduce some basics in classical game theory. An n-player game is a set of n players where each player has a finite set of pure strategies, and each player, *i*, has a payoff function, p_i, which maps the set of n-tuples of the pure strategies to real numbers. Each entry in n-tuple refers to an unique single player. A mixed strategy is convex-linear combination of the pure strategies. We define a mixed strategy for a player i to be $s_i=\sum_\alpha c_{i\alpha}\pi_{i\alpha}$ where $\sum_\alpha c_{i\alpha}=1$ and $c_{i\alpha}\geq 0$, and $\pi_{i\alpha}$ is the pure strategy. The geometric structure of the mixed strategy can be described as simplex with vertices $\pi_{i\alpha}$, and the mixed strategy s_i is the point in the simplex. This simplex is also a convex subset of real vector space. The payoff function for the mixed strategies is $p_i(s_1,s_2,\ldots,s_n)$, where each entry is the mixed strategy for the corresponding player. The n-tuples can be regarded as points in the vector space, and the mixed strategy is contained in the product space of vector spaces. The set of all n-tuples forms a convex polytope. In Nash's paper (Nash, 1950, 1951), he gave a substitution notation for computational convenience, i.e. $(\mathbf{s};t_i)=(s_1,s_2,\ldots,s_{i-1},t_i,s_{i+1},\ldots,s_n)$.

The Nash equilibrium point is an n-tuple if and only if for all *i*,

$$p_i(\mathbf{s})=\max_{\text{all}\,r_i}[p_i(\mathbf{s};r_i)] \tag{1}$$

The equilibrium point represents the mixed strategy for every player which maximizes the payoff when others fixed. It indicates that every mixed strategy is optimal.

Example 2.1 *For Prisoner's Dilemma, each player's optimal strategy is to defect for maximizing its own payoff. The Nash equilibrium is (D, D), and it is unique.*

2.2. Asano-Khrennikov-Ohya Model

For Prisoner's Dillemma, the rational behavoir for the player is to defect and there exists the unique Nash equilibrium point, i.e. (D,D). As was pointed, the players can demonstrate the irrational behavoir, namely, selecting the strategies different from the Nash equilibrium in real experiments. In this section, we introduce the Asano-Khrennikov-Ohya model (Asano *et al.*, 2011c; Khrennikov, 2011b) for the explanation of why it possibly happens.

Given a player A, it doesn't know anything about which decision the player B will make, but she may guess. We consider it as a quantum superposition state on Hilbert space $\mathcal{H}=\mathbb{C}^2$, named prediction state and denoted by $\phi_B=\alpha 0_B+\beta 1_B$ where $|\alpha|^2+|\beta|^2=1$. We call $\{0_B,1_B\}$ the prediction basis.

The player A can choose either 0 or 1. These choices are represented by the decision basis $\{0_A,1_A\}$. Coupled with the prediction state for the player B, there will be two consequences which is also in a quantum superposition state, called alternative state and denoted by $\Phi_0=\alpha 0_A 0_B+\beta 0_A 1_B=0_A\otimes\phi_B$ if it chooses 0, and $\Phi_1=\alpha 1_A 0_B+\beta 1_A 1_B=1_A\otimes\phi_B$ for choosing 1.

With two alternative states, there is a new quantum superposition state, called mental state, dentoed by $\Psi=x\Phi_0+y\Phi_1$. Formally (in the operational framework), this state can be treated as the state of a composite quantum system, i.e. $\mathbb{C}^2 \otimes \mathbb{C}^2$. The corresponding operator of the mental state is defined by

$$\Psi\Psi = |x|^2\,\Psi_0\Psi_0 + |y|^2\,\Psi_1\Psi_1 + xy^*\Psi_0\Psi_1 + yx^*\Psi_1\Psi_0, \quad (2)$$

which has the corresponding density matrix in the form $\rho_\Psi = \begin{pmatrix} |x|^2 & xy^* \\ yx^* & |y|^2 \end{pmatrix}$.

An important idea of the model is that the decision maker is a self-observer which enables the player to guess the other's decision. Therefore, the player's own mind is considered as an open quantum system. The self-imaging of the prediction for the other's decision is regarded as the environment which is also a quantum system, since the decision maker can generate the mental reservoir by itself. We consider the interaction is between the mental of the player's own decision and the mental self-image of the other's decision.

The Lindblad equation is in the form $\frac{d}{dt}\rho_S = \mathcal{L}\rho_S$, where

$$\mathcal{L}\rho_S = -i[H,\rho_S] + \sum_{k=1}^{N^2-1}\gamma_k (A_k\rho_S A_k^\dagger - \frac{1}{2}A_k^\dagger A_k \rho_S - \frac{1}{2}\rho_S A_k^\dagger A_k). \quad (3)$$

where ρ_S is the reduced density matrix, H denotes the Hamiltonian which is Hermitian operator, A_k denote orthonormal operator basis which are called Lindblad operators and γ_k are the contants which determine the dynamics .

Remark. *Since we are proceeding with finite dimensional case, we can deal with the matrix representations of the operators by fixing an orthogonal basis.*

The dynamics in an open quantum system is described by the Lindblad equation (3). By choosing the specific Lindblad operator which provides the stabilized solution, it can lead the density matrix to an equilibrium state. One can say it is the termination of the comparison process for a decision maker. This equilibrium density matrix describes the mixed strategy, where we can get the probability for each corresponding pure strategy by taking the quantum measurement. (See equations (32) and (33))

2.2.1 *Example of Lindblad Equation*

In this section, we present some examples of the Lindblad equation.

