Isomorphic Properties of m-Bipolar Fuzzy Graphs

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Abstract

Various concepts of fuzzy graph theory will find applications in many different fields such as image capturing, image segmentation, networking, data mining, planning (landscape connectivity, air lines connectivity etc.), scheduling, clustering, artificial intelligence, decision making, multi agent system and auto meta theory. In this article, we introduced the notion of order, size, free and busy vertices, etc. of m-bipolar fuzzy graphs (m-BPFGs) and showed that isomorphic m-BPFGs have same degree, size and order. In addition we studied various properties of busy vertices in isomorphism and weak isomorphism (w-isomorphism) of m-BPFGs. Comparative studies of operations and complements on m-BPFGs has been carried out with suitable examples.

Key Words: M-BPFG, busy vertices, isomorphism, complement

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Introduction

FSs have been used in different domains in order to resolve issues associated with doubt and uncertainty in day to day applications of living conditions as shown by Zadeh [16] in 1965. The constraints in earlier model can be overcomes with the introduction of BFS idea in 1994 by Zhang [17]. Some concepts were later revamped by Chen et al. [4] into m-PF set theory.

It is well known that, a "graph" is a collection of points (known as vertices) and the lines between those points (known as edges) which can be used to characterize a physical situation comprising discrete objects with a relationship. In a view of its simplicity, the graph theory has various applications like analysing data, image segmentation, networking, clustering, planning, communication etc. But, in some cases, these graphs are unable to accurately represent several practical phenomena due to the ambiguity of different attributes and vagueness of the systems. This has led to define the FGs to over-come efficiently most of the real-world problems. In "FGs", assigned values of vertices and edges removes uncertainty in the physical problems.

Rosenfeld [13] paved the path for the idea of fuzzy vertex, edges, path, subgraph and also complement of a FG. The works of Akram [1, 2] played a essential role in studying various major properties of BFGs, interval-valued FGs. Samanta and Pal [15] extended this FG theory technique to fuzzy planar graphs in order to other complex problems related to image segmentation using kernel contraction method. Ghorai and Pal [5–9] introduced the technique of generalized m-PFGs, planar graphs.

Talebi et al. [14] initiated edge regularity in m-polar interval-intuitionistic FGs. Further, Bera and Pal [3] studied about the statistical expressions like irregularity, regularity and density on these graphs. Mahapatra et al. [10, 11] investigated m-PF threshold graphs as well as their application on RPCS- resource power controlling system and interval-valued m-PF planar graphs. Ramakrishna et al.[12] gave the mark on the concept of m-BPFG, edge regularity on m-BPFGs with suitable examples. m-BPFGs and related definitions can refer in [12].

Preliminaries

The dimension of a graph is represented by its order and size. Generally, different operations are defined for different FGs. The order and size of a m-BPFG are defined below.

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Def 3.1. The order of the m-BPFG $G=(V,Q,R)$ is defined as

$$|V|=O(G)=\sum_{q\in V}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_Q^{+}(q)-P_h\bullet\overleftarrow{\Psi}_Q^{-}(q)}{2}.$$

The size of G is defined as $$|E|=S(G)=\sum_{qr\in E}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_R^{+}(qr)-P_h\bullet\overleftarrow{\Psi}_R^{-}(qr)}{2}.$$

Theorem 3.1. Two IMPC m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of G_1^* and G_2^* have same order and size.

Proof. Let Φ be an IMPS from G_1 onto G_2.

Then $P_h\bullet\overrightarrow{\Psi}_{Q_1}^{+}(q)=P_h\bullet\overrightarrow{\Psi}_{Q_2}^{+}(\Phi(q)),\ P_h\bullet\overleftarrow{\Psi}_{Q_1}^{-}(q)=P_h\bullet\overleftarrow{\Psi}_{Q_2}^{-}(\Phi(q))$ for all $q\in V_1$ and $P_h\bullet\overrightarrow{\Psi}_{R_1}^{+}(qr)=P_h\bullet\overrightarrow{\Psi}_{R_2}^{+}(\Phi(q)\Phi(r)),\ P_h\bullet\overleftarrow{\Psi}_{R_1}^{-}(qr)=P_h\bullet\overleftarrow{\Psi}_{R_2}^{-}(\Phi(q)\Phi(r)),$ for all $qr\in \mathrm{V}_1^2$, $h=1\ to\ m.$

Now $$O(G_1)=|V_1|=\sum_{q\in V_1}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_{Q_1}^{+}(q)-P_h\bullet\overleftarrow{\Psi}_{Q_1}^{-}(q)}{2}$$

$$=\sum_{\Phi(q)\in V_2}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_{Q_2}^{+}(\Phi(q))-P_h\bullet\overleftarrow{\Psi}_{Q_2}^{-}(\Phi(q))}{2}=O(G_2)$$ and

$$S(G_1)=|E_1|=\sum_{qr\in E_1}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_{R_1}^{+}(qr)-P_h\bullet\overleftarrow{\Psi}_{R_1}^{-}(qr)}{2}$$

$$=\sum_{\Phi(q)\Phi(r)\in E_2}\frac{1+\sum_{h=1}^{m}P_h\bullet\overrightarrow{\Psi}_{R_2}^{+}(\Phi(q)\Phi(r))-P_h\bullet\overleftarrow{\Psi}_{R_2}^{-}(\Phi(q)\Phi(r))}{2}=S(G_2).$$

Def 3.2. The busy value (BVL) of a vertex $q\in V$ of an m-BPFG G is defined as $D(q)=\left\langle\left[P_h\bullet\overrightarrow{D}^{+}(q),P_h\bullet\overleftarrow{D}^{-}(q)\right]_{h=1}^{m}\right\rangle$ where $P_h\bullet\overrightarrow{D}^{+}(q)=\sum_j\min\left\{P_h\bullet\overrightarrow{\Psi}_Q^{+}(q),P_h\bullet\overrightarrow{\Psi}_Q^{+}(q_j)\right\},$ $P_h\bullet\overleftarrow{D}^{-}(q)=\sum_j\max\left\{P_h\bullet\overleftarrow{\Psi}_Q^{-}(q),P_h\bullet\overleftarrow{\Psi}_Q^{-}(q_j)\right\};$ q_j are the neighbors of q. The BVL of G is defined as $D(G)=\sum_j D(q_j),\ q_j\in V.$

Def 3.3. A vertex $q\in V$ of G is said to be a busy vertex (BV) if $P_h\bullet\overrightarrow{\Psi}_Q^{+}(q)\le P_h\bullet\overrightarrow{d}_G^{+}(q)$ and $P_h\bullet\overleftarrow{\Psi}_Q^{-}(q)\ge P_h\bullet\overleftarrow{d}_G^{-}(q)$ for $h=1\ to\ m$ Or else, it is a free vertex (FV).

