TOPOLOGIES INDUCED BY GRAPH GRILLS ON VERTEX SET OF GRAPHS

K.LALITHAMBIGAI AND P. GNANACHANDRA

Abstract. This research aim at introducing the notion of graph grill and graph limit operator and presents some of their properties. A new kind of graph topology induced on vertex set of a graph by graph grill is defined and some of its characterisations are studied. Further, this paper analyses the graph grills that best fit the graph adjacency topology.

Keywords : *Graph adjacency topological space, graph grill, graph limit operator, closure operator*
AMS Subject classification *: 57M15, 54A10, 54H99, 05C20*

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1.Introduction

Choquet [5] was the first to introduce the notion of Grill on a topological space (X, τ). He defined grill G^* on X as a collection of nonempty subsets of X which satisfies: (i) A ∈ G^* and A □ B □ X □ B □ G^* and (ii) A,B □ X and A□ B ∈ G^* □ A ∈ G^* or B ∈ G^* . Like nets and filters, grill is also an important tool in dealing with most of the topological concepts quite effectively. Proximity spaces, closure spaces, the compactification theories and some other extension problems have been tackled efficiently by the use of grills [3],[4],[15]. B.Roy and M.N.Mukherjee [13] defined a new type of topology as an associated structure on a topological space X induced by a grill on X. A link between Graph Theory and Topology can be made by defining a relation on the graph. Graphs can be regarded as a one-dimensional topological space. When we talk about connected graphs or homeomorphic graphs, the adjectives have the same meaning as in topology. So Graph Theory can be regarded as a subset of the topology of, say, one-dimensional simplicial complexes. Diesto and Gervacio [6] constructed topology on vertex set of graphs using neighbourhoods. Also, it was further studied in [8] and [9]. Nianga and Canoy in [11], [12], [1] used the hop neighbourhoods to generate topology on vertex set of graphs and studied topologies induced by some unary, binary operations on graphs. Shokry Nada et al.[14] defined a relation R on vertex set V (G) of a graph G by R = { ($(2m_x+n_x)_x$, $(2m_y+n_y)_y$) : x, y ∈ V (G)}, where m_x and m_y are the number of loops of vertices x and y, respectively and n_x and n_y are the number of multiple edges of vertices x and y, respectively.

Then he defined the post class for each v_i as the open neighbourhood of v_i in R which is denoted as v_iR and constructed a subbase for a topology by S_G = ∪{v_iR : v_i □ V (G)}. P.Gnanachandra et al. [7], introduced the method of generating sub basis for different topologies on vertex set of simple undirected graphs and studied the basic properties of closure, interior, exterior and boundary of vertex induced subgraphs of a graph with respect to graph adjacency topology. This paper presents the method of generating graph topologies on vertex set of graphs using graph grills.

2. Preliminaries

In this section, sub basis for topologies on vertex set of graphs using the relations adjacency, non-adjacency, incidence, non-incidence on the vertex set of graphs are generated. Definitions in this section are observed as in [7]. Throughout the paper, the graphs under discussion is the simple undirected graph which is not a star graph.

Definition 2.1. Let *G* = (*V* (*G*),*E*(*G*)) be a graph. For *v* ∈ *V* (*G*), the neighbourhood set N_v of *v* is defined as N_v = *{u* ∈ *V* (*G*) : *uv* ∈ *E*(*G*)*}* and the non-neighbourhood set NA_v of *v* as NA_v = *{u* ∈ *V* (*G*) : *uv* ∉*E*(*G*)*}*. For *e* ∈ *E*(*G*), define *I*(*e*) as the set of all vertices incident with *e* and *Ni*(*e*) as the set of all vertices not incident

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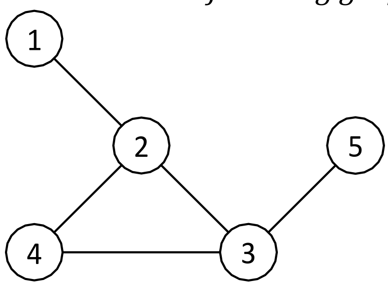
with e.

Definition 2.2. Let $G = (V(G), E(G))$ be a graph without isolated vertices. Define S_N as the family of N_v for all v ⍰ $V(G)$, i.e., $S_N = \{N_v : v \in V(G)\}$. Then S_N forms a sub base for a topology T_A on $V(G)$ and the pair $(V(G), T_A)$ is called graph adjacency topological space. Define S_I as the family of $I(e)$ for all $e \in E(G)$, i.e., $S_I = \{I(e) : e \in E(G)\}$. Then S_I forms a subbase for a topology T_I on $V(G)$. The pair $(V(G), T_I)$ is called graph incidence topological space. For $|E(G)| > 2$, define S_{Ni} as the family of $Ni(e)$ for all $e \in E(G)$, i.e., $S_{Ni} = \{Ni(e) : e \in E(G)\}$. Then S_{Ni} forms a subbase for a topology T_{Ni} on $V(G)$. The pair $(V(G), T_{Ni})$ is called graph non-incidence topological space. If $|V(G)| = n$ and $0 \le d(v) \le n-2$ for all $v \in V(G)$, define S_{NA} as the family of NA_v for all $v \in V(G)$, i.e., $S_{NA} = \{NA_v : v \in V(G)\}$. Then S_{NA} forms a subbase for a topology T_{NA} on $V(G)$ and the pair $(V(G), T_{NA})$ is called graph non-adjacency topological space.

