ON ASYMPTOTICALLY I-EQUIVALENT SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract
This study aims to introduce the asymptotically I- equivalence, strongly asymptotically I-equivalence, asymptotically I^*- equivalence, and strongly asymptotic I^*-equivalence of two non-negative sequences of some multiple λ, where I is an ideal on $\mathbb{N}$, in intuitionistic fuzzy normed space. We've also included some fascinating examples to go along with the definitions. Furthermore, several theorems have been examined to demonstrate how the different ideas are connected.

Keywords:-Asymptotic equivalent sequences, I-Convergence, Intuitionistic fuzzy normed spaces, t-norm, t-conorm.
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1 Introduction

The proposed work is inspired by [1], where B. Hazarika developed the notion of asymptotically I −equivalence of two non-negative sequences $[x] = (x_p)$ and $[y] = (y_p)$. We will frequently extend some definitions and result from [1] to intuitionistic fuzzy normed spaces. In 2006, Saadati and Park [2] presented the idea of intuitionistic fuzzy normed space, which is nothing more than a generalised study of the intuitionistic fuzzy set, which was initially analyzed by Atanassov [3]. Many authors further examined the principles of [2] in their own ways, such as the idea of statistical convergence of single sequences in intuitionistic fuzzy normed spaces in [4], and for double sequences in [5]. In certain contexts, determining the vector's norm is impossible, hence the intuitionistic fuzzy norm is more appropriate in such cases. The notion of the fuzzy norm has been employed in various literature [6, 7], to cope with the inexactness of the norm in certain circumstances. 502

M.S.Marouf [8] presented literature on asymptotically equivalent sequences of real numbers and asymptotic regular matrices. Later, R.F.Patterson [9] investigated the notion for asymptotic statistically equivalent sequences and natural regularity requirements for non-negative summability matrices. Since then, progress has been made on the concept examined in [8].

The idea of statistical convergence evolved as a helpful tool in examining the convergence of numerical sequences using the natural density concept. Fast[10, 11], and Schoenberg were the first to investigate the concept. Fridy [12] later investigated the notion for the sequence space and expanded it to the summability theory. The natural density of set T, a subset of the set $\mathbb{N}$, is denoted by $\delta(A)$ and defined by $\delta(T) = \lim_{p\to\infty} \frac{1}{p}|\{q \le p : q \in \mathbb{N}\}|$, where $\mathbb{N}$ signifies a set of positive integer. In 2001, Kostyrko [13] proposed the idea of ideal convergence as a generalisation of statistical convergence. Later, in 2004, T. Salat, Tripathy, and Ziman[14] examined various $I-$convergence characteristics. In 2010, B.C Tripathy investigated $I-$ acceleration convergence of sequences[15]. In 2012, A. J. Dutta and B.C Tripathy further extended the theory to $I-$ acceleration convergence of sequence of fuzzy real numbers [16].

An ideal I on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that meet the requirements, (i) $\phi \in I$, (ii) $X_1, Y_1 \in I$ then $X_1 \cup Y_1 \in I$, (iii) $X_1 \in I$ and $Y_1 \subseteq X_1$ then $Y_1 \in I$. Ideal I is called non-trivial if $\mathbb{N}$ doesn't lies in I. There is another family of subsets of set $\mathbb{N}$, we denote it by I^* and termed a filter on $\mathbb{N}$ associated to ideal I, that meets the requirements, (i)$\phi \notin I^*$, (ii) $X_1, Y_1 \in I^*$ then $X_1 \cap Y_1 \in I^*$, (iii)$X_1 \in I^*$ and $Y_1 \supseteq X_1$ then $Y_1 \in I^*$. Some types of non trivial admissible ideals (see [13]), defined as (i) $I_\delta = \{k \in \mathbb{N} : \delta(A) = 0\}$ and (ii) $I_c = \{k \in \mathbb{N} : \sum_{k\in A} k^{-1} < \infty\}$, are frequently used in the work.

We now outline the proposed work in the article as follows. Section 1 is the introduction part containing brief history and some concepts, section 2 contains the basic definitions and results which are useful in the work, section 3 contains some new definitions with examples, and section 4 is devoted to some new theorems and results that established the relationship between studied notions in section 3.

Throughout the article $\mathbb{R}$ signifies collection of all real numbers, $\mathbb{N}$ stands for set of positive integers, $[x] = (x_p)$ and $[y] = (y_p)$ are for the sequences of non-negative terms, U and U^2 denotes the unit interval $[0,1]$ and $[0,1]^2$, respectively. The considered ideal I in the study is an admissible ideal on $\mathbb{N}$.