Def 3.4. If $P_h\bullet\overrightarrow{\Psi_R^+}(q_1r_1)=\min\left\{P_h\bullet\overrightarrow{\Psi_Q^+}(q_1),P_h\bullet\overrightarrow{\Psi_Q^+}(r_1)\right\}$, $P_h\bullet\overleftarrow{\Psi_R^-}(q_1r_1)=\max\left\{P_h\bullet\overleftarrow{\Psi_Q^-}(q_1),P_h\circ\overleftarrow{\Psi_Q^-}(r_1)\right\}$, $h=1\ to\ m$ for $q_1r_1\in E$, then it is called an effective edge of G.

Def 3.5. Let $q\in V$ be a vertex of the m-BPFG $G=(V,Q,R)$.

q is said to be a partial FV if q is a FV of G and $\overline{G}$

q is said to be a fully FV if it q is a FV of G and q is a BV of $\overline{G}$

q is said to be a partial BV if q is a BV of G and $\overline{G}$

q is said to be a fully BV if q is a BV of G and it is a FV of $\overline{G}$.

Ex 3.1. Consider two BPFG $G=(V,Q,R)$ of G^* where $V=\{q,r,s\}$, $E=\{qr,rs,sq\}$,

$$Q=\left\{\frac{\langle[0.8,-0.3],[0.4,-0.6]\rangle}{q},\frac{\langle[0.4,-0.3],[0.6,-0.8]\rangle}{r},\frac{\langle[0.9,-0.1],[0.6,-0.8]\rangle}{s}\right\}$$

and

$$R=\left\{\frac{\langle[0.2,-0.2],[0.3,-0.5]\rangle}{qr},\frac{\langle[0.3,-0.1],[0.5,-0.7]\rangle}{rs},\frac{\langle[0.4,-0.1],[0.3,-0.2]\rangle}{sq}\right\}.$$

Then Figure 1., we have $D(q)=\langle[1.2,-0.4],[0.8,-1.2]\rangle$, $D(r)=\langle[0.8,-0.4],[1.0,-1.4]\rangle$, $D(s)=\langle[1.2,-0.2],[1.0,-1.4]\rangle$.

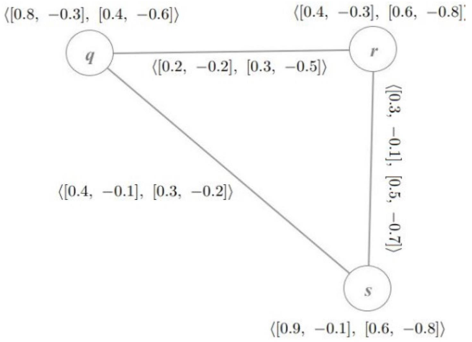


Figure.1. 2-BPFG G and BV of its vertices

Theorem 3.2. Let Φ be an IMPS from $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$. Then, $d_{G_1}(q)=d_{G_2}(\Phi(q))$ for all $q\in V_1$.

Proof. Here Φ is an IMPS between G_1 and G_2, we get $P_h\bullet\overrightarrow{\Psi_{Q_1}^+}(q)=P_h\bullet\overrightarrow{\Psi_{Q_2}^+}(\Phi(q))$, $P_h\bullet\overleftarrow{\Psi_{Q_1}^-}(q)=P_h\bullet\overleftarrow{\Psi_{Q_2}^-}(\Phi(q))$ for all $q\in V_1$ and $P_h\bullet\overrightarrow{\Psi_{R_1}^+}(qr)=P_h\bullet\overrightarrow{\Psi_{R_2}^+}(\Phi(q)\Phi(r))$, $P_h\Box\overleftarrow{\Psi_{R_1}^-}(qr)=P_h\bullet\overleftarrow{\Psi_{R_2}^-}(\Phi(q)\Phi(r))$ for all $qr\in\overleftrightarrow{V_1^2}$, $h=1\ to\ m.$

Therefore,

$$P_h\bullet\overrightarrow{d_{G_1}^+}(q)=\sum_{\substack{q\neq r,\\ qr\in E_1}}P_h\bullet\overrightarrow{\Psi_{R_1}^+}(qr)=\sum_{\substack{\Phi(q)\neq\Phi(r),\\ \Phi(q)\Phi(r)\in E_2}}P_h\bullet\overrightarrow{\Psi_{R_2}^+}(\Phi(q)\Phi(r))=P_h\bullet\overrightarrow{d_{G_2}^+}(\Phi(q)).$$

Similarly, $P_h\bullet\overleftarrow{d_{G_1}^-}(q)=P_h\bullet\overleftarrow{d_{G_2}^-}(\Phi(q))$ for $q\in V_1$, $h=1\ to\ m.$

Hence, $d_{G_1}(q)=d_{G_2}(\Phi(q))$ for all $q \in V_1$.

Theorem 3.3. If Φ is an IMPS between G_1 onto G_2 and q is a BV of G_1, then $\Phi(q)$ is a BV of G_2.

Proof. Here Φ is an IMPS between G_1 and G_2, we get $P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) = P_h \bullet \overrightarrow{\Psi_{Q_2}^{+}}(\Phi(q))$, $P_h \bullet \overleftarrow{\Psi_{Q_1}^{-}}(q)=P_h \bullet \overleftarrow{\Psi_{Q_2}^{-}}(\Phi(q))$ for all $q \in V_1$ and $P_h \bullet \overrightarrow{\Psi_{R_1}^{+}}(qr)=P_h \bullet \overrightarrow{\Psi_{R_2}^{+}}(\Phi(q)\Phi(r))$, $P_h \bullet \overleftarrow{\Psi_{R_1}^{-}}(qr)=P_h \bullet \overleftarrow{\Psi_{R_2}^{-}}(\Phi(q)\Phi(r))$ for all $qr \in \overleftrightarrow{V_1^2}$, $h=1 \textit{ to } m.$

If q is BV of G_1 then $P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) \leq P_h \bullet \overrightarrow{d_{G_1}^{+}}(q)$ and $P_h \bullet \overleftarrow{\Psi_{Q_1}^{-}}(q) \geq P_h \bullet \overleftarrow{d_{G_1}^{-}}(q)$, for $h=1 \textit{ to } m.$

Then, by the above theorem 3.2, $P_h \bullet \overrightarrow{\Psi_{Q_2}^{+}}(\Phi(q))=P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) \leq P_h \bullet \overrightarrow{d_{G_1}^{+}}(q) = P_h \bullet \overrightarrow{d_{G_2}^{+}}(\Phi(q))$, $P_h \bullet \overleftarrow{\Psi_{Q_2}^{-}}(\Phi(q))=P_h \bullet \overleftarrow{\Psi_{Q_1}^{-}}(q) \geq P_h \bullet \overleftarrow{d_{G_1}^{-}}(q) = P_h \bullet \overleftarrow{d_{G_2}^{-}}(\Phi(q))$, for $h=1 \textit{ to } m.$

Hence, $\Phi(q)$ is a BV in G_2.

Theorem 3.4. Let the two m-BPFGs G_1 and G_2 be w-isomorphic. If $q \in V_1$ is a BV of G_1, then the image of q under the w- IMPS is also BV in G_2.