Example 2.2 *Let us consider a two level atom for the spontaneous emission, and let us assume the ground state is* $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$., *the exicited state is* $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *and the Hamiltonian is* $H = -\frac{\omega_0}{2}\sigma_z$ *where* σ_z *is the Pauli matrix and* ω_0 *is the energy difference between the ground state and excited state. Then let us assume the Lindblad operator* $A = \sqrt{\Gamma}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *which indicates the relaxation process from* $|1\rangle$ *to* $|0\rangle$. *Therefore, we have the corresponding Lindblad equation*

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = -i[H,\rho] + A\rho A^\dagger - \frac{1}{2}(A^\dagger A\rho + \rho A^\dagger A) \quad (4)$$

$$= i\omega_0 \begin{pmatrix} 0 & \rho_{01} \\ -\rho_{10} & 0 \end{pmatrix} + \Gamma \begin{pmatrix} \rho_{11} & -\frac{1}{2}\rho_{01} \\ -\frac{1}{2}\rho_{10} & -\rho_{11} \end{pmatrix} \quad (5)$$

The analytic solution has the form

$$\rho_{00}(t) = \rho_{00}(0) + \rho_{11}(0)(1 - \exp(-\Gamma t)) \quad (6)$$

$$\rho_{01}(t) = \rho_{01}(0)\exp(i\omega_0 t - \frac{\Gamma}{2}t) \quad (7)$$

$$\rho_{10}(t) = \rho_{10}(0)\exp(-i\omega_0 t - \frac{\Gamma}{2}t) \quad (8)$$

$$\rho_{11}(t) = \rho_{11}(0)\exp(-\Gamma t). \quad (9)$$

Remark. *Given* $\frac{d}{dt}\rho_{00}(t) = \Gamma\rho_{11}(t)$, *it implies* $\rho_{00}(t) = C - \rho_{11}(0)\exp(-\Gamma t)$. *We can find* $C = \rho_{00}(0) + \rho_{11}(0)$. *By substituting it back, we obtain the equation (6).*

We assume $\omega_0 = \Gamma_+ = \Gamma_- = \Gamma_z = 1$ for the computational simplicity. With the assumed symmetric initial condition

$\rho(0)=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, we present the graph for the solutions as $t\to\infty$ in the Figure 1.

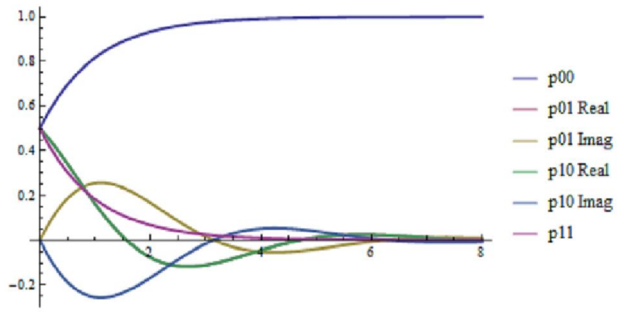

Figure 1. Solutions of Lindblad equation for spotaneous emission.

Analytically, we can prove $\rho(t)$ is a diagonal matrix as $t\to\infty$ as follows

$$\lim_{t\to\infty}\rho_{01}(t)=\lim_{t\to\infty}(\rho_{01}(0)\frac{\cos(\omega_0 t)}{e^{\frac{\Gamma}{2}t}}+\rho_{01}(0)i\frac{\sin(\omega_0 t)}{e^{\frac{\Gamma}{2}t}})=0 \quad (10)$$

$$\lim_{t\to\infty}\rho_{10}(t)=\lim_{t\to\infty}(\rho_{10}(0)\frac{\cos(\omega_0 t)}{e^{\frac{\Gamma}{2}t}}-\rho_{10}(0)i\frac{\sin(\omega_0 t)}{e^{\frac{\Gamma}{2}t}})=0. \quad (11)$$

We present the graph of the von Neumann entropy for the density matrix in the Figure 2.

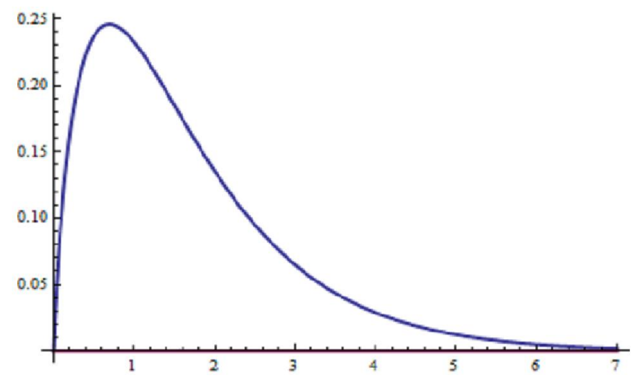

Figure 2. von Neumann entropy of the density matrix for spotaneous emission

Remark. *Note that the von Neumann entropy is decreasing down to 0 with as $t\to\infty$. Additionally, one of the diagonal entry in the density matrix tends to 0 which means the limiting density matrix describes the pure states. This situation will happen again in the Example 2.4.*

Example 2.3 *Let us consider the Bloch equation in Nuclear Magnetic Resonance (NMR) and we assume the Hamiltonian is $H=-\frac{\omega_0}{2}\sigma_z$, and three Lindblad operators are $A_+=\sqrt{\Gamma_+}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $A_-=\sqrt{\Gamma_-}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ and $A_z=\sqrt{\Gamma_z}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ where A_+ is the relaxation process from the excited state to the ground state, A_- is the exictation process from the ground state to the exited state and A_z is the dephasing process such that there is no energy transmission for the spin with the environment.*