2 Background

Definition 1.[2] An intuitionistic fuzzy normed space (abbreviated as **IFNS**) is a five tuple object of the form ($\mathbb{E}$,ψ, ϕ,$\circ$,$\diamond$), where $\mathbb{E}$ is a linear space, $\circ$ and $\diamond$ are t-norm and t-conorm, respectively on $\mathbb{E}$, ψ and ϕ define fuzzy sets on $\mathbb{E} \times \mathbb{E} \times (0,\infty)$ that fulfils the following subsequent requirements. If $\forall\ y, w, z \in \mathbb{E}$ and $t, \tau \in (0,\infty)$

(i). $0 < \psi(y,\tau) + \phi(y,\tau) \le 1$
(ii). $\psi(y,\tau): (0,\infty) \to (0,1]$ is a continuous function of τ
(iii). $0 < \psi(y,\tau) \le 1$
(iv). $\psi(y,\tau) = 1$ for all τ, iff $y = 0$
(v). $\psi(y,\alpha\tau) = \psi(y,\frac{\tau}{|\alpha|})$ for non-zero scalar, α
(vi). $\psi(y,t)\ \ o\ \ \psi(w,\tau) \le \psi(y+w, t+\tau)$
(vii). $\lim_{\tau\to\infty}\psi(y,\tau) = 1$ and $\lim_{\tau\to 0}\psi(y,\tau) = 0.$
(viii). $0 \le \phi(y,\tau) < 1$
(ix). $\phi(y,\tau): (0,\infty) \to [0,1)$ is a continuous function of τ
(x). $\psi(y,\alpha\tau) = \phi(y,\frac{\tau}{|\alpha|})$ for non-zero scalar, α
(xi). $0 \le \phi(y,\tau) < 1$

(xii). $\phi(y,\tau) = 0$ for all τ, iff $y = 0$
(xiii). $\phi(y,t) \diamond \phi(w,\tau) \geq \phi(y+w, t+\tau)$
(xiv). $\lim_{\tau\to\infty}\phi(y,\tau) = 0$ and $\lim_{\tau\to 0}\phi(y,\tau) = 1$.

The fuzzy norm on $\mathbb{E}$ is defined by the pair (ψ,ϕ). The degrees of belongings and non-belongings of u to $\mathbb{E}$, with regard to τ are defined by the functions $\psi(u,\tau)$ and $\phi(u,\tau)$.

Definition 2.[7] A binary operation $\circ: U^2 \to U$, meeting the requirement.

(i). $\circ$ is continuous.
(ii). $\circ$ is commutative.
(iii). $\circ$ is associative.
(iv) $\circ$ is monotonically increasing, i.e. for all $g_1, g_2, g_3, g_4 \in [0,1]$ we have $g_1 \circ g_2 \leq g_3 \circ g_4$ whenever $g_1 \leq g_3$ and $g_2 \leq g_4$.
(v). $g \circ 1 = g$ for all $g \in [0,1]$

is defined as continuous t −norm.

Definition 3.[7] A binary operation. $\diamond: U^2 \to U$, meeting the requirements. [(i)]

(i). $\diamond$ is continuous.
(ii). $\diamond$ is commutative.
(iii). $\diamond$ is associative.
(iv). $\diamond$ is monotonically increasing, i.e. for all $g_1, g_2, g_3, g_4 \in [0,1]$ we have $g_1 \diamond g_2 \leq g_3 \diamond g_4$ whenever $g_1 \leq g_3$ and $g_2 \leq g_4$.
(v). $g \diamond 0 = g$ for all $g \in [0,1]$

is defined as continuous t −conorm.

Remark 1.[17] On IFNS $\mathbb{E}$, $\psi(u,\tau)$ and $\phi(u,\tau)$ are non-decreasing and non-increasing functions of τ, respectively.

Remark 2.[17] For any $\gamma_1, \gamma_2 \in (0,1)$ we can find $0 < \gamma < 1$ such that $(1-\gamma_1) \circ (1-\gamma_2) \geq 1-\gamma$ and $\gamma_1 \diamond \gamma_2 \leq \gamma$.

Definition 4. [8] Two sequences $[x] = (x_p)$ and $[y] = (y_p)$ of real numbers are defined as asymptotically equivalent, if $lim_{p\to\infty}\frac{x_p}{y_p} = 1$. Symbolically, we state the case as, $[x] \sim [y]$.