Proof. Let $\Phi : V_1 \to V_2$ be a w- IMPS between G_1 and G_2. Then $P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) = P_h \bullet \overrightarrow{\Psi_{Q_2}^{+}}(\Phi(q))$, $P_h \bullet \overleftarrow{\Psi_{Q_1}^{-}}(q)=P_h \bullet \overleftarrow{\Psi_{Q_2}^{-}}(\Phi(q))$ for all $q \in V_1$ and $P_h \bullet \overrightarrow{\Psi_{R_1}^{+}}(qr) \leq P_h \bullet \overrightarrow{\Psi_{R_2}^{+}}(\Phi(q)\Phi(r))$, $P_h \bullet \overleftarrow{\Psi_{R_1}^{-}}(qr) \geq P_h \bullet \overleftarrow{\Psi_{R_2}^{-}}(\Phi(q)\Phi(r))$ for all $qr \in \overleftrightarrow{V_1^2}$, $h=1 \textit{ to } m.$

Let $q \in V_1$ be a BV.

Then, $P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) \leq P_h \bullet \overrightarrow{d_{G_1}^{+}}(q)$, $P_h \bullet \overleftarrow{\Psi_{Q_1}^{-}}(q) \geq P_h \bullet \overleftarrow{d_{G_1}^{-}}(q)$ for $h=1,2,\ldots,m.$

Now, by the above for $h=1 \textit{ to } m$, $P_h \bullet \overrightarrow{\Psi_{Q_2}^{+}}(\Phi(q))=P_h \bullet \overrightarrow{\Psi_{Q_1}^{+}}(q) \leq P_h \bullet \overrightarrow{d_{G_1}^{+}}(q)$

$$= \sum_{\substack{q \neq r, \\ qr \in E_1}} P_h \bullet \overrightarrow{\Psi_{R_1}^{+}}(qr) \leq \sum_{\substack{\Phi(q) \neq \Phi(r), \\ \Phi(q)\Phi(r) \in E_2}} P_h \bullet \overrightarrow{\Psi_{R_2}^{+}}(\Phi(q)\Phi(r)) = P_h \bullet \overrightarrow{d_{G_2}^{+}}(\Phi(q)).$$

Similarly, $P_h \bullet \overleftarrow{\Psi_{Q_2}^{-}}(\Phi(q))=P_h \bullet \overleftarrow{d_{G_2}^{-}}(\Phi(q)).$

Hence, $\Phi(q)$ is a BV in G_2.

4.Complement and IMPS in m-BPFGs

Some properties of w- IMPS, co w- IMPS and IMPS corresponding to complement are discussed in this section.

Def 4.1. The union $G_1 \cup G_2=(V_1 \times V_2, Q_1 \cup Q_2, R_1 \cup R_2)$ of 2 m-BPFGs $G_1=(V_1, Q_1, R_1)$ and $G_2=(V_2, Q_2, R_2)$ of G_1^* and G_2^* respectively and is defined as: for $h=1 \textit{ to } m.$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(q_1,q_2)=\begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q)\, if\ q\in V_1-V_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q)\, if\ q\in V_2-V_1, \\ \max\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q)\right\}, if\ q\in V_1\cap V_2 \end{cases}$$

$$P_h \bullet \overleftarrow{\Psi^{-}_{(Q_1\cup Q_2)}}(q_1,q_2)=\begin{cases} P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q)\, if\ q\in V_1-V_2 \\ P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(q)\, if\ q\in V_2-V_1 \\ \min\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q), P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(q)\right\}, if\ q\in V_1\cap V_2 \end{cases}$$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\cup R_2)}}(qr)=\begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr)\, if\ qr\in E_1-E_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}(qr)\, if\ q\in E_2-E_1 \\ \max\left\{P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr), P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}(qr)\right\}, if\ qr\in E_1\cap E_2 \end{cases}$$

$$P_h \bullet \overleftarrow{\Psi^{-}_{(R_1\cup R_2)}}(qr)=\begin{cases} P_h \bullet \overleftarrow{\Psi^{-}_{R_1}}(qr)\, if\ qr\in E_1-E_2 \\ P_h \bullet \overleftarrow{\Psi^{-}_{R_2}}(qr)\, if\ qr\in E_2-E_1 \\ \min\left\{P_h \bullet \overleftarrow{\Psi^{-}_{R_1}}(qr), P_h \bullet \overleftarrow{\Psi^{-}_{R_2}}(qr)\right\}, if\ qr\in E_1\cap E_2 \end{cases}$$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\cup R_2)}}(qr)=0,\ P_h \bullet \overleftarrow{\Psi^{-}_{(R_1\cup R_2)}}(qr)=0\ \ if\ \ qr\in \overleftrightarrow{V_1\times V_2^2}-E_1\cup E_2.$$

Def 4.2. The join $G_1+G_2=(V_1\cup V_2, Q_1+Q_2, R_1+R_2)$ of 2 m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of G_1^* and G_2^* respectively and is defined as: $h=1\ to\ m$

$P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1+Q_2)}}(q)=P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(q),\ P_h \bullet \overleftarrow{\Psi^{-}_{(Q_1+Q_2)}}(q)=P_h \bullet \overleftarrow{\Psi^{-}_{(Q_1\cup Q_2)}}(q)$ if $q\in V_1\cup V_2$.

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1+R_2)}}(qr)=P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\cup R_2)}}(qr), P_h \bullet \overleftarrow{\Psi^{-}_{(R_1+R_2)}}(qr)=P_h \bullet \overleftarrow{\Psi^{-}_{(R_1\cup R_2)}}(qr)$ if $qr\in E_1\cup E_2$.

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1+R_2)}}(qr)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\},\ P_h \bullet \overleftarrow{\Psi^{-}_{(R_1+R_2)}}(qr)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q), P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(r)\right\}$
$if\ \ qr\in E'$ where E' is the set of each and every one of the edges of the combination of the vertices of V_1 and V_2 assuming that $V_1\cap V_2=\phi$.

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1+R_2)}}(qr)=0, P_h \bullet \overleftarrow{\Psi^{-}_{(R_1+R_2)}}(qr)=0$ if $qr\in \overleftrightarrow{V_1\times V_2^2}-E_1\cup E_2\cup E'$.