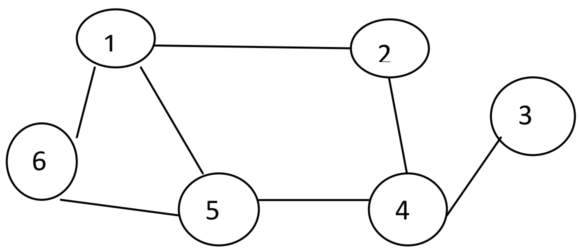
Example 2.3. *Consider the following graph.*



$S_N = \{\{2\}, \{1, 3, 4\}, \{2, 5, 4\}, \{2, 3\}\{3\}\}$
B $= \{\phi, \{2\}, \{1, 3, 4\}, \{2, 5, 4\}, \{2, 3\}, \{4\}, \{3\}\}$
$T_A = \{\phi, \{2\}, \{1, 3, 4\}, \{2, 5, 4\}, \{2, 3\}, \{4\}, \{3\}, \{1, 2, 3, 4\}, \{2, 4\}, \{1, 2, 3, 4, 5\}, \{2, 3, 4, 5\}, \{2, 3, 4\}, \{3, 4\}\}$
$S_I = \{\{1, 2\}, \{2, 4\}, \{3, 4\}, \{2, 3\}, \{3, 5\}\}$
B $= \{\phi, \{1, 2\}, \{2, 4\}, \{3, 4\}, \{2, 3\}, \{3, 5\}, \{2\}, \{4\}, \{3\}\}$
$T_I = \{\phi, \{1, 2\}, \{2, 4\}, \{3, 4\}, \{2, 3\}, \{3, 5\}, \{2\}, \{3\}, \{4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 3, 5\}, \{2, 3, 4\}, \{2, 4, 3, 5\}, \{3, 4, 5\}, \{2, 3, 5\}, \{1, 2, 3, 4, 5\}\}$
$S_{Ni} = \{\{3, 4, 5\}, \{1, 3, 5\}, \{1, 2, 5\}, \{1, 4, 5\}, \{1, 2, 4\}\}$
B $= \{\phi, \{3, 4, 5\}, \{1, 3, 5\}, \{1, 2, 5\}, \{1, 4, 5\}, \{1, 2, 4\}, \{3, 5\}, \{5\}, \{4, 5\}, \{4\}, \{1, 5\}, \{1\}, \{1, 2\}, \{1, 4\}\}$
$T_{Ni} = \{\phi, \{3, 4, 5\}, \{1, 3, 5\}, \{1, 2, 5\}, \{1, 4, 5\}, \{1, 2, 4\}, \{3, 5\}, \{5\}, \{4, 5\}, \{4\}, \{1, 5\}, \{1\}, \{1, 2\}, \{1, 4\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}\}$.
$S_{NA} = \{\{3, 4, 5\}, \{5\}, \{1\}, \{1, 5\}, \{1, 2, 4\}\}$
B $= \{\phi, \{3, 4, 5\}, \{5\}, \{1\}, \{1, 5\}, \{1, 2, 4\}, \{4\}\}$
$T_{NA} = \{\phi, \{3, 4, 5\}, \{5\}, \{1\}, \{1, 5\}, \{1, 2, 4\}, \{4\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 2, 4, 5\}, \{4, 5\}, \{1, 4\}, \{1, 4, 5\}\}$

Definition 2.4. Let $G = (V(G), E(G))$ be a graph without isolated vertices and $(V(G), T_A)$ ($(V(G), T_I)$ respectively.) be a graph adjacency topological space (graph incidence topological space). Let W be a vertex induced subgraph of G. Then the closure of $V(W)$ is defined by $cl(V(W)) = V(W) \cup \{v \in V(G) : N_v \cap V(W) \neq \phi\}$ ($cl(V(W)) = V(W) \cup \{v \in V(G) : v \in (I(e)), I(e) \cap V(W) \neq \phi\}$) and the interior of $V(W)$ is defined by $int(V(W)) = \{v \in V(G) : N_v \subseteq V(W)\}$ ($int(V(W)) = \{v \in I(e) : I(e) \subseteq V(W)\}$).

Example 2.5. Consider the following graph



$S_N = \{\{2, 5, 6\}, \{1, 4\}, \{4\}, \{3, 2, 5\}, \{4, 1, 6\}, \{1, 5\}\}$.
$cl(\{1, 4, 3\}) = \{1, 2, 3, 4, 5, 6\}$,
$cl(\{3, 6\}) = \{1, 3, 4, 5, 6\}$,
$int(\{1, 4, 3\}) = \{2, 3\}$, $int(\{1, 6\}) = \phi$.
$S_I = \{\{1, 2\}, \{2, 4\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{1, 5\}\}$.
$cl(\{1, 4, 3\}) = \{1, 2, 3, 4, 5, 6\}$,
$int(\{1, 4, 3\}) = \{3, 4\}$.

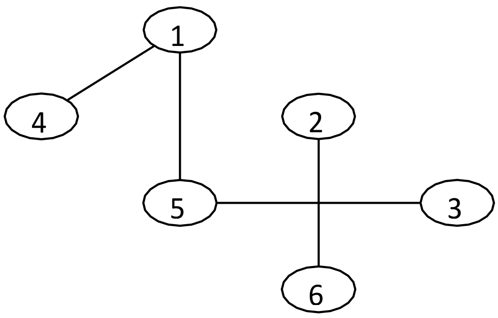
3. Graph Grill and Graph Limit Operator

In this section, graph grill and graph limit operator in graph adjacency topological space are defined with illustrations.

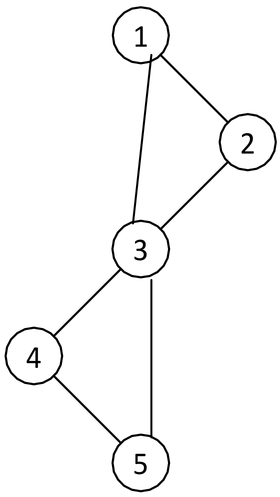
Definition 3.1. Let G = (V,E) be a graph. Then $\mathcal{G}$ = {G': G' = (V',E') is a vertex induced subgraph of G, where V'⊆ V , E'⊆ E} is said to be a graph grill on the graph adjacency topological space (V (G), TA) if it satisfies the following two conditions:

i) $G' \in \mathcal{G}$ and G' is a vertex induced sub graph of $G'' \Rightarrow G'' \in \mathcal{G}$ and

ii) G', G'' are vertex induced sub graphs of G and G'▯ $G'' \in \mathcal{G}$ ▯ $G' \in \mathcal{G}$ or $G'' \in \mathcal{G}$.



Example 3.2. Consider the following graph



$\mathcal{G}$ = {{2}, {3}, {1, 2}, {1, 3}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {1, 2, 3, 4, 5}} is a graph grill.

Definition 3.3. Let $(V(G), T_A)$ be a graph adjacency topological space. Let v ∈ $V(G)$. The open neighbourhood system at v denoted by N(v) is defined as N(v) = {U ∈ T_A : v ∈ U}

Definition 3.4. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$. Let H be a vertex induced subgraph of G. Then $(V(H))^l(\mathcal{G}, T_A)$ = {v ∈ $V(G)$: for every U ∈ N(v), $V(H)$ ∩ U = V(G') for some $G' \in \mathcal{G}$ } is called the graph limit function of $V(H)$ with respect to $\mathcal{G}$ and T_A.