We present the corresponding Linblad equation

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}=i\omega_0\begin{pmatrix}0 & \rho_{01}\\ -\rho_{10} & 0\end{pmatrix}+\Gamma_+\begin{pmatrix}\rho_{11} & -\frac{1}{2}\rho_{01}\\ -\frac{1}{2}\rho_{10} & -\rho_{11}\end{pmatrix}$$
$$+\Gamma_-\begin{pmatrix}-\rho_{00} & -\frac{1}{2}\rho_{01}\\ -\frac{1}{2}\rho_{10} & \rho_{00}\end{pmatrix}+\Gamma_z\begin{pmatrix}0 & -2\rho_{01}\\ -2\rho_{10} & 0\end{pmatrix}. \quad (12)$$

The analytic solution has the form

$$\rho_{00}(t)=\rho_{00}(0)\frac{\exp(-(\Gamma_++\Gamma_-)t)\Gamma_-+\Gamma_+}{\Gamma_++\Gamma_-} -\rho_{11}(0)\frac{\exp(-(\Gamma_++\Gamma_-)t)\Gamma_+-\Gamma_+}{\Gamma_++\Gamma_-} \quad (13)$$

$$\rho_{01}(t)=\rho_{01}(0)\exp((i\omega_0-\frac{\Gamma_+}{2}-\frac{\Gamma_-}{2}-2\Gamma_z)t) \quad (14)$$

$$\rho_{10}(t)=\rho_{10}(0)\exp((-i\omega_0-\frac{\Gamma_+}{2}-\frac{\Gamma_-}{2}-2\Gamma_z)t) \quad (15)$$

$$\rho_{11}(t)=\rho_{00}(0)\frac{-\exp(-(\Gamma_++\Gamma_-)t)\Gamma_-+\Gamma_-}{\Gamma_++\Gamma_-} +\rho_{11}(0)\frac{\exp(-(\Gamma_++\Gamma_-)t)\Gamma_++\Gamma_-}{\Gamma_++\Gamma_-}. \quad (16)$$

We assume $\omega_0=\Gamma_+=\Gamma_-=\Gamma_z=1$ for the computational simplicity. With the assumed symmetric initial condition $\rho(0)=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, we present the graph for the solutions as $t \to \infty$ in the Figure 2.3., and for the asymmetric initial condition

$$\rho(0) = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \otimes (\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}})$$ in the Figure 2.3.

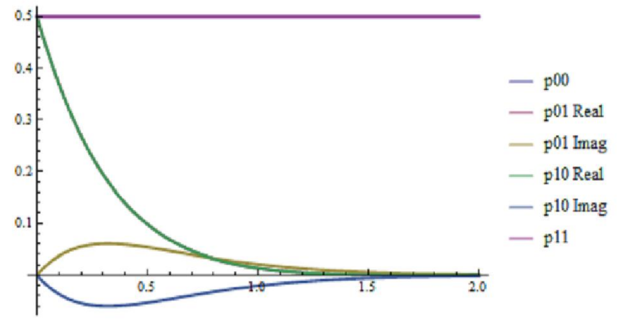

Figure 3. Solutions of Lindblad equation for Bloch equation in NMR with symmetric initial condition.

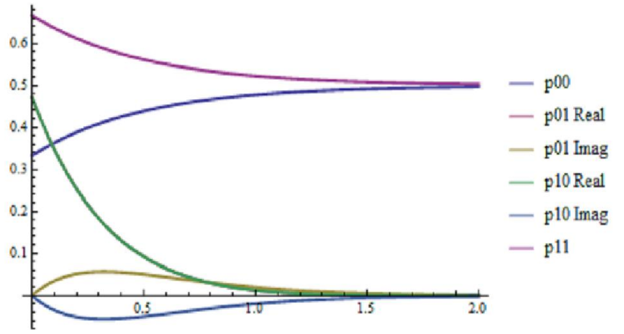

Figure 4. Solutions of Lindblad equation for Bloch equation in NMR with asymmetric initial condition.

Remark. *We present an analytic proof that the ρ_{00}, ρ_{11} are approaching to the same limit when it has the asymmetric initial condition as $t \to \infty$. Let us substitute the asymmetric initial condition into the equation (13) and equation (16), we have*

$$\lim_{t\to\infty} \rho_{00}(t) = \frac{1}{2} - \frac{e^{-2t}}{6} = \frac{1}{2} \quad (17)$$

$$\lim_{t\to\infty} \rho_{11}(t) = \frac{1}{2} + \frac{e^{-2t}}{6} = \frac{1}{2} \quad (18)$$

It is proven that ρ_{00} and ρ_{11} are approaching the same limit at 0.5 as $t \to \infty$.

Analytically, we can prove $\rho(t)$ is a diagonal matrix as $t \to \infty$ as follows

$$\lim_{t\to\infty} \rho_{01}(t) = \lim_{t\to\infty} (\rho_{01}(0) \frac{\cos(\omega_0 t)}{e^{(\frac{\Gamma_+}{2} + \frac{\Gamma_-}{2} + 2\Gamma_z)t}} + \rho_{01}(0) i \frac{\sin(\omega_0 t)}{e^{(\frac{\Gamma_+}{2} + \frac{\Gamma_-}{2} + 2\Gamma_z)t}}) = 0 \quad (19)$$

$$\lim_{t\to\infty} \rho_{10}(t) = \lim_{t\to\infty} (\rho_{10}(0) \frac{\cos(\omega_0 t)}{e^{(\frac{\Gamma_+}{2} + \frac{\Gamma_-}{2} + 2\Gamma_z)t}} - \rho_{10}(0) i \frac{\sin(\omega_0 t)}{e^{(\frac{\Gamma_+}{2} + \frac{\Gamma_-}{2} + 2\Gamma_z)t}}) = 0. \quad (20)$$

We present the graph of the von Neumann entropy for the density matrix in the Figure 5.