Def 4.3. The composition $G_1[G_2]$ of 2 m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of G_1^* and G_2^* respectively and is defined as a triplet $(V_1\times V_2, Q_1\circ Q_2, R_1\circ R_2)$ such that for $h=1\ to\ m$

$P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q_1,q_2)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q_2)\right\}$

$P_h \bullet \overleftarrow{\Psi^{-}_{(Q_1 \circ Q_2)}}(q_1,q_2)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q_1), P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(q_2)\right\}$ for all $(q_1,q_2)\in V_1 \times V_2$,

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1 \circ R_2)}}\left((q,q_2)(q,r_2)\right)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}(q_2 r_2)\right\}$

$P_h \bullet \overleftarrow{\Psi^{-}_{(R_1 \circ R_2)}}\left((q,q_2)(q,r_2)\right)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q), P_h \bullet \overleftarrow{\Psi^{-}_{R_2}}(q_2 r_2)\right\}$ for all $q\in V_1, q_2 r_2 \in E_2$,

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1 \circ R_2)}}\left((q_1,s)(r_1,s)\right)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(q_1 r_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(s)\right\}$

$P_h \bullet \overleftarrow{\Psi^{-}_{(R_1 \circ R_2)}}\left((q_1,s)(r_1,s)\right)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{R_1}}(q_1 r_1), P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(s)\right\}$, for all $s\in V_2, q_1 r_1 \in E_1$,

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1 \circ R_2)}}\left((q_1,q_2)(r_1,r_2)\right)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q_2), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r_2), P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(q_1 r_1)\right\}$

$P_h \bullet \overleftarrow{\Psi^{-}_{(R_1 \circ R_2)}}\left((q_1,q_2)(r_1,r_2)\right)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(q_2), P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(r_2), P_h \bullet \overleftarrow{\Psi^{-}_{R_1}}(q_1 r_1)\right\}$, for all $\left((q_1,q_2)(r_1,r_2)\right)\in E^0 - E$, where $E=\left\{(q,q_2)(q,r_2): q\in V_1, q_2 r_2 \in E_2\right\}\cup\left\{(q_1,s)(r_1,s): s\in V_2, q_1 r_1 \in E_1\right\}$ and $E^0 = E \cup \left\{(q_1,q_2)(r_1,r_2): q_1 r_1 \in E_1, q_2 \neq r_2\right\}$,

$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1 \circ R_2)}}\left((q_1,q_2)(r_1,r_2)\right)=0, P_h \bullet \overleftarrow{\Psi^{-}_{(R_1 \circ R_2)}}\left((q_1,q_2)(r_1,r_2)\right)=0,$ for all $(q_1,q_2)(r_1,r_2)\in \overleftrightarrow{V_1 \times V_2^2} - E^0$.

Def 4.4. An m-BPFG $G=(V,Q,R)$ of G^* is strong if for all $qr\in E$ and $h=1\ to\ m$ holding $P_h \bullet \overrightarrow{\Psi^{+}_{R}}(qr)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q}}(r)\right\}$, $P_h \bullet \overleftarrow{\Psi^{-}_{R}}(qr)=\max\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q}}(q), P_h \bullet \overleftarrow{\Psi^{-}_{Q}}(r)\right\}$.

Def 4.5. Let $G=(V,Q,R)$ be an m-BPFG of G^*. Then the complement of an m-BPFG G is $\overline{G}=\left(V,\overline{Q},\overline{R}\right)$ of $\overline{G^*}=\left(V,\overleftrightarrow{V^2}\right)$ such that $\overline{Q}=Q$, $\overline{R}$ is stated as $\overline{R}(qr)=\left\langle\left[P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R}}}(qr), P_h \bullet \overleftarrow{\Psi^{-}_{\overline{R}}}(qr)\right]_{h=1}^{m}\right\rangle$,

$P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R}}}(qr)=\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q}}(q) \wedge P_h \bullet \overrightarrow{\Psi^{+}_{Q}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R}}(qr)$,

$P_h \bullet \overleftarrow{\Psi^{-}_{\overline{R}}}(qr)=\left\{P_h \bullet \overleftarrow{\Psi^{-}_{Q}}(q) \vee P_h \bullet \overleftarrow{\Psi^{-}_{Q}}(r)\right\} - P_h \bullet \overleftarrow{\Psi^{-}_{R}}(qr)$ for every $qr\in\overleftrightarrow{V^2}$ and $h=1\ to\ m.$

Theorem 4.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs of G_1^* and G_2^*. If G_1 is ISMP to G_2, then $\overline{G_1}$ is ISMP to $\overline{G_2}$.

Proof. Let G_1 is ISMP to G_2. Then there is an IMPS $\Phi: V_1 \to V_2$ such that $P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q)=P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(\Phi(q))$, $P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q)=P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(\Phi(q))$ for all $q\in V_1$ and $P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr)=P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}(\Phi(q)\Phi(r))$, $P_h \bullet \overleftarrow{\Psi^{-}_{R_1}}(qr)=P_h \bullet \overleftarrow{\Psi^{-}_{R_2}}(\Phi(q)\Phi(r))$ for all $qr\in \overleftrightarrow{V_1^2}$ and $h=1\ to\ m$

Now, $P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1}}}(q)=P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q)=P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(\Phi(q))=P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_2}}}(\Phi(q))$ and $P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_1}}}(q)=P_h \bullet \overleftarrow{\Psi^{-}_{Q_1}}(q)=P_h \bullet \overleftarrow{\Psi^{-}_{Q_2}}(\Phi(q))=P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_2}}}(\Phi(q))$ for all $q\in V_1$. Also, for $h=1\ to\ m$ and $qr\in\overleftrightarrow{V_1^2}$ we have, $P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_1}}}(qr)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr)$

$=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}\left(\Phi(q)\right), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}\left(\Phi(r)\right)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}\left(\Phi(q)\Phi(r)\right) = P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_2}}}\left(\Phi(q)\Phi(r)\right).$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{\overline{R_1}}}(qr) = P_h \bullet \overleftarrow{\Psi^{-}_{\overline{R_2}}}\left(\Phi(q)\Phi(r)\right).$

Hence, Φ is an IMPS from $\overline{G_1}$ to $\overline{G_2}$ i.e. $\overline{G_1}$ is ISMP to $\overline{G_2}$.

Remark 4.1. Assume that there is a w- IMPS between 2 m-BPFGs G_1 and G_2. Then there may not be a w- IMPS between $\overline{G_1}$ and $\overline{G_2}$. This can be proved by the below Example 4.1.

Example 4.1. Consider the 2-BPFGs $G_1=(V_1,Q_1,R_1)$ of $G_1^*=(V_1,E_1)$ where $V_1=\{a,b,c\}$, $E_1=\{ab,ac,bc\}$,

$$Q_1=\left\{\frac{\langle[0.2,-0.9],[0.7,-0.9]\rangle}{a},\frac{\langle[0.3,-0.8],[0.6,-0.8]\rangle}{b},\frac{\langle[0.3,-0.4],[0.5,-0.4]\rangle}{c}\right\}$$

$$R_1=\left\{\frac{\langle[0.2,-0.3],[0.5,-0.3]\rangle}{ab},\frac{\langle[0.15,-0.3],[0.4,-0.3]\rangle}{ac},\frac{\langle[0.2,-0.3],[0.4,-0.3]\rangle}{bc}\right\},$$ and

$G_2=(V_2,Q_2,R_2)$ of $G_2^*=(V_2,E_2)$ where $V_2=\{\mathrm{q},r,s\}$, $E_2=\{\mathrm{qs},qr,sr\}$,

$$Q_2=\left\{\frac{\langle[0.2,-0.9],[0.7,-0.9]\rangle}{q},\frac{\langle[0.3,-0.8],[0.6,-0.8]\rangle}{r},\frac{\langle[0.3,-0.4],[0.5,-0.4]\rangle}{s}\right\}$$

$$R_2=\left\{\frac{\langle[0.2,-0.4],[0.5,-0.4]\rangle}{qr},\frac{\langle[0.2,-0.4],[0.5,-0.4]\rangle}{qs},\frac{\langle[0.3,-0.4],[0.5,-0.4]\rangle}{sr}\right\}.$$ Then