Definition 5. [1] Assume I is an admissible ideal on $\mathbb{N}$. Two sequences $[x] = (x_p)$ and $[y] = (y_p)$ are called asymptotically I- equivalent of multiple λ if for every $\varepsilon_0 > 0$ the set

$$\left\{p: \left|\frac{x_p}{y_p} - \lambda\right| \geq \varepsilon_0\right\} \text{ lies in } I.$$

The two sequences are said to be simply asymptotic $I-$ equivalent, if $\lambda = 1$

Symbolically, we state the case as, $[x] \overset{I_\lambda}{\sim} [y]$.

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3 Main definitions

Definition 6. Two sequences $[x]=(x_p)$ and $[y]=(y_p)$ of the elements of IFNS $(\mathbb{E},\psi,\phi,\circ,\diamond)$ are known as asymptotically equivalent of multiple λ with regard to IFN (ψ,ϕ) if for all $\tau>0$, we have

$$\lim_{p\to\infty}\psi\left(\frac{x_p}{y_p}-1,\tau\right)\to 1 \quad \text{and} \quad \lim_{p\to\infty}\phi\left(\frac{x_p}{y_p}-1,\tau\right)\to 0.$$

Symbolically, we denote the case as $(\psi,\phi)-[x]\sim[y]$.

Definition 7. Two sequences $[x]=(x_p)$ and $[y]=(y_p)$ of the elements of IFNS $(\mathbb{E},\psi,\phi,\circ,\diamond)$ are known as asymptotic I- equivalent of multiple λ with regard to IFN (ψ,ϕ) if for all $\tau>0$ and for every $0<\varepsilon<1$ the set,

$$\left\{p\ :\ \psi\left(\frac{x_p}{y_p}-\lambda,\tau\right)\le 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_p}{y_p}-\lambda,\tau\right)\ge\varepsilon\right\} \quad \text{lies in} \quad I.$$

Symbolically, we state the case as $(\psi,\phi)-[x]\overset{I_\lambda}{\sim}[y]$.

The two sequences are called simply asymptotic $I-$ equivalent with regard to IFN (ψ,ϕ) if $\lambda=1$.

Example 1. *Assume $I=\{B\subseteq\mathbb{N}\ :\delta(B)=0\}$(see [13]) is an ideal on $\mathbb{N}$ and sequences $[x]=(x_r)$ and $[y]=(y_r)$ on IFNS $(\mathbb{R},\psi,\phi,\circ,\diamond)$ are defined as*

$$(x_r)=\begin{cases} r & \text{if } r=p^2 \quad ; \quad p\in\mathbb{N} \\ 2r^2 & \text{otherwise}\end{cases}$$

$$(y_r)=\begin{cases} \sqrt{r} & \text{if } r=p^2 \quad ; \quad p\in\mathbb{N} \\ r^2 & \text{otherwise}\end{cases}$$

Define $\psi(w,\tau)=\frac{\tau}{\tau+|w|}$ and $\psi(w,\tau)=\frac{|w|}{\tau+|w|}$; then $(\psi,\phi)-[x]\overset{I_2}{\sim}[y]$.

Proof. When, $r=p^2$ where $p\in\mathbb{N}$, then for $\tau>0$ and $0<\varepsilon<1$ we have,

$$\lim_{p\to\infty}\psi(\frac{x_r}{y_r}-2,\tau)=\lim_{p\to\infty}\frac{\tau}{\tau+|\frac{x_r}{y_r}-2|}=\lim_{p\to\infty}\frac{\tau}{\tau+|\sqrt{r}-2|}\le 1-\varepsilon \quad (1)$$

or,

$$\lim_{p\to\infty}\phi(\frac{x_r}{y_r}-2,\tau)=\lim_{p\to\infty}\frac{|\frac{x_r}{y_r}-2|}{\tau+|\frac{x_r}{y_r}-2|}=\lim_{p\to\infty}\frac{|\sqrt{r}-2|}{\tau+|\sqrt{r}-2|}\ge\varepsilon. \quad (2)$$

we therefore, from equation (3.1) and (3.2) obtained,

$$\left\{r\ :\ \psi\left(\frac{x_r}{y_r}-\lambda,t\right)\le 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_r}{y_r}-\lambda,t\right)\ge\varepsilon\right\}=\{r=p^2:\ p\in\mathbb{N}\}=B(\text{say}).$$