Example 3.5. Consider the following graph

S_N = {{4, 5}, {6}, {5}, {1}, {1, 3}, {2}}.
B = {ϕ,{4,5},{6},{5},{1},{1,3},{2}}.
T_A = {ϕ,{4,5},{6},{5}, {1},{1,3},{2},{4,5,6}, {4,5,1},{1,3,4,5},{2,4,5},{5,6}, {1,6},{1,3,6}, {2,6},{1,5}, {1,3,5},{2,5},{1,2},{1,5,6}, {1,4,5},{1,4,5,6},{1,5,6},{1,3,5,6}, {1,3,4,5,6}, {2,4,5,6},{2,4,5,1}, {1,2,3,4,5}, {2,5,6},{1,2,6}, {1,2,3,6},{1,2,5},{1,2,3,5}, {1,2,5,6},{1,2,4,5}, {1,2,4,5,6}, {1,2,5,6},{1,2,3,5,6},{1,2,3,4,5,6}}.
Let $\mathcal{G}$ = {{2}, {3}, {1, 2}, {1, 3}, {2, 3}, {2, 4}, {2, 5}, {2, 6}, {3, 4}, {3, 5}, {3, 6}, {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {2, 3, 4}, {2, 3, 5}, {2, 3, 6}, {2, 4, 5}, {2, 4, 6}, {2, 5, 6}, {3, 4, 5}, {3, 4, 6}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {2, 3, 4, 5}, {2, 3, 4, 6}, {1, 3, 4, 5}, {2, 3, 5, 6}, {1, 3, 4, 6}, {3, 4, 5, 6}, {1, 4, 2, 5}, {1, 4, 2, 6}, {4, 5, 2, 6}, {1, 2, 5, 6}, {1, 5, 6, 3}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 5, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6}, {2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}} be a graph grill.
If $V(W)$ = {1, 5, 6}, then $(V(W))^l(\mathcal{G}, T_A)$ = ϕ
If $V(W)$ = {3, 4, 5}, then $(V(W))^l(\mathcal{G}, T_A)$ = {3}

The following theorems present some basic facts concerning the graph limit function which is useful in the study of generating new topology from old one.

Theorem 3.6. Let $(V(G), T_A)$ be a graph adjacency topological space with graph grills $\mathcal{G}_1$ and $\mathcal{G}_2$ and let H and W be vertex induced subgraphs of G. Then

(1) $V(H) \subseteq V(W) \Rightarrow (V(H))l \subseteq (V(W))l$

(2) $\mathcal{G}_1 \subseteq \mathcal{G}_2 \Rightarrow (V(H))^l(\mathcal{G}_1, T_A) \subseteq (V(H))^l(\mathcal{G}_2, T_A)$

(3) $V(H) \neq V(G')$ for any $G' \in G \Rightarrow (V(H))l = \phi$

(4) $(V(H \cup W))^l = (V(H))^l \cup (V(W))^l$

Proof. 1. Let $v \in (V(W))^l$.
Hence, for all $U \in N(v)$, $V(W) \cap U = V(G')$ for some $G' \in \mathcal{G}$.
Since $V(H) \subseteq V(W)$, $V(H) \cap U \subseteq V(W) \cap U$ for all $U \in N(v)$.
So, by definition of graph grill, for all $U \in N(v)$, $V(H) \cap U = V(G'')$ for some $G'' \in \mathcal{G}$.
So $v \in (V(H))^l$. Hence $(V(H))^l \subseteq (V(W))^l$.
2. Let $v \in (V(H))^l(\mathcal{G}_1, T_A)$.
Hence, for every $U \in N(v)$, $V(H) \cap U = V(G')$ for some $G' \in \mathcal{G}_1$. Since $\mathcal{G}_1 \subseteq \mathcal{G}_2$, $V(H) \cap U = V(G')$ for some $G' \in \mathcal{G}_2$. So, $v \in (V(H))^l(\mathcal{G}_2, T_A)$.
Hence $(V(H))^l(\mathcal{G}_1, T_A) \subseteq (V(H))^l(\mathcal{G}_2, T_A)$
3. Let $v \in (V(H))^l$. Hence for every $U \in N(v)$, $V(H) \cap U = V(G')$ for some $G' \in G$. So $V(H) \cap V(G) = V(G')$ for some $G' \in \mathcal{G}$.
Hence $V(H) = V(G')$ for some $G' \in \mathcal{G}$ which contradicts the hypothesis. So, $(V(H))^l = \phi$
4. Since $V(H) \subseteq V(H) \cup V(W)$ and $V(W) \subseteq V(H) \cup V(W)$, by (1), $(V(H))^l \cup (V(W))^l \subseteq (V(H) \cup V(W))^l$.
Now, $v \in (V(H) \cup V(W))^l \Rightarrow$ For every $U \in N(v)$, $(V(H) \cup V(W)) \cap U = V(G')$ for some $G' \in \mathcal{G}$.
$\Rightarrow$ For every $U \in N(v)$, $(V(H) \cap U) \cup (V(W) \cap U) = V(G')$ for some $G' \in \mathcal{G}$.
$\Rightarrow$ For every $U \in N(v)$, $V(H) \cap U = V(G'')$ for some $G'' \in \mathcal{G}$ or $V(W) \cap U = V(G''')$ for some $G''' \in \mathcal{G}$.
$\Rightarrow v \in (V(H))^l \cup (V(W))^l$.
So $(V(H) \cup V(W))^l \subseteq (V(H))^l \cup (V(W))^l$.
Hence, $(V(H \cup W))^l = (V(H))^l \cup (V(W))^l$. □

Theorem 3.7. Let $(V(G), T_A)$ be a graph adjacency topological space such that $N_v = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$. Let $\mathcal{G}$ be a graph grill and H be a vertex induced subgraph of G. Then $((V(H))^l)^l \subseteq (V(H))^l = cl((V(H))^l) \subseteq cl(V(H))$.