$\overline{G_1}=\left(V_1,Q_1,\overline{R_1}\right)$ where

$$\overline{R_1}=\left\{\frac{\langle[0.0,-0.5],[0.1,-0.5]\rangle}{\overline{ab}},\frac{\langle[0.05,-0.1],[0.1,-0.1]\rangle}{\overline{ac}},\frac{\langle[0.1,-0.1],[0.1,-0.1]\rangle}{\overline{bc}}\right\}$$ and $\overline{G_2}=\left(V_2,Q_2,\overline{R_2}\right)$

where $$\overline{R_2}=\left\{\frac{\langle[0.0,-0.4],[0.1,-0.4]\rangle}{\overline{qr}}\right\}.$$

Let us characterize $\Phi:V_1\to V_2$ such that $\Phi(a)=u,\Phi(b)=v,\Phi(c)=w.$ Then, Φ is a w- IMPS between $\overline{G_1}$ onto $\overline{G_2}$. So, there is no w- IMPS between $\overline{G_1}$ onto $\overline{G_2}$. (see Figure 3).

Since $P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_2}}}\left(uw=\Phi(a)\Phi(c)\right) \le P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1}}}(ac)$, $P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_2}}}\left(uw=\Phi(a)\Phi(c)\right) \ge P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_1}}}(ac)$ and

$P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_2}}}\left(vw=\Phi(b)\Phi(c)\right) \le P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1}}}(bc)$, $P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_2}}}\left(vw=\Phi(b)\Phi(c)\right) \ge P_h \bullet \overleftarrow{\Psi^{-}_{\overline{Q_1}}}(bc)$.

Remark 4.2. In the same manner, we build an example to prove that if there is a co w- IMPS from 2 m-BPFGs G_1 and G_2 then there might not be a co w- IMPS from $\overline{G_1}$ and $\overline{G_2}$.

Theorem 4.2. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs of G_1^* and G_2^* such that $V_1\cap V_2=\phi.$ Then $\overline{G_1+G_2}\cong\overline{G_1}\cup\overline{G_2}.$

Proof. To prove that $\overline{G_1+G_2}\cong\overline{G_1}\cup\overline{G_2}$, we show that there exists an IMPS between $\overline{G_1+G_2}$ and $\overline{G_1}\cup\overline{G_2}$.

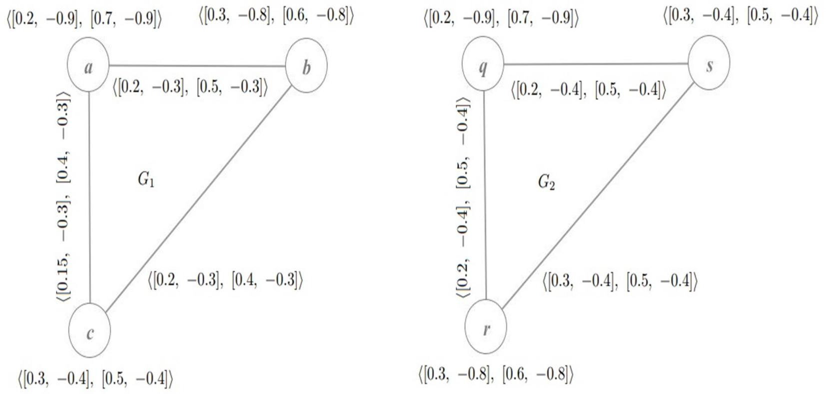


Figure 2. W-isomorphic 2-BPFG fuzzy graphs G_1 and G_2

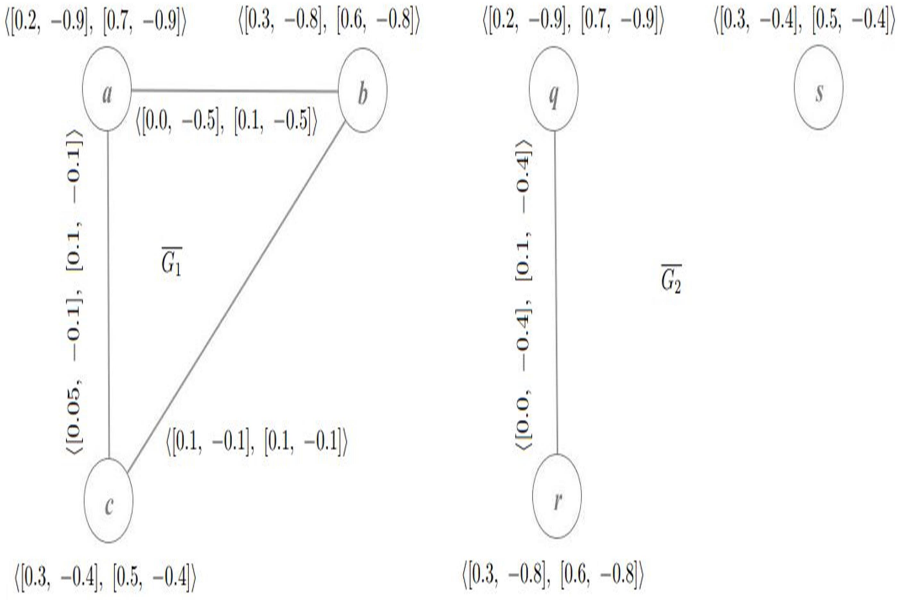


Figure 3. Example of w-isomorphic 2-BPFGs whose complement is not w-isomorphic

We shall prove that the identity map $I: V_1 \cup V_2 \to V_1 \cup V_2$ is the necessary IMPS between them. For this, we shall prove the following: $P_h \bullet \overrightarrow{\Psi^+_{(\overline{Q_1+Q_2})}}(q) = P_h \bullet \overrightarrow{\Psi^+_{(\overline{Q_1} \cup \overline{Q_2})}}(q)$, $P_h \bullet \overleftarrow{\Psi^-_{(\overline{Q_1+Q_2})}}(q) = P_h \bullet \overleftarrow{\Psi^-_{(\overline{Q_1} \cup \overline{Q_2})}}(q)$ for all $q \in V_1 \cup V_2$, and $P_h \bullet \overrightarrow{\Psi^+_{(\overline{R_1+R_2})}}(qr) = P_h \bullet \overrightarrow{\Psi^+_{(\overline{R_1} \cup \overline{R_2})}}(qr)$, $P_h \bullet \overleftarrow{\Psi^-_{(\overline{R_1+R_2})}}(qr) = P_h \bullet \overleftarrow{\Psi^-_{(\overline{R_1} \cup \overline{R_2})}}(qr)$ for all $qr \in \overleftrightarrow{V_1 \times V_2^2}$, $h = 1$ *to m*