We now have to prove that $B\in I$ for this requirement, it is suffice to prove $\delta(B)=0$.
Now using the fact,

$$0\le\delta(B)\le\lim_{r\to\infty}\frac{\sqrt{r}}{r}=0.\Rightarrow B\in I. \quad \text{Hence} \quad (\psi,\phi)-[x]\overset{I_2}{\sim}[y]$$

Definition 8. Two sequences $[x]=(x_p)$ and $[y]=(y_p)$ of the elements of IFNS $(\mathbb{E},\psi,\phi,\circ,\diamond)$

are known as strongly asymptotic I- equivalent of multiple λ with regard to IFN (ψ,ϕ) if $\forall$ $\tau>0$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{p\in A_n}\psi\left(\frac{x_p}{y_p}-\lambda,\tau\right)\to 1 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{p\in A_n}\phi\left(\frac{x_p}{y_p}-\lambda,\tau\right)\to 0$$

for every $A_n\in I$. Symbolically, we state the case as $(\psi,\phi)-[x]\overset{N_\lambda^I}{\sim}[y]$.

The two sequences are known as simply strongly asymptotic $I-$ equivalent with respect to IFN (ψ,ϕ) if $\lambda=1$

Example 2. *Assume I is an ideal on $\mathbb{N}$ and two sequences $[x]=(x_p)$ and $[y]=(y_p)$ on IFNS $(\mathbb{R},\psi,\phi,\circ,\diamond)$ are defined as $[x]=(x_p)=2p^2+p$ and $[y]=(y_p)=p^2$ and let $\psi(w,\tau)=\frac{\tau}{\tau+|w|}$. Define $\phi(w,\tau)=\frac{|w|}{\tau+|w|}$, then $(\psi,\phi)-[x]\overset{N_2^I}{\sim}[y]$.*

Proof. Let $A_n\in I$ such that $|A_n|=n$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(\frac{x_r}{y_r}-2,\tau\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(2+\frac{1}{r}-2,\tau\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(\frac{1}{r},\tau\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\frac{\tau}{\tau+\left|\frac{1}{r}\right|}. \quad (3)$$

Now there exist some $r_i\in A_n$ such that for all $r\geq r_i$ we have, $\frac{1}{r}\approx 0$ which implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\frac{\tau}{\tau+\left|\frac{1}{r}\right|}=\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}^{r_i}\frac{\tau}{\tau+\left|\frac{1}{r}\right|}+\lim_{n\to\infty}\frac{1}{n}\sum_{r\geq r_i;r\in A_n}\frac{\tau}{\tau+\left|\frac{1}{r}\right|}=\lim_{n\to\infty}\frac{1}{n}(finite)+\lim_{n\to\infty}\frac{1}{n}(n-|\{r_1,r_2,r_3,..r_i\}|=0+\lim_{n\to\infty}\left(1-\frac{finite}{n}\right)=1$$

And

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\phi\left(\frac{x_r}{y_r}-2,\tau\right)=1-\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(2+\frac{1}{r}-2,\tau\right)=1-\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(\frac{1}{r},\tau\right)=1-1=0. \quad (4)$$

Now equations (3) and (4) yield, $(\psi,\phi)-[x]\overset{N_2^I}{\sim}[y]$.

Definition 9. Sequences $[x]=(x_p)$ and $[y]=(y_p)$ are asymptotically I^*- equivalent of multiple λ if there exist a set $T\in I$ such that for $K=\mathbb{N}-T=\{k_1,k_2,k_3,\dots\}$ we have,

$$\lim_{p\to\infty}\frac{x_{k_p}}{y_{k_p}}=\lambda$$

Symbolically, we state the case as, $[x]\overset{I_\lambda^*}{\sim}[y]$.

Definition 10. Sequences $[x]=(x_p)$ and $[y]=(y_p)$ of the elements of IFNS $(\mathbb{E},\psi,\phi,\circ,\diamond)$ are known as asymptotic I^*- equivalent of multiple λ with regard to IFN (ψ,ϕ) if there exist a set $T\in I$ such that for $K=\mathbb{N}-T=\{k_1,k_2,k_3,\dots\}$ we have,

$$\lim_{p\to\infty}\psi\left(\frac{x_{k_p}}{y_{k_p}}-\lambda,\tau\right)\to 1 \quad \text{or} \quad \lim_{p\to\infty}\phi\left(\frac{x_{k_p}}{y_{k_p}}-\lambda,\tau\right)\to 0$$

Symbolically, we state the case as, $(\psi,\phi)-[x]\overset{I_\lambda^*}{\sim}[y]$.