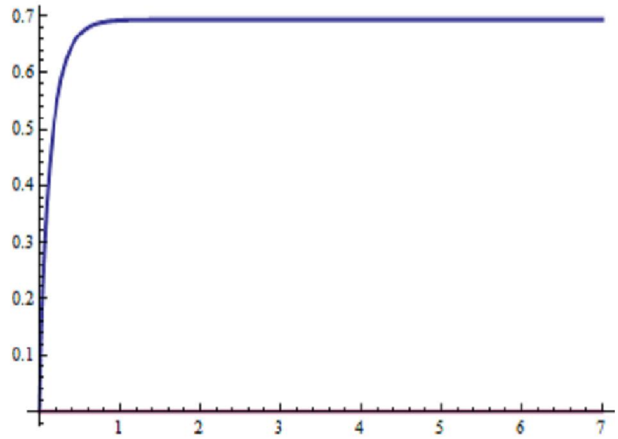

Figure 5. von Neumann entropy of the density matrix for Bloch equation.

Remark. *Note that the von Neumann entropy increases and being convergent after some time. This is because ρ_{00} and ρ_{11} are approaching to the same limit and stable after some time. Therefore, the limiting density matrix describes the quantum system in a mixed state. Since the probabilities for ρ_{00} and ρ_{11} are the same with 50%. According to our cognitive model (See Section 2.2), the player has the same probability of either choosing 0 or choosing 1. Probabilistically, each of the decisions can be made by the player.*

Example 2.4 *Let us assume the Hamiltonian is $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and Lindblad operator is $A = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$, then we can write the corresponding Lindblad equation*

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix} \quad (21)$$

$$= -i[H,\rho] + A\rho A^{\dagger} - \frac{1}{2}(A^{\dagger}A\rho + \rho A^{\dagger}A)$$

$$= -i\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\rho - \rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\} + \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}\rho\begin{pmatrix}0 & 0\\ -i & 0\end{pmatrix}$$

$$-\frac{1}{2}\left\{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\rho + \rho\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \quad (22)$$

$$= \begin{pmatrix}\rho_{11} & -\frac{1}{2}\rho_{01}\\ -\frac{1}{2}\rho_{10} & -\rho_{11}\end{pmatrix} \quad (23)$$

The analytic solution has the form

$$\rho_{00}(t) = \rho_{00}(0) + \rho_{11}(0) - \rho_{11}(0)\exp(-t) \quad (24)$$

$$\rho_{01}(t) = \rho_{01}(0)\exp(-\frac{1}{2}t) \quad (25)$$

$$\rho_{10}(t) = \rho_{10}(0)\exp(-\frac{1}{2}t) \quad (26)$$

$$\rho_{11}(t) = \rho_{11}(0)\exp(-t). \quad (27)$$

With the assumed symmetric initial condition $\rho(0) = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, we present the graph for the solutions as $t \to \infty$ in the Figure 2.4.

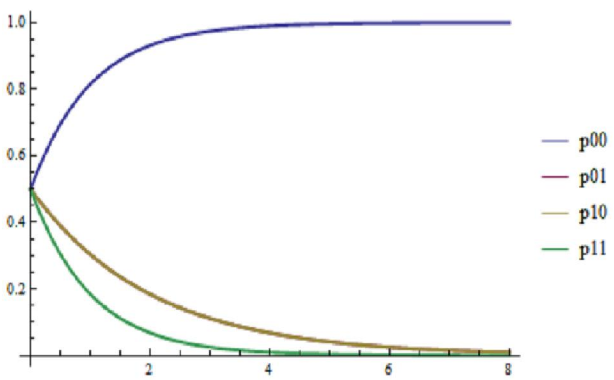

Figure 6. Solutions of Lindblad equation.

Analytically, we can prove $\rho(t)$ is a diagonal matrix as $t \to \infty$ as follows

$$\lim_{t\to\infty}\rho_{01}(t) = \lim_{t\to\infty}(\rho_{01}(0)\frac{1}{e^{\frac{1}{2}t}}) = 0 \quad (28)$$

$$\lim_{t\to\infty}\rho_{10}(t) = \lim_{t\to\infty}(\rho_{10}(0)\frac{1}{e^{\frac{1}{2}t}}) = 0. \quad (29)$$

We present the graph of the von Neumann entropy for the density matrix in the Figure 2.4

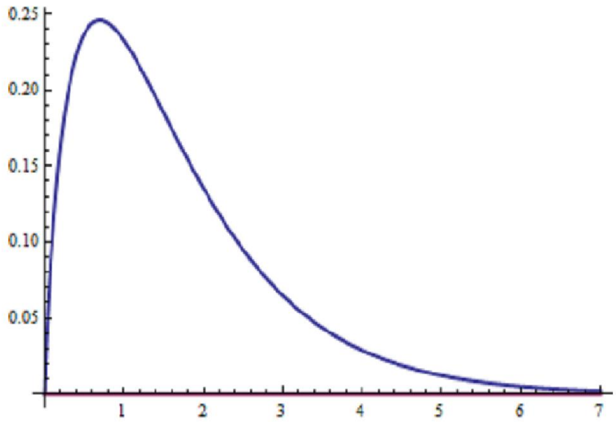

Figure 7. von Neumann entropy of the density matrix.