Let $q \in V_1 \cup V_2$. Then

$$
\begin{aligned}
P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1+Q_2}}}(q) &= P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1+Q_2)}}(q) \\
&= P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(q) \\
&= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q)\, if\ q\in V_1-V_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q)\, if\ q\in V_2-V_1 \end{cases} \\
&= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1}}}(q)\, if\ q\in V_1-V_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_2}}}(q)\, if\ q\in V_2-V_1 \end{cases} \\
&= P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{Q_1}\cup\overline{Q_2})}}(q).
\end{aligned}
$$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{Q_1+Q_2})}}(q) = P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{Q_1}\cup\overline{Q_2})}}(q).$ Now for each $h=1\ to\ m$ and $qr\in \overleftrightarrow{V_1\times V_2^2}$ we have,

$$
\begin{aligned}
P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1+R_2})}}(qr) &= \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1+Q_2)}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1+Q_2)}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{(R_1+R_2)}}(qr) \\
&= \begin{cases} \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\cup R_2)}}(qr), if\ qr\in E_1\cup E_2 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\cup Q_2)}}(r)\right\} - \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\}, if\ qr\in E' \end{cases}
\end{aligned}
$$

$$
= \begin{cases} \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr)\, if\ qr\in E_1-E_2 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_2}}(qr)\, if\ qr\in E_2-E_1 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} if\ qr\in E' \end{cases}
$$

$$
\begin{aligned}
&= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_1}}}(qr)\, if\ qr\in E_1-E_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_2}}}(qr)\, if\ qr\in E_2-E_1 \\ 0, if\ qr\in E' \end{cases} \\
&= P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1}\cup\overline{R_2})}}(qr).
\end{aligned}
$$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{R_1+R_2})}}(qr) = P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{R_1}\cup\overline{R_2})}}(qr).$

Theorem 4.3. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be two m-BPFGs of G_1^* and G_2^* such that $V_1\cap V_2=\phi.$ Then $\overline{G_1\cup G_2}$ is IMPC to $\overline{G_1}+\overline{G_2}.$

Proof. Take the identity function $I: V_1\cup V_2 \to V_1\cup V_2$. Now we shall prove that I is the necessary IMPS between $\overline{G_1\cup G_2}$ and $\overline{G_1}+\overline{G_2}$.

For this, we shall prove the subsequent: $P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{Q_1\cup Q_2})}}(q) = P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{Q_1}+\overline{Q_2})}}(q),$
$P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{Q_1\cup Q_2})}}(q) = P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{Q_1}+\overline{Q_2})}}(q)$ for all $q\in V_1\cup V_2$ and $P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1\cup R_2})}}(qr) = P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1}+\overline{R_2})}}(qr),$

$P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{R_1 \cup R_2}\right)}}(qr) = P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{R_1}+\overline{R_2}\right)}}(qr),$ for all $qr \in \overleftrightarrow{V_1 \times V_2^2},\ h = 1\ to\ m$

Let $q \in V_1 \cup V_2$. Then

$$P_h \bullet \overrightarrow{\Psi^{+}_{\left(\overline{Q_1 \cup Q_2}\right)}}(q) = P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1 \cup Q_2)}}(q)$$

$$= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q)\, if\ q \in V_1 - V_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q)\, if\ q \in V_2 - V_1 \end{cases}$$

$$= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_1}}}(q)\, if\ q \in V_1 - V_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{\overline{Q_2}}}(q)\, if\ q \in V_2 - V_1 \end{cases} = P_h \bullet \overrightarrow{\Psi^{+}_{\left(\overline{Q_1} \cup \overline{Q_2}\right)}}(q).$$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{Q_1 \cup Q_2}\right)}}(q) = P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{Q_1} \cup \overline{Q_2}\right)}}(q)$ and

$$P_h \bullet \overrightarrow{\Psi^{+}_{\left(\overline{R_1 \cup R_2}\right)}}(qr) = \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1 \cup Q_2)}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1 \cup Q_2)}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{(R_1 \cup R_2)}}(qr)$$

$$= \begin{cases} \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr), if\ qr \in E_1 - E_2 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - P_h \bullet \overrightarrow{\Psi^{+}_{R_1}}(qr), if\ qr \in E_2 - E_1 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - 0\, if\ q \in V_1, r \in V_2 \end{cases}$$

$$= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_1}}}(qr)\, if\ qr \in E_1 - E_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_2}}}(qr)\, if\ qr \in E_2 - E_1 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - 0\ if\ q \in V_1, r \in V_2, \end{cases}$$

$$= \begin{cases} P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_1}}}(qr)\, if\ qr \in E_1 - E_2 \\ P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_2}}}(qr)\, if\ qr \in E_2 - E_1 \\ \min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r)\right\} - 0\ if\ qr \in E', \end{cases}$$

$$= P_h \bullet \overrightarrow{\Psi^{+}_{\left(\overline{R_1}+\overline{R_2}\right)}}(qr).$$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{R_1 \cup R_2}\right)}}(qr) = P_h \bullet \overleftarrow{\Psi^{-}_{\left(\overline{R_1}+\overline{R_2}\right)}}(qr).$

Theorem 4.4. If $G_1 = (V_1, Q_1, R_1)$ and $G_2 = (V_2, Q_2, R_2)$ are two strong m-BPFGs of G_1^* and G_2^* respectively. Then $\overline{G_1 \circ G_2}$ is IMPC to $\overline{G_1} \circ \overline{G_2}$.

Proof. Let $G_1 \circ G_2 = (V_1 \times V_2, Q_1 \circ Q_2, R_1 \circ R_2)$ be an m-BPFG of $G^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{(q, q_2)(q, r_2) : q \in V_1, q_2 r_2 \in E_2\} \cup \{(q_1, t)(r_1, t) : t \in V_2, q_1 r_1 \in E_1\} \cup \{(q_1, q_2)(r_1, r_2) : q_1 r_1 \in E_1, q_2 \neq r_2\}$. We prove the identity map I is the required IMPS between the graphs $\overline{G_1 \circ G_2}$ and $\overline{G_1} \circ \overline{G_2}$.

Let us define the identity function $I:V_1\times V_2\rightarrow V_1\times V_2$. In such a way to prove that I is the necessary IMPS, we prove that $P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(qr)=P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(qr)$ for all $qr\in\overleftrightarrow{(V_1\times V_2)}^2,\ h=1\ to\ m$

This will be done in seven cases.