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The two sequences are known as simply asymptotic $I^* -$ equivalent if $\lambda = 1$

Example 3. *Assume* $I = \{A \subseteq \mathbb{N} : \sum_{k \in A} k^{-1} < \infty\}$*(see [13]) be an ideal on* $\mathbb{N}$*, the two sequences* $[x] = (x_p)$ *and* $[y] = (y_p)$ *on IFNS* $(\mathbb{R}, \psi, \phi, \circ, \diamond)$ *are defined as*

$$(x_r) = \begin{cases} r & \text{if } r = p^2 \quad ; \quad p \in \mathbb{N} \\ 3r^2 + 2 & \text{otherwise} \end{cases}$$

$$(y_r) = \begin{cases} \sqrt{r} & \text{if } r = p^2 \quad ; \quad p \in \mathbb{N} \\ r^2 & \text{otherwise} \end{cases}$$

Define $\psi(w, \tau) = \frac{\tau}{\tau + |w|}$ and $\psi(w, \tau) = \frac{|w|}{\tau + |w|}$; then $(\psi, \phi) - [x] \overset{I_2}{\sim} [y]$.

Proof. For every $\tau > 0$ and $0 < \varepsilon < 1$ there exist a set $T = \{p^2 : p \in \mathbb{N}\}$ such that for $K = \mathbb{N} - T = \{k_1, k_2, k_3, \dots\}$ we have,

$$\lim_{p \to \infty} \psi\left(\frac{x_{k_p}}{y_{k_p}} - 3, \tau\right) = \lim_{n \to \infty} \frac{\tau}{\tau + \left|\frac{3k_p^2 + 2}{k_p^2} - 3\right|} = \lim_{p \to \infty} \frac{\tau}{\tau + \left|\frac{2}{k_p^2}\right|} = 1 \tag{5}$$

And,

$$\lim_{p \to \infty} \phi\left(\frac{x_{k_p}}{y_{k_p}} - 3, \tau\right) = \frac{\left|\frac{3k_p^2 + 2}{k_p^2} - 3\right|}{\tau + \left|\frac{3k_p^2 + 2}{k_p^2} - 3\right|} = \lim_{p \to \infty} \frac{\frac{2}{k_p^2}}{\tau + \left|\frac{2}{k_p^2}\right|} = 0. \tag{6}$$

the two equations (5) and (6) simultaneously yield, $(\psi, \phi) - [x] \overset{I_3^*}{\sim} [y]$.

Definition 11. Two sequences $[x] = (x_p)$ and $[y] = (y_p)$ of the elements of IFNS$(\mathbb{E}, \psi, \phi, \circ, \diamond)$ are known as strongly asymptotic I^*- equivalent of multiple λ with regard to IFM (ψ, ϕ) if there exist a set $T \in I$ such that for $K = \mathbb{N} - T = \{k_1, k_2, k_3, \dots\}$ we have,

$$\lim_{p \to \infty} \frac{1}{K_p} \sum_{k_p \in K_p} \psi\left(\frac{x_{k_p}}{y_{k_p}} - \lambda, \tau\right) \to 1 \quad \text{and} \quad \lim_{p \to \infty} \frac{1}{K_p} \sum_{k_p \in K_p} \phi\left(\frac{x_{k_p}}{y_{k_p}} - \lambda, \tau\right) \to 0,$$

where $K_p \subseteq K$ containing p elements of K.

Symbolically, we state the case as, $(\psi, \phi) - [x] \overset{N_\lambda^{I^*}}{\sim} [y]$.

For $\lambda = 1$, the two sequences are called simply strongly asymptotically $I^* -$ equivalent

4 Main theorems

Theorem 1 *Let* I *be an ideal on* $\mathbb{N}$ *and the two sequences* $[x] = (x_p)$ *and* $[y] = (y_p)$ *are defined on IFNS* $(\mathbb{E}, \psi, \phi, \circ, \diamond)$*, if* $[x]$ *and* $[y]$ *are asymptotically equivalent of multiple* λ *with regard to* (ψ, ϕ) *then* $(\psi, \phi) - [x] \overset{I_\lambda}{\sim} [y]$

Proof. For every $\tau > 0$ and $0 < \varepsilon < 1$ we have,

$$\lim_{r \to \infty} \psi\left(\frac{x_r}{y_r} - \lambda, \tau\right) = 1 \quad \text{or} \quad \lim_{r \to \infty} \phi\left(\frac{x_r}{y_r} - \lambda, \tau\right) = 0 \tag{7}$$

hence, there exist $r_0 \in \mathbb{N}$ such that for all $r \geq r_0$ we have

$$\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right) > 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_r}{y_r}-\lambda,\tau\right) < \varepsilon \tag{8}$$

which gives

$$\left\{k \ : \ \psi\left(\frac{x_k}{y_k}-\lambda,\tau\right) \le 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_k}{y_k}-\lambda,\tau\right) \ge \varepsilon\right\} = \{1,2,3\ldots r_0\} \in I$$

$\Rightarrow \ (\psi,\phi) - [x] \overset{I_\lambda}{\sim} [y]$.