Remark. *Note that the von Neumann entropy increases at the beginning since the non-diagonal entries vanished slower than the diagonal entries. But after some time, the final behavoir is that all entries except* ρ_{00} *decreases down to* 0. *Therefore, the limiting density matrix describes the quantum system in a pure state.*

Example 2.5 *Let us assume the Hamiltonian is* $H = \tau\sigma_x$ *where* $\tau > 0$ *and* σ_x *is a Pauli matrix* $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, *and the Linblad operator is* $A = \tau\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. *Then we write the corresponding Lindblad equation*

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix} = -i[H,\rho] + A\rho A^{\dagger} - \frac{1}{2}(A^{\dagger}A\rho + \rho A^{\dagger}A) \quad (30)$$

$$= -\tau i\begin{pmatrix}\rho_{10}-\rho_{01} & \rho_{11}-\rho_{00}\\ \rho_{00}-\rho_{11} & \rho_{01}-\rho_{10}\end{pmatrix} + \tau^2\begin{pmatrix}\rho_{11} & -\frac{1}{2}\rho_{01}\\ -\frac{1}{2}\rho_{10} & -\rho_{11}\end{pmatrix} \quad (31)$$

Note that the analytic solutions are too complicated to deal with, then it is better to present the numerical solutions and the von Neumann entropy is still possible to be calculated numerically.

With the assumed symmetric initial condition $\rho(0)=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, we present the graph for the solutions as $t\to\infty$ in the Figure 2.5. We present the graph of the von Neumann entropy for the density matrix in the Figure 2.5

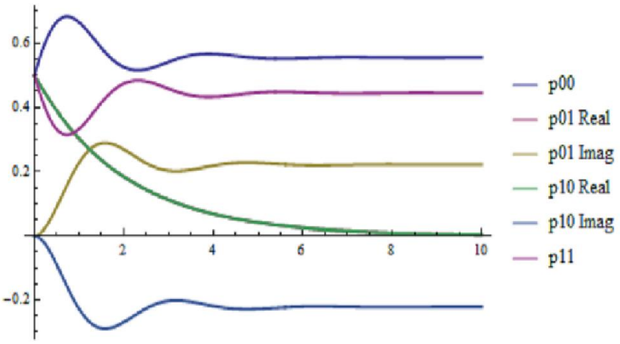

Figure 8. Solutions of Lindblad equation.

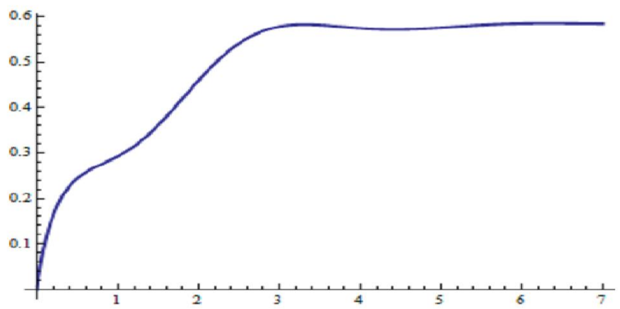

Figure 9. von Neumann entropy of the density matrix.

Remark. *Note that the von Neumann entropy increases, it means the limiting density matrix describes the quantum system in a mixed state, as it shown in the Figure 2.5. In our cognitive model, we can consider the corresponding pure state with the higher probability after a quantum measurement will refer to the decision of the player.*

2.2.2. Quantum Decoherence and Quantum Darwinism

For an open quantum system, the quantum decoherence leads to the collapse of the wave function. The coherence of superpositions has a very short decay time scale. In Copenhagen interpretation of quantum mechanics, the wave function collapse allows the observer gets the appearance of the pure states probabilistically. In addition, it also provides a process of selecting pointer basis. In quantum mechanics, it is still an open problem about how the wave function collapse really happens which is called the measurement problem. It is infeasible to observe internally, therefore many different interpretations of quantum mechanics are provided.

A canonical example is a thought experiment called Schrödinger's Cat such that there are a cat, a glass of poisonous gas and a radioactive switch for the glass in an isolated box. If a single atom decay is detected, then the radioactive switch will be opened and the poisonuous gas will be released. As a result, the cat will be killed. According to the Copenhagen interpretation, after some time, the cat will be in a quantum superposition of two pure states, i.e. alive and dead. Due to the quantum decoherence, the cat will not persist forever on the superposition. It will be destroyed, as the observation in our classical world, the cat can only be in either alive or dead, but cannot be both. Then the question is, when and how exactly the quantum superposition is destroyed, i.e. wave function collapse, where the quantum state vectors are evolved in time by the Schrödinger equation.

Zurek proposed (Wheeler and Zurek, 2014) that the process of selecting the quantum states which leads to the stabilization of the pointer states is analogous to the Darwinian natural selection. It provides a possible explanation for the emergence of the observation in the classical world from the quantum world. It is called Quantum Darwinism.

In an open quantum system, a selection process so-called einselection (Environment-induced superselection) transforms the superpositions of a quantum system to a reduced set of the pointer states, and the preferred basis after decoherence is the pointer basis which interprets the classical observation.