Case(i): Let $f=(q,q_2)(q,r_2)$ where $q\in V_1, q_2r_2\in E_2$. Then $f\in E.$

Since $G_1\circ G_2$ is strong m-BPFG, we have for each $h=1\ to\ m$

$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=0$ and $P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q),P_h\bullet\overrightarrow{\Psi^{+}_{\overline{R_2}}}(q_2r_2)\right\}=0$

(Since G_2 is strong and $q_2r_2\in E_2$, we have $P_h\bullet\overrightarrow{\Psi^{+}_{\overline{R_2}}}(q_2r_2)=0$ for each $h=1\ to\ m$)

Case(ii): Let $f=(q,q_2)(q,r_2)$ where $q_2\neq r_2, q_2r_2\notin E_2$. Then $f\notin E.$

So, for each $h=1\ to\ m$

$P_h\bullet\overrightarrow{\Psi^{+}_{(R_1\circ R_2)}}(f)=0$ and

$$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q,q_2),P_h\bullet\overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q,r_2)\right\}$$
$$=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(q_2),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(r_2)\right\}.$$

Again since $q_2r_2\in\overline{E_2}$, therefore

$$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q),P_h\bullet\overrightarrow{\Psi^{+}_{\overline{R_2}}}(q_2r_2)\right\}$$
$$=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(q_2),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(r_2)\right\}$$ for each $h=1\ to\ m$

Case(iii): $f=(q_1,z)(r_1,z)$ where $q_1r_1\in E_1, z\in V_2$. Then $f\in E$

$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=0$ for each $h=1\ to\ m$ as in case(i).

Also, since $q_1r_1\notin E_1$ therefore $P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=0$, for each $h=1\ to\ m.$

Case(iv): Let $f=(q_1,z)(r_1,z)$ where $q_1r_1\notin E_1, z\in V_2$. Then $f\notin E$

Hence, $P_h\bullet\overrightarrow{\Psi^{+}_{(R_1\circ R_2)}}(f)=0$ for each $h=1\ to\ m$

$$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q_1,z),P_h\bullet\overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(r_1,z)\right\}$$
$$=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q_1),P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(r_1),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(z)\right\}$$ and

$$P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(z),P_h\bullet\overrightarrow{\Psi^{+}_{\overline{R_1}}}(q_1r_1)\right\}$$
$$=\min\left\{P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(q),P_h\bullet\overrightarrow{\Psi^{+}_{Q_1}}(r_1),P_h\bullet\overrightarrow{\Psi^{+}_{Q_2}}(z)\right\}$$ (G_1 being strong).

Case(v): Let $f=(q_1,q_2)(r_1,r_2)$ where $q_1r_1\in E_1, q_2\neq r_2$. Then $f\in E$

So, we have $P_h\bullet\overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=0$ for each $h=1\ to\ m$ as in case (i).

Also, since $q_1r_1 \in E_1$ we have $P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=0$ for each $h=1 \text{ to } m$

Case(vi): Let $f=(q_1,q_2)(r_1,r_2)$ where $q_1r_1 \notin E_1, q_2 \neq r_2.$ Then $f \notin E$ and hence for each $h=1 \text{ to } m$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\circ R_2)}}(f)=0$$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q_1,q_2), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(r_1,r_2)\right\}$$

$$=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(r_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q_2), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r_2)\right\}$$

and since $q_1r_1 \in E_1,$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(f)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q_2), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r_2), P_h \bullet \overrightarrow{\Psi^{+}_{\overline{R_1}}}(q_1r_1)\right\}$$

$$=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(q_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_1}}(r_1), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(q_2), P_h \bullet \overrightarrow{\Psi^{+}_{Q_2}}(r_2)\right\}.$$

Case(vii): Finally, let $f=(q_1,q_2)(r_1,r_2)$ where $q_1r_1 \notin E_1, q_2r_2 \notin E_2$
Then $f \notin E$ and hence for each $h=1 \text{ to } m$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(R_1\circ R_2)}}(f)=0,$$

$$P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(f)=\min\left\{P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(q_1,q_2), P_h \bullet \overrightarrow{\Psi^{+}_{(Q_1\circ Q_2)}}(r_1,r_2)\right\}.$$

Now, $q_1r_1 \in \overline{E_1}$ and $q_2=r_2=z,$ then we have Case (iv).

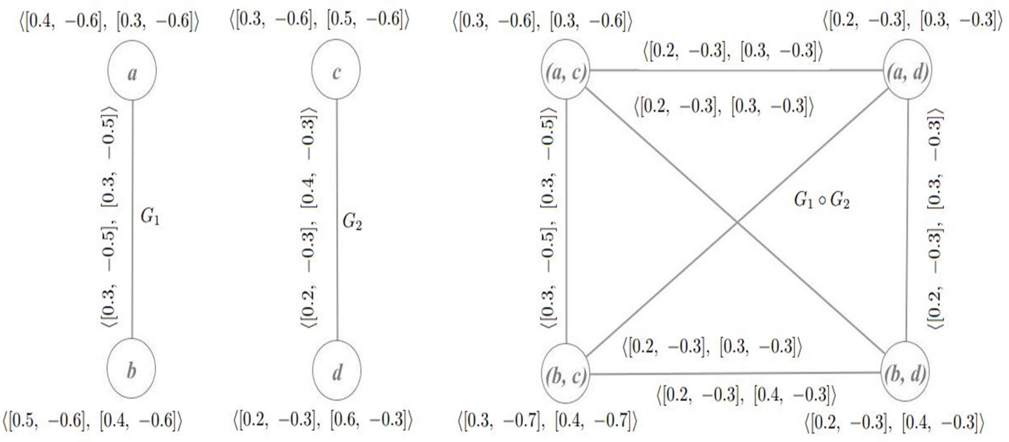
Again, $q_1r_1 \in \overline{E_1}$ and $q_2 \neq r_2,$ then we have Case (vi).

Thus, from above discussions we get that, $P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1\circ R_2})}}(qr)=P_h \bullet \overrightarrow{\Psi^{+}_{(\overline{R_1}\circ\overline{R_2})}}(qr).$

Similarly, $P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{R_1\circ R_2})}}(qr)=P_h \bullet \overleftarrow{\Psi^{-}_{(\overline{R_1}\circ\overline{R_2})}}(qr)$ for all $qr \in \overleftrightarrow{V_1 \times V_2^2}$, $h=1,2,\ldots,m$

Hence $\overline{G_1\circ G_2} \cong \overline{G_1}\circ\overline{G_2}.$

Remark 4.3. If G_1 and G_2 are not strong m-BPFGs then $\overline{G_1\circ G_2}$ is not IMPC to $\overline{G_1}\circ\overline{G_2}$ always. For instance, consider 2-BPFGs G_1 and G_2 which are not strong as shown in figure 4.Then $\overline{G_1\circ G_2}$ is not IMPC to $\overline{G_1}\circ\overline{G_2}$ are not isomorphic as shown in example 4.2.