Theorem 2. *If* I *be an ideal on* $\mathbb{N}$ *and* $[x] = (x_p)$ *and* $[y] = (y_p)$ *are two sequences of the elements of IFNS* $(\mathbb{E},\psi,\phi,\circ,\diamond)$ *then* $(\psi,\phi) - [x] \overset{N_\lambda^I}{\sim} [y] \Rightarrow (\psi,\phi) - [x] \overset{I_\lambda}{\sim} [y]$

Proof. Given I is an ideal on $\mathbb{N}$ and $(\psi,\phi) - [x] \overset{N_\lambda^I}{\sim} [y]$ then for $A_n \in I$ where $|A_n| = n$ we have,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right) \to 1 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\phi\left(\frac{x_r}{y_r}-\lambda,\tau\right) \to 0 \tag{9}$$

Now for,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\left\{1-\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right)\right\} = 0 \tag{10}$$

We decompose $A_n = X(\varepsilon) \cup Y(\varepsilon)$ and for every $0 < \varepsilon < 1$ and $\tau > 0$ we put.

$$X(\varepsilon) = \left\{k : 1-\psi\left(\frac{x_k}{y_k}-\lambda,\tau\right) \ge \varepsilon\right\} \quad \text{and} \quad Y(\varepsilon) = \left\{k: 1-\psi\left(\frac{x_k}{y_k}-\lambda,\tau\right) < \varepsilon\right\}$$

. such that $X(\varepsilon) \cap Y(\varepsilon) = \phi$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\left\{1-\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right)\right\} = \lim_{n\to\infty}\frac{1}{n}\sum_{r\in X(\varepsilon)}\left\{1-\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right)\right\} + \lim_{n\to\infty}\frac{1}{n}\sum_{r\in Y(\varepsilon)}\left\{1-\psi\left(\frac{x_r}{y_r}-\lambda,\tau\right)\right\} \tag{11}$$

The second limit in R.H.S is sufficiently small (≈ 0) and first limit force us to write $X(\varepsilon) \in I$ (Since, $X(\varepsilon) \subseteq A_n \in I$). Now on continuing the same reasoning on

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r\in A_n}\phi\left(\frac{x_r}{y_r}-\lambda,\tau\right) = 0 \tag{12}$$

Thus equations (11) and (12) simultaneously conclude the theorem,

i.e $\left\{r: \psi\left(\frac{x_r}{y_r}-\lambda,\tau\right) \le 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_r}{y_r}-\lambda,\tau\right) \ge \varepsilon\right\} \in I$

Theorem 3. *If* I *be an ideal on* $\mathbb{N}$ *and* $[x] = (x_p)$ *and* $[y] = (y_p)$ *are two sequences of the elements of IFNS* $(\mathbb{E},\psi,\phi,\circ,\diamond)$ *then* $(\psi,\phi) - [x] \overset{I_\lambda^*}{\sim} [y] \Rightarrow (\psi,\phi) - [x] \overset{I_\lambda}{\sim} [y]$

Proof. Given that $(\psi,\phi) - [x] \overset{I_\lambda^*}{\sim} [y]$, then there exist $M \in I$ such that for $K = \mathbb{N} - M = \{k_1, k_2, k_3, \ldots\}$ we have,

$$\lim_{r\to\infty}\psi\left(\frac{x_{k_r}}{y_{k_r}}-\lambda,\tau\right) = 1 \quad or \quad \lim_{r\to\infty}\phi\left(\frac{x_{k_r}}{y_{k_r}}-\lambda,\tau\right) = 0. \tag{13}$$

Which implies that for every $0 < \varepsilon < 1$ and $\tau > 0$ there exist $k_{r_0} \in K$ such that for all $k_r \geq k_{r_0}$ we have

$$\psi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) > 1 - \varepsilon \quad or \quad \phi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) < \varepsilon \tag{14}$$

which follows,

$$\left\{k_r : \psi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) \leq 1 - \varepsilon \quad or \quad \phi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) \geq \varepsilon\right\} \subseteq M \cup \{k_1, k_2, k_3, \dots\} \tag{15}$$

Since, $M \cup \{k_1, k_2, k_3, \dots\} \in I$. Therefore

$$\left\{k_r : \psi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) \leq 1 - \varepsilon \quad or \quad \phi\left(\frac{x_{k_r}}{y_{k_r}} - \lambda, \tau\right) \geq \varepsilon\right\} \in I. \tag{16}$$

Thus theorem is established.