These pointer states are selected in the way analogous to the Darwinian natural selection. Let us introduce the Darwinian algorithm as follows

1. Reproduction: Implementing copies to generate descendants.

2. Selection: For the enriched trait in the population after generations, it is preferred to be selected over other traits.

3. Variation: Herited trait difference affecting fitness

Analogously, we present the Quantum Darwinism as follows

1. Copies consisting of pointer states

2. Evolution of pointer states is continuous and predictable such that the trait inheritance for the descendants is from ancestor states.

3. Environmental interactions provide evolution and the survived states corresponds to the predictable observations in the classical world.

In our quantum-like decision making model, we expect the player's final decision is made by the quantum measurement on the pointer basis, where the quantum decoherence leads to the stabilization of the density matrix which is evolved by the Lindblad equation.

When the density matrix is stabilized at an equilibrium state and being diagonal in the limit as $t \to \infty$, we can consider the comparison process in the player's mental dynamics is terminated. In the ideal mathematical model, the non-diagonal elements approach zero only in the limit $t \to \infty$. This situation physically and psychologically correspond to the presence of two time scales determined by the process of decision making, the very fine time scale of the processing of the mental state and the rough time scale of the conscious decision making, respectively. Then the final decision is made probabilistically by performing a quantum measurement as follows

$$|\Phi_0|^2 = \mathrm{tr}\{0_A 0_A | \rho_{out} | 0_A 0_A\} \quad (32)$$

$$|\Phi_1|^2 = \mathrm{tr}\{1_A 1_A | \rho_{out} | 1_A 1_A\}, \quad (33)$$

where $|\Phi_0|^2, |\Phi_1|^2$ are the probabilities for choosing 0 and 1, respectively.

Addtionally, the mental dynamics in the comparison process can explain why there are irrational behaviors.

Now let us reformulate the presented examples of the Lindblad equation in the last chapter as the system of linear differential equations and explain the meaning to our corrsponding cognitive models.

For the example 2.2, we present the system of differential equations

$$\frac{\partial}{\partial t}\rho_{00}(t) = \Gamma \rho_{11}(t) \quad (34)$$

$$\frac{\partial}{\partial t}\rho_{01}(t) = (i\omega_0 - \frac{\Gamma}{2})\rho_{01}(t) \quad (35)$$

$$\frac{\partial}{\partial t}\rho_{10}(t) = (-i\omega_0 - \frac{\Gamma}{2})\rho_{10}(t) \quad (36)$$

$$\frac{\partial}{\partial t}\rho_{11}(t) = -\Gamma \rho_{11}(t) \quad (37)$$

and the limiting density matrix

$$\lim_{t\to\infty}\begin{pmatrix}\rho_{00}(t) & \rho_{01}(t)\\ \rho_{10}(t) & \rho_{11}(t)\end{pmatrix} = \begin{pmatrix}\rho_{00}(0)+\rho_{11}(0) & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \quad (38)$$

In this system of differential equations with the initial conditions as it shown in the section of examples, the corresponding probability for ρ_{11} will decrease down to 0, and the corresponding probability for ρ_{00} will increase, where the speed slows down when time t increases. For ρ_{01}, ρ_{10}, their imaginary parts will fluctuate at the beginning and stabilize after some time t. Their real parts will decrease down to 0 respect to time t. This dynamical process is shown in the Figure 1, and it can be regarded as the player's mental dynamics for the comparison process of all possible consequences. In the last part of the example 2.2, we have proven that $\rho(t)$ will be a diagonal matrix as $t \to \infty$. As it shown in the equation (38), the limiting density matrix is describing the pure state. When we consider the comparison process is terminated, we can see that after performing the quantum measurement by applying the equation (32) and the equation (33), the player will have 100% to make the decision corresponding to the choice 0 and 0% for the decision corresponding to the choice 1.

For the example 2.3, we present the system of differential equations

$$\frac{\partial}{\partial t}\rho_{00}(t) = -\Gamma_{-}\rho_{00}(t) + \Gamma_{+}\rho_{11}(t) \quad (39)$$

$$\frac{\partial}{\partial t}\rho_{01}(t) = (i\omega_0 - \frac{\Gamma_{+}}{2} - \frac{\Gamma_{-}}{2} - 2\Gamma_z)\rho_{01}(t) \quad (40)$$

$$\frac{\partial}{\partial t}\rho_{10}(t) = (-i\omega_0 - \frac{\Gamma_{+}}{2} - \frac{\Gamma_{-}}{2} - 2\Gamma_z)\rho_{10}(t) \quad (41)$$

$$\frac{\partial}{\partial t}\rho_{11}(t) = \Gamma_{-}\rho_{00}(t) - \Gamma_{+}\rho_{11}(t) \quad (42)$$

and the limiting density matrix

$$\lim_{t\to\infty}\begin{pmatrix}\rho_{00}(t) & \rho_{01}(t)\\ \rho_{10}(t) & \rho_{11}(t)\end{pmatrix} = \begin{pmatrix}\frac{\rho_{00}(0)+\rho_{11}(0)}{2} & 0\\ 0 & \frac{\rho_{00}(0)+\rho_{11}(0)}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \quad (43)$$

In this system of differential equations with the initial conditions as it shown in the section of examples, the corresponding probability for ρ_{00} and ρ_{11} is analogous to the chemical equilibrium. The increment of the corresponding probability for ρ_{00} is proportional to the value of ρ_{11} and for ρ_{11} it is inversely proportional to the value of ρ_{00} respect to the time t. This will finally lead to an equilibrium. For ρ_{01}, ρ_{10}, their imaginary parts will fluctuate at the beginning and stabilize after some time t and their real parts will decrease down to 0 respect to the time t. This dynamical process is shown in the Figure 2.3 and it can be regarded as the player's mental dynamics for the comparison process of all possible consequences. In the last part of the example 2.3, we have proven that $\rho(t)$ will be a diagonal matrix as $t \to \infty$. As it shown in the equation (43), the limiting density matrix is describing the mixed state. When we consider the comparison process is terminated, we can see that after performing the quantum measurement by applying the equation (32) and the equation (33), the player will have 50% to make the decision corresponding to the choice 0 and 50% for the decision corresponding to the choice 1.