Proof. Now, $v \notin cl(V(H)) \Rightarrow V(H) \cap Nv = \phi \Rightarrow$ For some $U \in N(v)$, $V(H) \cap U = \phi \neq V(G')$ for any $G' \in \mathcal{G} \Rightarrow v \notin (V(H))^l$. So $(V(H))^l \subseteq cl(V(H))$.
Let $v \in cl((V(H))^l)$. So $(V(H))^l \cap N_v \neq \phi$.
Hence there exists $u \in (V(H))^l \cap N_v$.
$u \in (V(H))^l \Rightarrow$ For all $U \in N(u)$, $V(H) \cap U = V(G')$ for some $G' \in \mathcal{G}$.
$u \in Nv \Rightarrow V(H) \cap N_v = V(G')$ for some $G' \in \mathcal{G}G$.
Since $v \in N_v$, for all $U \in N(v)$, $V(H) \cap U = V(G')$ for some $G' \in \mathcal{G}$. So, $v \in (V(H))^l$
Hence $cl((V(H))^l) \subseteq (V(H))^l$. Also, $(V(H))^l \subseteq cl((V(H))^l)$.
So $(V(H))^l = cl((V(H))^l)$. Also, $((V(H))^l)^l \subseteq cl((V(H))^l) = (V(H))^l$.
Hence $((V(H))^l)^l \subseteq (V(H))^l = cl((V(H))^l) \subseteq cl(V(H))$. □

4. Topology Generated by Graph Grill

In this section, a new topology that inherits from the old one using Kuratowski closure operator cl^l is generated.

Definition 4.1. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$. Define $cl^l{}_G(V(W)) = V(W) \cup (V(W))l$ for all vertex induced subgraphs W of a graph G. When there is no ambiguity, it is denoted by cl^l.

The following theorems show the operator cl^l satisfies the conditions of Kuratowski's closure operator.

Theorem 4.2. For any induced subgraphs H and W of a graph G,

(1) $V(H) \subseteq cl^l(V(H))$

(2) $cl^l(V(H) \cup V(W)) = cl^l(V(H)) \cup cl^l(V(W))$

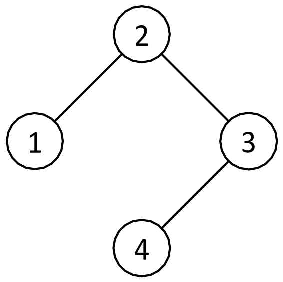
Proof. The proof follows directly from the definition of the operator cl^l. □

Theorem 4.3. Let $(V(G), T_A)$ be a graph adjacency topological space such that $N_v = \{v\}$ ⍰ $\{u \in V(G) : uv \in E(G)\}$. Let $\mathcal{G}$ be a graph grill and H be a vertex induced subgraph of G. Then $cl^l(cl^l(V(H))) = cl^l(V(H))$.

Proof. $cl^l(cl^l(V(H))) = cl^l[V(H)$ ⍰ $(V(H))l] =$ $[V(H)$⍰ $(V(H))^l]$⍰ $[V(H)$⍰ $(V(H))^l]^l = V(H)$⍰ $(V(H))^l$ ⍰ $((V(H))^l)^l = V(H)$ ⍰ $(V(H))^l =$ $cl^l(V(H))$. □

Definition 4.4. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$. Corresponding to the graph grill $\mathcal{G}$, there exists a unique topology $T_A{}^l(\mathcal{G}) = \{ V(H)$ ⍰ $V(G) : cl^l(V(G) - V(H)) = V(G) - V(H)\}$. $T_A{}^l(\mathcal{G})$ is called the graph adjacency topology generated by cl^l. When there is no ambiguity, it is denoted as $T_A{}^l$.

Example 4.5. Consider the following graph



$S_N = \{\{2\}, \{1,3\}, \{2,4\},\{3\}\}$
$B = \{\phi, \{2\}, \{1,3\}, \{2,4\}, \{3\}\}$
$T_A = \{\phi, \{2\}, \{1,3\}, \{2,4\}, \{3\}, \{1,2,3\}, \{2,3\},\{1,2,3,4\},\{2,3,4\}\}$
$\mathcal{G} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{3\}, \{2, 3\}, \{3, 4\}, \{2, 3, 4\}\}$
$N(1) = \{\{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
$N(2) = \{\{2\}, \{2, 4\}, \{1, 2, 3\}, \{2, 3\}, \{1, 2, 3, 4\}, \{2, 4, 3\}\}$
$N(3) = \{\{1, 3\}, \{3\}, \{1, 2, 3\}, \{2, 3\}, \{1, 2, 3, 4\}, \{2, 3, 4\}\}$
$N(4) = \{\{2, 4\}, \{1, 2, 3, 4\}, \{2, 3, 4\}\}$
$(\{1\})^l = (\{1, 2\})^l = (\{1, 4\})^l = (\{1, 2, 4\})^l = \{1\}$, $(\{2\})^l = (\{4\})^l = (\{2, 4\})^l = \phi$,
$(\{3\})^l = (\{1, 3\})^l = (\{2, 3\})^l = (\{3, 4\})^l = (\{1, 2, 3\})^l = (\{1, 3, 4\})^l = (\{2, 3, 4\})^l = (\{1, 2, 3, 4\})^l = \{1, 3\}$
$cl^l(\{1\}) = \{1\}$, $cl^l(\{2\}) = \{2\}$, $cl^l(\{3\}) = \{1, 3\}$, $cl^l(\{4\}) = \{4\}$, $cl^l(\{1, 2\}) = \{1, 2\}$,
$cl^l(\{1, 3\}) = \{1, 3\}$, $cl^l(\{1, 4\}) = \{1, 4\}$,
$cl^l(\{2, 3\}) = \{1, 2, 3\}$, $cl^l(\{2, 4\}) = \{2, 4\}$,
$cl^l(\{3, 4\}) = \{1, 3, 4\}$, $cl^l(\{1, 2, 3\}) = \{1, 2, 3\}$, $cl^l(\{1, 2, 4\}) = \{1, 2, 4\}$,
$cl^l(\{1, 3, 4\}) = \{1, 3, 4\}$, $cl^l(\{2, 3, 4\}) = \{1, 2, 3, 4\}$, $cl^l(\{1, 2, 3, 4\}) = \{1, 2, 3, 4\}$
$T^l{}_A = \{\phi, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3\}, \{3, 4\}, \{2, 4\}, \{2, 3\}, \{1, 3\}, \{4\}, \{3\}, \{1, 2, 3, 4\}, \{2\}\}$

The following theorems present the properties of $T_A{}^l(\mathcal{G})$.