Theorem 4. *If I be an admissible ideal on $\mathbb{N}$ and $[x] = (x_p)$ and $[y] = (y_p)$ are two sequences of the elements of IFNS $(\mathbb{E}, \psi, \phi, \circ, \diamond)$ then following are equivalent.*

(i). $(\psi, \phi) - [x] \overset{I_{\alpha\lambda}}{\sim} [y]$

(ii). $(\psi, \phi) - [x] \overset{I_{\lambda}}{\sim} \alpha[y]$

(iii). $(\psi, \phi) - \frac{1}{\alpha}[x] \overset{I_{\lambda}}{\sim} [y]$

***Proof*.** We prove (i)⇒(ii). For every $t > 0$ and $0 < \varepsilon < 1$ we have

$$X = \left\{k : \psi\left(\frac{x_k}{y_k} - \alpha\lambda, t\right) \leq 1 - \varepsilon \quad \text{or} \quad \phi\left(\frac{x_k}{y_k} - \alpha\lambda, t\right) \geq \varepsilon\right\} \in I$$

Let us put, $$Y = \left\{p : \psi\left(\frac{x_p}{\alpha y_p} - \lambda, t\right) \leq 1 - \varepsilon \quad \text{or} \quad \phi\left(\frac{x_p}{\alpha y_p} - \lambda, t\right) \geq \varepsilon\right\}$$

We now prove that $Y \subseteq X$, let $q \in Y \Rightarrow \psi(\frac{x_q}{\alpha y_q} - \lambda, t) \leq 1 - \varepsilon$

using the properties (v) of **definition 1,** we have

$$\psi\left(\frac{x_q}{\alpha y_q} - \lambda, t\right) = \psi\left(\frac{x_q}{y_q} - \alpha\lambda, |\alpha|t\right) = \psi\left(\frac{x_q}{y_q} - \alpha\lambda, \tau\right) \leq 1 - \varepsilon$$

where $\tau = |\alpha|t$ Similarly,

$$\phi\left(\frac{x_q}{y_q} - \alpha\lambda, \tau\right) \geq \varepsilon, \text{where} \quad \tau = |\alpha|t \Rightarrow q \in X$$

Thus we get $Y \subseteq X$ moreover $Y \in I$, since $X \in I$.

Now, (ii)⇒(iii).

Let us put

$$Z = \left\{k : \psi\left(\frac{\frac{1}{\alpha}x_k}{y_k} - \lambda, t\right) \leq 1 - \varepsilon \quad \text{or} \quad \phi\left(\frac{\frac{1}{\alpha}x_k}{y_k} - \lambda, t\right) \geq \varepsilon\right\} \tag{17}$$

. Now it is suffice to prove that $Z \subseteq Y$. Let $p \in Z$, then

$$\psi\left(\frac{\frac{1}{\alpha}x_p}{y_p} - \lambda, t\right) = \psi\left(\frac{x_p}{\alpha y_p} - \lambda, t\right) \leq 1 - \varepsilon \tag{18}$$

Also,

$$\phi\left(\frac{\frac{1}{\alpha}x_p}{y_p} - \lambda, t\right) = \phi\left(\frac{x_p}{\alpha y_p} - \lambda, t\right) \geq \varepsilon, \tag{19}$$

From equations (18) and (19) we reach $p \in Y \Rightarrow Z \subseteq Y$

Now, (iii)⇒(i). For this, we have to prove that $X \subseteq Z$. Let $k \in X$, and

$$\psi\left(\frac{x_k}{y_k} - \alpha\lambda, t\right) = \psi\left(\frac{\frac{1}{\alpha}x_k}{y_k} - \lambda, t|\alpha|\right) = \psi\left(\frac{\frac{1}{\alpha}x_k}{y_k} - \lambda, \tau\right) \leq 1 - \varepsilon, \quad (20)$$

because of (iii).