For the example 2.4, we present the system of differential equations

$$\frac{\partial}{\partial t}\rho_{00}(t) = \rho_{11}(t) \quad (44)$$

$$\frac{\partial}{\partial t}\rho_{01}(t) = -\frac{1}{2}\rho_{01}(t)$$
(45)

$$\frac{\partial}{\partial t}\rho_{10}(t) = -\frac{1}{2}\rho_{10}(t) \quad (46)$$

$$\frac{\partial}{\partial t}\rho_{11}(t) = -\rho_{11}(t) \quad (47)$$

and the limiting density matrix

$$\lim_{t\to\infty}\begin{pmatrix}\rho_{00}(t) & \rho_{01}(t)\\ \rho_{10}(t) & \rho_{11}(t)\end{pmatrix} = \begin{pmatrix}\rho_{00}(0)+\rho_{11}(0) & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad (48)$$

The situation for this example is similar to that of the example 2.2. For the example 2.5, we present the system of differential equations

$$\frac{\partial}{\partial t}\rho_{00}(t) = \tau i\rho_{01}(t) - \tau i\rho_{10}(t) + \tau^2\rho_{11}(t) \quad (49)$$

$$\frac{\partial}{\partial t}\rho_{01}(t) = \tau i\rho_{00}(t) - \frac{\tau^2}{2}\rho_{01}(t) - \tau i\rho_{11}(t) \quad (50)$$

$$\frac{\partial}{\partial t}\rho_{10}(t) = -\tau i\rho_{00}(t) - \frac{\tau^2}{2}\rho_{10}(t) + \tau i\rho_{11}(t) \quad (51)$$

$$\frac{\partial}{\partial t}\rho_{11}(t) = -\tau i\rho_{01}(t) + \tau i\rho_{10}(t) - \tau^2\rho_{11}(t) \quad (52)$$

and the approximated limiting density matrix from the numerical simulation with the symmetric initial condition (since we do not have the analytic solution)

$$\lim_{t\to\infty}\begin{pmatrix}\rho_{00}(t) & \rho_{01}(t)\\ \rho_{10}(t) & \rho_{11}(t)\end{pmatrix} \approx \begin{pmatrix}0.555294 & 0.0055545+0.222162i\\ 0.0055545-0.222162i & 0.444706\end{pmatrix} \quad (53)$$

In this system of differential equations with the initial conditions as it shown in the section of examples, for the real part, it is also similar to that of the example 2.2 and example 2.4. For the imaginary parts of each one, there will be fluctuation at the beginning and stabilize after some time t. This dynamical process is shown in the Figure 2.5 and it can be regarded as the player's mental dynamics for the comparison process of all possible consequences. Since the analytic solution for this system is too complicated to deal with, we can refer to the graphically numerical solutions to predict the dynamical behaviors as $t \to \infty$ .As it shown in the equation (53), the limiting density matrix is describing the mixed state. When we consider the comparison process is terminated, we can see that after performing the quantum measurement by applying the equation (32) and the equation (33), the player will approximately have 56% to make the decision corresponding to the choice 0 and 46% for the decision corresponding to the choice 1.

Remark. *The decision making process is closely related to the quantum decoherence where the stabilized solution is required. Therefore, not all the Lindblad operators will be useful in general. For an equilibrium state of* *the mental dynamics, we need the Lindblad equation with the stabilizable solution.*

3. Conclusion

The main aim of this paper is to analyze a decision making model proposed by Asano, Khrennikov and Ohya (Asano *et al.*, 2011c; Khrennikov, 2011b), and based on the representation of processing of mental information with the aid of the mathematical formalism of the theory of open quantum systems. For Prisoner's Dilemma game, we model the processes of decision making by using the Lindblad equation to describe the dynamics of player's mental state("belief-state").

Then we illustrate the functioning of the Asano-Khrennikov-Ohya decision making model for the game of Prisoner's Dilemma by using the result of the numerical and analytical studies of the solutions of concrete Lindblad equations. We note that a decision maker is considered as a self-observer such that it can guess the other's decision. This information structure is regarded as an open quantum system(information). Such a mental reservoir can be generated by itself, and the (information)interaction is between the own decision and mental self-image of possible actions of another player. The dynamics in this open quantum system is described by the Lindblad equation, and only the specific Lindblad operators will be useful if they provide the stabilization of solutions. When the quantum decoherence leads to a thermal equilibrium state, it is considered as the termination of the comparison process by the player. Then the player's pure strategy can be selected probabilistically by performing the quantum measurement in the pointer basis. Such quantum decoherence is crucial and essential for the decision making process, thus in general not all the Lindblad operators will work.