Theorem 4.6. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two graph grills such that $\mathcal{G}_1$ ⍰ $\mathcal{G}_2$ then $T_A{}^l(\mathcal{G}_2)$ ⍰ $T_A{}^l(\mathcal{G}_1)$.

Proof. Now, $V(H) \in T^l{}_A(\mathcal{G}_2)$ ⍰ $cl^l\, \mathcal{G}_2(V(G) - V(H)) = V(G) - V(H)$ ⍰ $(V(G) - V(H))$ ⍰ $(V(G) - V(H))^l(\mathcal{G}_2, T_A) = V(G) - V(H)$ ⍰ $(V(G) - V(H))^l(\mathcal{G}_2, T_A)$ ⍰ $V(G) - V(H)$. Also, by theorem 3.6, $(V(G)-V(H))^l(\mathcal{G}_1)$ ⍰ $(V(G)-V(H))^l(\mathcal{G}_2, T_A)$. Hence $(V(G)-V(H))$⍰ $(V(G)-V(H))^l(\mathcal{G}_1, T_A) = V(G)-V(H)$ and $cl^l\, \mathcal{G}_1(V(G)-V(H)) = V(G) - V(H)$. So, $V(H) \in T^l{}_A(\mathcal{G}_1)$ and $T^l{}_A(\mathcal{G}_2)$ ⍰ $T^l{}_A(\mathcal{G}_1)$. □

Theorem 4.7. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$. Then (1) If H is an induced subgraph of G such that $V(H) \neq V(G')$ for any $G' \in \mathcal{G}$, then $V(H)$ is $T_A{}^l(\mathcal{G})$-closed.

(2) For any vertex induced subgraph H of G, $(V(H))^l$ is $T_A{}^l(\mathcal{G})$-closed if $Nv = \{v\}$ ⍰ $\{u \in V(G) : uv \in E(G)\}$.

Proof. 1. By theorem 3.6, $V(H) \neq V(G')$ for any $G' \in \mathcal{G}$ ⍰ $(V(H))^l = \phi$.
Now, $cl^l(V(H)) = V(H)$ ⍰ $(V(H))^l = V(H)$.
Hence $V(H)$ is $T_A{}^l(\mathcal{G})$-closed.

2. Now, by theorem 3.7, $cl^l((V(H))^l) =$

$((V(H))^l) \subseteq ((V(H))^l)^l = (V(H))^l$. So, $(V(H))^l$ is $T_A^l(\mathcal{G})$-closed. $\square$

Theorem 4.8. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$. Then $B(\mathcal{G}, T^l_A(G)) = \{U - V(H) : U \in T_A$ and $V(H) \neq V(G')$ for any $G' \in \mathcal{G}\}$ is an open base for $T^l_A(G)$. Consequently, $T_A \subseteq B(\mathcal{G}, T^l_A(\mathcal{G})) \subseteq T^l_A(\mathcal{G})$.

Proof. Let $V(W) \in T^l_A(\mathcal{G})$ and $v \in V(W)$. Then $cl^l(V(G) - V(W)) = V(G) - V(W)$ and hence $(V(G) - V(W))^l \subseteq (V(G) - V(W))$. Since $v \in V(W)$, $v \notin (V(G) - V(W))^l$. So there exists $U \in N(v)$ such that $(V(G) - V(W)) \cap U \neq V(G')$ for any $G' \in \mathcal{G}$. Let $V(H) = (V(G) - V(W)) \cap U$. Then $v \notin V(H)$ and $V(H) \neq V(G')$ for any $G' \in \mathcal{G}$. Thus $v \in U - V(H) = U - [(V(G) - V(W)) \cap U] = U - (V(G) - V(W)) \subseteq V(W)$; where $U - V(H) \in B(G, T^l_A(\mathcal{G}))$.

It now suffices to prove that $T^l_A(\mathcal{G}))$ is closed under finite intersections.

Let $U_1 - V(H_1)$ and $U_2 - V(H_2) \in B(\mathcal{G}, T^l_A(\mathcal{G}))$. Then $U_1 \cap U_2 \in T_A$ and $V(H_1) \cup V(H2) \neq V(G')$ for any $G' \in \mathcal{G}$. Also, $(U_1 - V(H_1)) \cap (U_2 - V(H_2)) = (U_1 \cap U_2) - (V(H_1) \cup V(H_2))$. So, $(U_1 - V(H_1)) \cap (U_2 - V(H_2)) \in B(\mathcal{G}, T^l_A(\mathcal{G}))$. Hence $B(\mathcal{G}, T^l_A(\mathcal{G}))$ is an open base for $T^l_A(\mathcal{G})$. $\square$

Theorem 4.9. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$.
(i) If $U \in T_A$, then $U \cap (V(H))^l = U \cap (U \cap V(H))^l$, for any subgraph H of G.
(ii) If $T_A - \{\phi\} \subseteq \mathcal{G}$, then for all subgraphs W with $V(W) \in TA$, $V(W) \subseteq (V(W))l$.

Proof. (i). Since $(U \cap V(H)) \subseteq V(H)$, $(U \cap V(H))^l \subseteq (V(H))l$ which implies $U \cap (U \cap V(H))^l \subseteq U \cap (V(H))^l$. Let $v \in U \cap (V(H))^l$ and $X \in N(v)$. Then $(U \cap X) \in N(v)$. Since $v \in (V(H))^l$, $(U \cap X) \cap V(H) = V(G')$ for some $G' \in \mathcal{G}$. Hence $(U \cap V(H)) \cap X = V(G')$ for some $G' \in \mathcal{G}$ G and $X \in N(v)$ implies $v \in (U \cap V(H))^l$. So $v \in U \cap (U \cap V(H))^l$. Hence $U \cap (V(H))^l \subseteq U \cap (U \cap V(H))^l$.
(ii). Since $T_A - \{\phi\} \subseteq \mathcal{G}$, $(V(G))^l = V(G)$. For any subgraph W with $V(W) \in T_A - \{\phi\}$, by (i), $V(W) \cap (V(G))^l = V(W) \cap (V(W) \cap V(G))^l = V(W) \cap (V(W))^l$. So $V(W) = V(W) \cap V(G) = V(W) \cap (V(G))^l = V(W) \cap (V(W))^l$. Hence $V(W) \subseteq (V(W))^l$ $\square$

Theorem 4.10. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$.
(i) If H and W are two induced subgraphs of G then $(V(H))^l - (V(W))^l = (V(H) - V(W))^l - (V(W))^l$.
(ii) If H and W are two induced subgraphs of G with $V(W) \neq V(G')$ for any $G' \in \mathcal{G}$, then $(V(H) \cup V(W))^l = (V(H))^l = (V(H) - V(W))^l$.