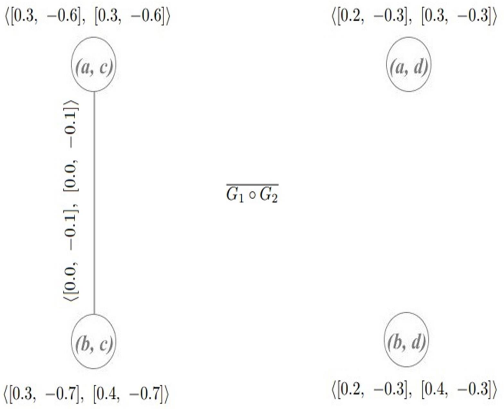


Figure 4. G_1, G_2, $G_1 \circ G_2$, $\overline{G_1 \circ G_2}$

Example 4.2. Consider the 2-BPFGs $G_1 = (V_1, Q_1, R_1)$ of $G_1^* = (V_1, E_1)$ where $V_1 = \{a, b\},\ E_1 = \{ab\}$,

$$Q_1 = \left\{ \frac{\langle [0.4,-0.6],[0.3,-0.6] \rangle}{a}, \frac{\langle [0.5,-0.6],[0.4,-0.6] \rangle}{b} \right\}\ R_1 = \left\{ \frac{\langle [0.3,-0.5],[0.3,-0.5] \rangle}{ab} \right\},$$

and $G_2 = (V_2, Q_2, R_2)$ of $G_2^* = (V_2, E_2)$ where $V_2 = \{c, d\},\ E_2 = \{cd\}$,

$$Q_2 = \left\{ \frac{\langle [0.3,-0.6],[0.5,-0.6] \rangle}{c}, \frac{\langle [0.2,-0.3],[0.6,-0.3] \rangle}{d} \right\}$$

and

$$R_2 = \left\{ \frac{\langle [0.2,-0.3],[0.4,-0.3] \rangle}{cd} \right\}.$$

Then $G_1 \circ G_2 = (V_1 \times V_2, Q_1 \circ Q_2, R_1 \circ R_2)$ where $V_1 \times V_2 = \{(a,c),(a,d),(\mathrm{b},c),(\mathrm{b},d)\}$,

$$Q_1 \circ Q_2 = \left\{ \frac{\langle [0.3,-0.6],[0.3,-0.6] \rangle}{(a,c)}, \frac{\langle [0.2,-0.3],[0.3,-0.3] \rangle}{(a,d)}, \frac{\langle [0.3,-0.7],[0.4,-0.7] \rangle}{(b,c)}, \frac{\langle [0.2,-0.3],[0.4,-0.3] \rangle}{(b,d)} \right\},$$

$$R_1 \circ R_2 = \left\{ \frac{\langle [0.2,-0.3],[0.3,-0.3] \rangle}{(a,c)(a,d)}, \frac{\langle [0.2,-0.3],[0.3,-0.3] \rangle}{(a,d)(b,d)}, \frac{\langle [0.2,-0.3],[0.4,-0.3] \rangle}{(b,d)(b,c)}, \right.$$

$$\left. \frac{\langle [0.3,-0.5],[0.3,-0.5] \rangle}{(b,c)(a,c)}, \frac{\langle [0.2,-0.3],[0.3,-0.3] \rangle}{(a,c)(b,d)}, \frac{\langle [0.2,-0.3],[0.3,-0.3] \rangle}{(a,d)(b,c)} \right\}$$

$\overline{G_1 \circ G_2} = \left(V_1 \times V_2, Q_1 \circ Q_2, \overline{R_1 \circ R_2}\right)$ where $\overline{R_1 \circ R_2} = \left\{ \frac{\langle [0.0,-0.1],[0.0,-0.1] \rangle}{\overline{(b,c)(a,c)}} \right\}$ shown in Figure 4.

From Figure 5. $\overline{G_1} = \left(V_1, Q_1, \overline{R_1}\right)$, where $\overline{R_1} = \left\{ \frac{\langle [0.1,-0.1],[0.0,-0.1] \rangle}{\overline{ab}} \right\}$ and $\overline{G_2} = \left(V_1, Q_1, \overline{R_2}\right)$, where $\overline{R_2} = \left\{ \frac{\langle [0.0,-0.0],[0.1,-0.0] \rangle}{\overline{cd}} \right\}$. Then $\overline{G_1} \circ \overline{G_2} = \left(V_1 \times V_2, Q_1 \circ Q_2, \overline{R_1} \circ \overline{R_2}\right)$, where

$$\overline{R_1} \circ \overline{R_2} = \left\{ \frac{\langle [0.0,-0.0],[0.1,-0.0] \rangle}{\overline{(\mathrm{a},c)(a,d)}}, \frac{\langle [0.1,-0.1],[0.0,-0.1] \rangle}{\overline{(\mathrm{a},d)(\mathrm{b},d)}}, \frac{\langle [0.0,-0.0],[0.1,-0.0] \rangle}{\overline{(\mathrm{b},d)(\mathrm{b},c)}}, \right.$$

$$\frac{\langle[0.1,-0.1],[0.0,-0.1]\rangle}{\overline{(\mathrm{b},c)(a,c)}},\frac{\langle[0.1,-0.1],[0.0,-0.1]\rangle}{\overline{(\mathrm{a},c)(\mathrm{b},d)}},\frac{\langle[0.1,-0.1],[0.0,-0.1]\rangle}{\overline{(\mathrm{a},d)(\mathrm{b},c)}}\Bigg\}$$

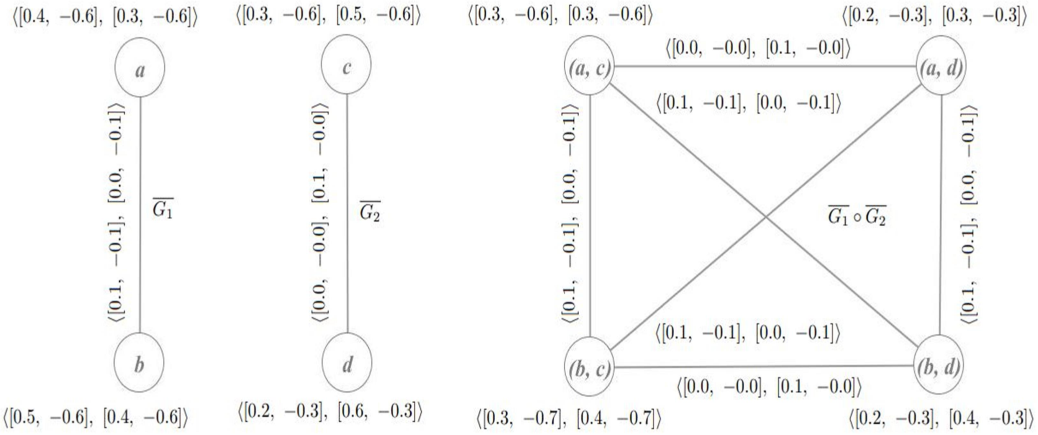


Figure 5. Example of 2-BPFGs G_1 and G_2 where $\overline{G_1 \circ G_2} \not\cong \overline{G_1} \circ \overline{G_2}$

5. Conclusion

The FG theory concepts are played a vital position in many fields with conclusion makings, computer science and chemical sciences. An m-BPFG can be useful to stand for real life problems which contain multi-agents, multi-attributes, multi-index, multi-objects, and mult-polar details with vagueness and uncertainty. Without bipolarity, reasonably measurable causality, mind-light-matter unity in AI/QI would be impossible. In this article, order, size, BV and FV in m-BPFGs are defined. In addition, a comparative learning of complement and operations has been finished under few conditions and explained them through examples.

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