Similarly, we obtain $$\phi\left(\frac{\frac{1}{\alpha}x_k}{y_k} - \lambda, \tau\right) \leq 1 - \varepsilon \quad (21)$$

$\Rightarrow k \in Z \quad \text{or} \quad X \subseteq Z \in I.$

Theorem 5. *If I be an ideal on $\mathbb{N}$ and $[x] = (x_r)$ and $[y] = (y_r)$ are two sequences of the elements of IFNS $(\mathbb{E}, \psi, \phi, \circ, \diamond)$ then for non-zero scalars α, λ the following are equivalent.*

(i). $(\psi, \phi) - [x] \overset{I_{\frac{\lambda}{\alpha}}}{\sim} [y]$,

(ii). $(\psi, \phi) - \frac{[x]}{\lambda} \overset{I}{\sim} \frac{[y]}{\alpha}$,

Proof is obvious by the results of **theorem 4**.

Theorem 6. *If I an admissible ideal on $\mathbb{N}$ and the three sequences $[x] = (x_r)$, $[y] = (y_r)$ and $[z] = (z_r)$ defined on IFNS $(\mathbb{E}, \psi, \phi, \circ, \diamond)$ such that $(\psi, \phi) - [x] \overset{I_\lambda}{\sim} [y]$ and $(\psi, \phi) - [z] \overset{I_\mu}{\sim} [y]$ then $(\psi, \phi) - [x] + [z] \overset{I_{\lambda+\mu}}{\sim} [y]$, where λ and μ are scalars.*

Proof. We are given, $(\psi, \phi) - [x] \overset{I_\lambda}{\sim} [y]$ and $(\psi, \phi) - [z] \overset{I_\mu}{\sim} [y]$ then for every $0 < \varepsilon_1, \varepsilon_2 < 1$ and $t > 0$ we have

$$X = \left\{k : \psi\left(\frac{x_k}{y_k} - \lambda, t\right) \leq 1 - \varepsilon_1 \quad \text{or} \quad \phi\left(\frac{x_k}{y_k} - \lambda, t\right) \geq \varepsilon_2\right\} \in I. \quad (22)$$

and

$$Y = \{k : \psi\left(\frac{z_k}{y_k} - \mu, \tau\right) \leq 1 - \varepsilon_2 \quad \text{or} \quad \phi\left(\frac{z_k}{y_k} - \mu, \tau\right) \geq \varepsilon_2\} \in I. \quad (23)$$

Now let $q \in (X \cap Y)^c$ or $q \in (\mathbb{N} - X) \cup (\mathbb{N} - Y)$. Now for every $0 < \varepsilon_1, \varepsilon_2 < 1$ we can find $0 < \varepsilon < 1$ such that $(1 - \varepsilon_1) \circ (1 - \varepsilon_2) > 1 - \varepsilon$ and $\varepsilon_1 \diamond \varepsilon_2 < \varepsilon$.(by **remark 2**), Now

$$\psi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) \geq \psi\left(\frac{x_q}{y_q} - \lambda, t\right) \circ \psi\left(\frac{z_q}{z_q} - \mu, \tau\right) \geq (1 - \varepsilon_1) \circ (1 - \varepsilon_2) > 1 - \varepsilon \quad (24)$$

and,

$$\phi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) \leq \phi\left(\frac{x_q}{y_q} - \lambda, t\right) \diamond \phi\left(\frac{z_q}{z_q} - \mu, \tau\right) \leq \varepsilon_1 \diamond \varepsilon_2 < \varepsilon. \quad (25)$$

Let

$$Z = \left\{q : \psi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) > 1 - \varepsilon \quad \text{and} \quad \phi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) < \varepsilon\right\} \quad (26)$$

Thus we reach that $(X \cap Y)^c \subseteq Z$. Which implies,

$$\left\{q : \psi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) \leq 1 - \varepsilon \quad \text{or} \quad \phi\left(\frac{x_q + z_q}{y_q} - (\lambda + \mu), t + \tau\right) \geq \varepsilon\right\} \subseteq X \cap Y \quad (27)$$

Hence,

$$\left\{q : \psi\left(\frac{x_q+z_q}{y_q} - (\lambda+\mu), t+\tau\right) \leq 1-\varepsilon \quad \text{or} \quad \phi\left(\frac{x_q+z_q}{y_q} - (\lambda+\mu), t+\tau\right) \geq \varepsilon\right\} \in I. \tag{28}$$

Thus theorem is established.

Data Availability

No data has been used in this work.

Conflict of Interest

The authors declare that there is no conflict of interest

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