Proof. (i) By theorem 3.6, $(V(H))^l = ((V(H) - V(W)) \cup (V(H) \cap V(W)))^l = (V(H) - V(W))^l \cup (V(H) \cap V(W))^l \subseteq (V(H) - V(W))^l \cup (V(W))^l$. So $(V(H))^l - (V(W))^l \subseteq (V(H) - V(W))^l - (V(W))^l$. Also, $(V(H) - V(W))^l \subseteq (V(H))^l$. So $(V(H) - V(W))^l - (V(W))^l \subseteq (V(H))^l - (V(W))^l$. Hence $(V(H))^l - (V(W))^l = (V(H) - V(W))^l - (V(W))^l$.
(ii) By theorem $(V(H) \cup V(W))^l = (V(H))^l \cup (V(W))^l = (V(H))^l$. Also, $(V(H) - V(W))^l \subseteq (V(H))^l$. By (i), $(V(H))^l - (V(W))^l \subseteq (V(H) - V(W))^l$, so that $(V(H))^l \subseteq (V(H) - V(W))^l$. Hence $(V(H))^l = (V(H) - V(W))^l$. $\square$

Theorem 4.11. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$ and $N_v = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$. If H is an induced subgraph of G such that $V(H) \subseteq (V(H))^l$ then $cl(V(H)) = T^l_A - cl(V(H)) = cl((V(H))^l) = (V(H))^l$.

Proof. Since $T_A \subseteq T^l_A$, $T^l_A - cl(V(H)) \subseteq cl(V(H))$. Now, $v \notin T^l_A - cl(V(H)) \Rightarrow$ there exist $U \in T_A$ and a subgraph W of G with $V(W) \neq V(G')$ for any $G' \in \mathcal{G}$ such that $v \in U - V(W)$ and $(U - V(W)) \cap V(H) = \phi \Rightarrow ((U - V(W)) \cap V(H))^l = \phi \Rightarrow ((U \cap V(H)) - V(W))^l = \phi \Rightarrow (U \cap V(H))^l = \phi \Rightarrow U \cap (V(H))^l =$

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ϕ ▯ $U \cap V(H) = \phi$ ▯ $v \notin cl(V(H))$. So $cl(V(H))$ ▯ $T^lA - cl(V(H))$. By theorem 3.7, $(V(H))^l = cl((V(H))^l)$ and $(V(H))^l$ ▯ $cl(V(H))$. Also, $(V(H))^l$ ▯ $cl(V(H))$ ▯ $cl(V(H))^l)$ ▯ $cl(cl(V(H))) = cl(V(H))$. Now, $V(H)$ ▯ $(V(H))^l$ ▯ $cl(V(H))$ ▯ $Cl((V(H))^l)$. So $cl(V(H)) = cl((V(H))^l) = (V(H))^l$. □

Theorem 4.12. Let $(V(G), T_A)$ be a graph adjacency topological space with a graph grill $\mathcal{G}$ and $N_v = \{v\}$ ▯ $\{u \in V(G) : uv\ \ E(G)\}$. Then the following are equivalent:
(i) For all vertex induced subgraphs H of G with $V(H) \in T_A$, $V(H)$ ▯ $(V(H))^l$.
(ii) For all vertex induced subgraphs H of G with U ▯ $V(H)$ ▯ $cl(U)$ for some $U \in T_A$, $V(H)$ ▯ $(V(H))^l$.

Proof. (i) ▯ (ii): Since $U \in T_A$, by (i) U ▯ $(U)^l$. By theorem 4.11, $cl(U) = (U)^l$. Since U ▯ $V(H)$, $(U)^l$ ▯ $(V(H))^l$. Also, $V(H)$ ▯ $cl(U)$. So, $V(H)$ ▯ $(V(H))^l$.
(ii) ▯ (i) is obvious. □

5. Graph Adjacency Topology that Fits a Graph Grill

This section presents the conditions satisfying by graph grills through which the graph grills become more suitable for graph adjacency topology on the vertex set of a graph.

Definition 5.1. Let $\mathcal{G}$ be a graph grill on a graph adjacency topological space $(V(G), TA)$. Then TA is said to fit the graph grill, if, for all induced subgraphs H of G, $V(H) - (V(H))^l \neq V(G')$, for any $G' \in \mathcal{G}$.

The following theorem describes some equivalent descriptions of the above definition.

Theorem 5.2. Let $\mathcal{G}$ be a graph grill on a graph adjacency topological space $(V(G), TA)$ and $N_v = \{v\}$ ▯ $\{u \in V(G) : uv \in E(G)\}$. Then the following are equivalent:
(i) T_A fits the graph grill $\mathcal{G}$
(ii) For any vertex induced subgraph H of G such that V(H) is $T_A{}^l(\mathcal{G})$-closed, $V(H) - (V(H))^l \neq V(G')$, for any $G' \in \mathcal{G}$.
(iii) If for any vertex induced subgraph H of G and every $v \in V(H)$ there exists some $U \in N(v)$ such that $U \cap V(H) \neq V(G')$, for any $G' \in \mathcal{G}$, then $V(H) \neq V(G')$, for any $G' \in \mathcal{G}$.
(iv) If H is a vertex induced subgraph of G such that $V(H) \cap (V(H))^l = \phi$, then $V(H) \neq V(G')$, for any $G' \in \mathcal{G}$