In classical game theory, a rational behavior of the player is to select the strategies corresponding to the Nash equilibrium. Specifically, in Prisoner's Dilemma, the behavoir of a player is considered as rational if the defect is decided. But in real experiments in cognitive psychology which were performed by Tversky and Shafir (Shafir and Tversky, 1992a, 1992b), the statistical data show that there exists irrational behavoir. Therefore, the Asano-Khrennikov-Ohya decision making model analyzed in this paper gives a possible explanation why it probably happens.

We remark that the model in this paper is based on only the mathematical apparatus of QM, but not the real quantum physical model. In fact, not only we do not present any corresponding relation to the neural basis for the Asano-Khrennikov-Ohya model, but also it is still an unsolved problem whether or not there really exists quantum phenomenon in the human brains. Such an argument started decades ago but still continue at present time. We refer to the works by Penrose and Hameroff (Penrose, 1989, 1994; Hameroff, 1994a, 1994b, 1996) who are considered as the representative supporters for the quantum brain theory. In their Orch-OR (Orchestrated Objective Reduction) model, they claim that the consciousness is generated by quantum gravity effects in the microtubules. As an opponent, Tegmark (Tegmark, 2000) argued that the degree of freedom of the humran brain should be regarded as a classical system, since according to the calculation, the quantum decoherence timescales are too much short to affect the brain function. However, we emphasize once again that the study in this paper has not any direct relation to quantum physical models of brain's functioning. The model under analysis is a purely informational model which is based on the assumption (confirmed by the experimental data from cognitive psychology) that information processing performed by complex cognitive systems can be modelled by using the mathematical formalism of theory of open quantum systems.

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Appendix

On (non)Stabilization of the solutions of the von Neumann equation

In this subsection, we study the conditions of the stabilization of the solutions of the von Neumann equation.

Theorem 4.1 *Let the time independent Hamiltonian H be a bounded Hermitian operator. If it commutes with the density matrix at intial time, i.e. H, i.e. $[H,\rho(0)]=0$, then the solution of the von Neumann equation is stabilized at all time scales.*

Proof. Given $[H,\rho(0)]=0$, we have

$$\frac{d}{dt}\rho(t)=-i[H,U(t,0)\rho(0)U^{\dagger}(t,0)] \quad (54)$$

$$=-iU(t,0)[H,\rho(0)]U^{\dagger}(t,0) \quad (55)$$

$$=0.$$

It implies that $\rho(t)=\rho(0)$ for any time t.

Remark. *Given $[H,\rho(0)]=0$, we have $H\rho(0)=\rho(0)H$. For the corresponding pure states $\psi(0)$, it follows $\sum_i w_i H\psi_i(0)\psi_i(0)=\sum_i w_i\psi_i(0)\psi_i(0)H$. And it implies each of the pure states is the eigenvector of the Hamiltonian H, e.g. $H\psi_i(0)=\lambda\psi_i(0)$.*

Theorem 4.2 *Let the Hamiltonian H be a bounded Hermitian operator. If $[H,\rho(0)]\neq 0$, then the solution of the von Neumann equation will never stabilize.*

Proof. For the simplification, we proceed in the case of the non-degenerate spectrum of the Hamiltonian H. The von Neumann equation has the form

$$\frac{d}{dt}\rho(t)=-ie^{-iHt}[H,\rho(0)]e^{iHt} \quad (56)$$

With the aid of the eigenstates (eigenvectors) n of the Hamiltonian, i.e. $Hn=E_n n$, where E_n denotes the energy eigenvalue corresponding to the eigenstate n, we can expand the density matrix as follows:

$$\rho(t)=e^{-iHt}(\sum_{m,n}\rho_{mn}(0)mn)e^{iHt} \quad (57)$$

$$=\sum_{m,n}\rho_{mn}(0)e^{-iHt}mne^{iHt} \quad (58)$$

$$=\sum_{m,n}\rho_{mn}(0)e^{-iE_m t}mne^{iE_n t} \quad (59)$$

$$=\sum_{m,n}\rho_{mn}(0)e^{-i(E_m-E_n)t}mn \quad (60)$$

where $\rho_{mn}(0)$ are the elements in the initial density matrix.

Given the condition that the Hamiltonian does not communte with the initial density matrix, i.e. $[H,\rho(0)]\neq 0$, it follows that

$$H\rho(0)=\sum_{m,n}\rho_{mn}(0)Hmn=\sum_{m,n}\rho_{mn}(0)E_m mn \quad (61)$$

$$\rho(0)H=\sum_{m,n}\rho_{mn}(0)mnH=\sum_{m,n}\rho_{mn}(0)mnE_n \quad (62)$$

Since we assume that the spectrum is non-degenerate, i.e. $E_m\neq E_n$ for $m\neq n$, we have the commutator in the form

$$\begin{aligned}&[H,\rho(0)]\\&=H\rho(0)-\rho(0)H\\&=\sum_{m\neq n}(E_m-E_n)\rho_{mn}(0)mn\end{aligned} \quad (63)$$

where $(E_m-E_n)\neq 0$. Since $[H,\rho(0)]\neq 0$, it implies that at least one coefficient $\rho_{mn}(0)$ with $m\neq n$ has to be nonzero. The corresponding phase factor in the equation (60) will oscillate forever at the frequency E_m-E_n.

We conclude that the solution of the von Neumann equation will never stabilize.

Remark. *Theorem 4.1 and Theorem 4.2 show that Schrödinger-von Neumann dynamics cannot be used to model the process of approaching of some stationary state from a non-stationary ones. Oscillating behavior described in 4.2 is one of the reasons to use the theory of open quantum systems based on the Lindblad equation and leading (for some generators) to stabilization regimes.*

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