Proof. (i) ▯ (ii) follows from definition.
(ii) ▯ (iii) : Let H be a vertex induced subgraph of G such that for every $v \in V(H)$ there exists some $U \in N(v)$ such that $U \cap V(H) \neq V(G')$, for any $G' \in \mathcal{G}$. Then $v \notin (V(H))^l$ and so $V(H) \cap (V(H))^l = \phi$. Since $V(H)$ ▯ $(V(H))^l$ is $T_A{}^l(G)$-closed, by (ii), $(V(H)$ ▯ $(V(H))^l) - (V(H)$ ▯ $(V(H))^l) \neq V(G')$, for any $G' \in \mathcal{G}$, i.e., $(V(H)$▯$(V(H))^l) - [(V(H))^l$▯ $((V(H))^l)^l] \neq V(G')$, for any $G' \in \mathcal{G}$, i.e., $(V(H)$ ▯ $(V(H))^l) - (V(H))^l \neq V(G')$, for any $G' \in \mathcal{G}$, i.e., $V(H) \neq V(G')$, for any $G' \in \mathcal{G}$.
(iii) ▯ (iv) : If H is a vertex induced subgraph of such that $V(H) \cap (V(H))^l = \phi$, then $V(H)$ ▯ $V(G) - (V(H))^l$. Let $v \in V(H)$. Then $v \notin (V(H))^l$. So there exists some $U \in N(v)$ such that $U \cap V(H) \neq V(G')$, for any $G' \in \mathcal{G}$. Hence by (iii), $V(H) \neq V(G')$, for any $G' \in \mathcal{G}$.
(iv) ▯ (i) : Let H be a vertex induced subgraph of G and $v \in [V(H)-(V(H))^l] \cap [V(H)-(V(H))^l]^l$. $v \in [V(H) - (V(H))^l]$ ▯ $v \in V(H)$ and $v \notin (V(H))^l$ which implies there exists $U \in N(v)$ such that $U \cap V(H) \neq V(G')$, for any $G' \in \mathcal{G}$. Further, $U \cap (V(H)-(V(H))^l)$ ▯ $U \cap V(H)$. So $U \cap (V(H) - (V(H))^l) \neq V(G')$, for any $G' \in \mathcal{G}$ and $v \notin (V(H)-(V(H))^l)^l$ which contradicts $v \in [V(H)-(V(H))^l] \cap [V(H)-(V(H))^l]^l$. So $[V(H) - (V(H))^l] \cap [V(H) - (V(H))^l]^l = \phi$. So by (iv), $V(H) - (V(H))^l \neq V(G')$, for any $G' \in \mathcal{G}$. □

Theorem 5.3. The following are equivalent for

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a graph grill $\mathcal{G}$ on the graph adjacency topological space $(V(G), T_A)$, and each is a necessary condition for the topology T_A to fit the graph grill $\mathcal{G}$.

(i) For any vertex induced subgraph H of G, $V(H) \cap (V(H))^l = \phi$ ⊠ $(V(H))^l = \phi$

(ii) For any vertex induced subgraph H of G, $(V(H) - (V(H))^l)^l = \phi$

(iii) For any vertex induced subgraph H of G, $(V(H) \cap (V(H))^l)^l = (V(H))^l$

Proof. The equivalence of (i), (ii) and (iii) is proved first.

(i) ⊠ (ii) : Since $[V(H) - (V(H))^l] \cap (V(H) - (V(H))^l) = \phi$, for any vertex induced subgraph H of G, it follows that $(V(H) - (V(H))^l)^l = \phi$

(ii) ⊠ (iii) : Now, $V(H) = [V(H) - (V(H) \cap (V(H))^l)]$⊠ $[V(H) \cap (V(H))^l]$ implies $(V(H))^l = [V(H) - (V(H) \cap (V(H))_l)]_l$⊠ $[V(H) \cap (V(H))^l]^l = (V(H) - (V(H))^l)^l$ ⊠ $(V(H) \cap (V(H))^l)^l = (V(H) \cap (V(H))^l)^l$

(iii) ⊠ (i) : Let H be a vertex induced subgraph of G with $V(H) \cap (V(H))^l = \phi$. By (iii), $(V(H))^l = [V(H) \cap (V(H))^l]^l = \phi$.

The rest of theorem follows from theorem 3.6. □

Corollary 5.4. If a topology TA fits the graph grill G, then $((V(H))l)l = (V(H))l$.

Proof. By theorem 5.3 and 3.6, $(V(H))l = (V(H) \cap (V(H))l)l$ ⊠ $((V(H))l)l$. From theorem 3.6, $((V(H))l)l$ ⊠ $(V(H))l$. So $((V(H))l)l = (V(H))l$. □

Theorem 5.5. Let G be a graph grill on a graph adjacency topological space $(V(G), T_A)$ such that T_A fits the graph grill $\mathcal{G}$ and $N_v = \{v\}$ ⊠ $\{u \in V(G) : uv \in E(G)\}$. Let H be a vertex induced subgraph of G. Then $V(H)$ is $T_A{}^l(\mathcal{G})$-closed if and only if it can be expressed as a union of a vertex induced subgraphs of G whose vertex set is closed in $(V(G), T_A)$ and a vertex induced subgraph of G whose vertex set is not equal to vertex set of any subgraph belong to G.

Proof. Let $V(H)$ be $T_A{}^l(\mathcal{G})$-closed. Then $(V(H))^l$ ⊠ $V(H)$. Further, $V(H) = (V(H))^l$ ⊠ $[V(H) - (V(H))^l]$. Since T_A fits the graph grill G, $[V(H) - (V(H))^l] \neq V(G')$, for any $G' \in \mathcal{G}G$, and by theorem 3.6, $(V(H))^l$ is TA-closed. Conversely, assume that $V(H) = V(W_1)$ ⊠ $V(W_2)$, where $V(W_1)$ is $T_A{}^l(\mathcal{G})$-closed and $V(W_2) \neq V(G')$, for any $G' \in \mathcal{G}$. Then by theorem 3.7, $(V(H))^l = (V(W_1)$ ⊠ $V(W_2))^l = (V(W_1))^l$ ⊠ $cl(V(W_1)) = V(W_1)$ ⊠ $V(H)$. Hence $V(H)$ is $T_A{}^l(\mathcal{G})$-closed. □

Corollary 5.6. Let G be a graph grill on a graph adjacency topological space $(V(G), T_A)$ such that T_A fits the graph grill $\mathcal{G}$. Then $B(G, T'{}^l_A(\mathcal{G}))$ is a topology on the vertex set of G and hence $B(G, T'{}^l_A(\mathcal{G})) = T_A{}^l(\mathcal{G})$. 433

Proof. Let $U \in T_A{}^l(\mathcal{G})$. Then by theorem 5.5, $V(G) - U = V(W_1)$ ⊠ $V(W_2)$, where $V(W_1)$ is $T_A{}^l(\mathcal{G})$-closed and $V(W_2) \neq V(G')$, for any $G' \in \mathcal{G}$. So $U = (V(G) - V(W_1)) \cap (V(G) - V(W_2)) = (V(G) - V(W_1)) - V(W_2)) = V(W_3) - V(W_2))$, where $V(W_3) = (V(G) - V(W_1)) \in T_A$. Thus every $T_A{}^l(\mathcal{G})$-open set is of the form $V(W_3) - V(W_2)$, where $V(W_3) \in T_A$ and $V(W_2) \neq V(G')$, for any $G' \in \mathcal{G}$. The rest follows from theorem 4.8 □

Theorem 5.7. Let G be a graph grill on a graph adjacency topological space $(V(G), T_A)$ such that $T_A - \{\phi\}$ ⊠ $\mathcal{G}$ and T_A fits the graph grill $\mathcal{G}$ and $Nv = \{v\}$ ⊠ $\{u \in V(G) : uv \in E(G)\}$. Let H be a vertex induced subgraph of G such that V(H) is $T_A{}^l(\mathcal{G})$-open and $V(H) = V(W_1) - V(W_2)$ where $V(W_1) \in T_A$ and $V(W_2) \neq V(G')$, for any $G' \in \mathcal{G}$. Then $T'{}^l_A - cl(V(H)) = cl(V(H)) = (V(H))^l = (V(W_1))^l = cl(V(W_1)) = T'{}^l_A - cl(V(W_1))$.

Proof. Since $T_A - \{\phi\}$ ⊠ $\mathcal{G}$, by theorem 4.9, $V(W_1)$ ⊠ $(V(W_1))^l$. Hence by theorem 4.11, $(V(W1))^l = cl(V(W_1)) = T'{}^l_A - cl(V(W_1))$. Since V(H) is $T_A{}^l(\mathcal{G})$-open , it is necessary to prove $V(H)$ ⊠ $(V(H))^l$. In-fact, $T'{}^l_A - cl(V(G) - V(H)) = V(G) - V(H)$ ⊠ $(V(G) - V(H))^l$ ⊠

$V(G) - V(H)$ ⍰ $(V(G))^l - (V(H))^l$ ⍰ $V(G) - V(H)$ ⍰ $V(G) - (V(H))^l$ ⍰ $V(G) - V(H)$ ⍰ $V(H)$ ⍰ $(V(H))^l$. Hence by theorem 4.11, $(V(H))^l = cl(V(H)) = T^l{}_A- cl(V(H))$. Since $V(H)$ ⍰ $V(W_1)$, $(V(H))^l$ ⍰ $(V(W_1))^l$ and $(V(H))^l = (V(W_1) - V(W_2))^l$ ⍰ $(V(W_1))^l - (V(W_2))^l = (V(W_1))^l$. So $(V(W_1)^l = (V(H))^l$. Thus $T^l{}_A- cl(V(H)) = cl(V(H)) = (V(H))^l = (V(W_1))^l = cl(V(W_1)) = T^l{}_A - cl(V(W_1))$. □

Theorem 5.8. Let $\mathcal{G}$ be a graph grill on a graph adjacency topological space $(V(G), T_A)$ such that T_A fits the graph grill $\mathcal{G}$ and $N_v = \{v\}$ ⍰ $\{u \in V(G) : uv \in E(G)\}$. Then for every $U \in T_A$ and any vertex induced subgraph H, $(U \cap V(H))^l = (U \cap (V(H))^l)^l = cl(U \cap (V(H))^l)$.

Proof. Let $U \in T_A$. By theorem 4.9, $U \cap (V(H))^l = U \cap (U \cap V(H))^l$ ⍰ $(U \cap V(H))l$ and so $(U \cap (V(H))^l))^l$ ⍰ $((U \cap V(H))^l)^l = (U \cap V(H))^l$. By theorems 4.9 and 5.3, $(U \cap (V(H)-(V(H))^l))^l = U \cap (V(H)-(V(H))^l)^l = U \cap \phi = \phi$. Also, $(U \cap V(H))^l-(U \cap (V(H))^l)^l$ ⍰ $((U \cap V(H)) - (U \cap (V(H))^l)^l = (U \cap (V(H) - (V(H))^l))^1 = \phi$. Hence $(U \cap V(H))^l$ ⍰ $(U \cap (V(H))^l)^l$. So $(U \cap V(H))^l = (U \cap (V(H))^l)^l$. By theorem 3.7, $(U \cap V(H))^l = (U \cap (V(H))^l)^l$ ⍰ $cl(U \cap (V(H))^l)$. By theorem 4.9, $U \cap (V(H))^l$ ⍰ $(U \cap V(H))^l$ ⍰ $cl(U \cap (V(H))^l)$ ⍰ $cl((U \cap V(H))^l) = (U \cap V(H))^l$. Hence $(U \cap V(H))^l = cl(U \cap (V(H))^l)$. □

Theorem 5.9. Let $\mathcal{G}$ be a graph grill on a graph adjacency topological space $(V(G), T_A)$ such that T_A fits the graph grill $\mathcal{G}$ and $Nv = \{v\}$ ⍰ $\{u \in V(G) : uv \in E(G)\}$. If H is an induced subgraph of G such that $V(H) \in T_A - \mathcal{G}$, then $V(H)$ ⍰ $V(G) - (V(G))^l$.

Proof. By theorem 5.8, $(V(H) \cap V(G))^l = cl(V(H) \cap (V(G))l)$, i.e., $(V(H))^l = cl(V(H) \cap (V(G))^l)$, for all vertex induced subgraphs H such that $V(H) \in T_A$. If $V(H) \neq V(G')$, for any $G' \in \mathcal{G}$, then $(V(H))^l = \phi$. If $V(H) \in T_A - \mathcal{G}$, then $cl(V(H) \cap (V(G))^l) = \phi$. So $V(H) \cap (V(G))^l = \phi$ and $V(H)$ ⍰ $V(G) - (V(G))^l$. □

6. Conclusion

Graph Grill and Graph Grill operator are introduced and their characteristic properties are studied. A basis for a new topology from the old one via graph grill has been generated and the graph grill that fits the graph adjacency topology in a graph adjacency topological space is investigated. Further, the results in this paper is useful in the study of some new sets and topologies in graph adjacency topological space with graph grill. These concepts can be studied further by defining graph limit operator of induced subgraph of a graph with respect to graph adjacency topology and filter.

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