Fractional Heat Transfer Effect In Cylindrical Body And Its Associated Stresses

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Abstract

Thermoelastic deformation of a finite length hollow cylinder is examined with effect of convective heat transfer boundaries on upper and lower faces beside assumed heat equation is of Caputo type with time fractional order derivative. The prescribed jump surface temperature is assumed on the upper face, while the lower face is kept at zero, also the curved surfaces are kept at zero temperature. The governing heat conduction problem is solved analytically by employing step by step finite Marchi-Fasulo, Hankel and Laplace transformation method. The obtained thermal results in terms of temperature and stresses are illustrated numerical and presented graphically by considering the material properties of pure aluminum metal.

Keywords: Hollow Cylinder, Fractional order derivative, Temperature, Integral transform, Thermal stresses

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1. Introduction

Thermoelasticity based on Fractional theory involves the heat equation with fractional differential order derivatives. Whereas, differentiation of time-fractional order describes memory effects and the space-fractional differential operators deal with the long-range interaction. From last two decades fractional calculus successfully implemented in various other fields like physics, solid mechanics, geology, chemistry, rheology, engineering, bioengineering, robotics, etc.

The Riemann–Liouville derivative of the fractional order α with its Laplace transformation was defined in [1, 2 and 3]. Classical heat equation successfully replaced by non-classical due to its utility and applications in various physical processes as shown in [4, 5 and 7]. Fractional models used in linear theory of viscoelasticity and found in agreement with experimental studies as in Caputo [8, 9] and Caputo and Mainardi [10, 11]. Thermoelastic analysis in cylindrical coordinate system was analyzed by Povstenko [12, 13] by application of integral transformation method and obtained non-axisymmetric solutions. Also, in [14] he studied fractional heat equation in half space. Magneto-thermoelasticity and electro-thermoelasticity modelling was constructed by Ezzat and Karamany [15, 16 and 17] with time-fractional order derivatives and investigation is done for perfect conducting half-space elastic material. Also, Ezzat [18] determined thermoelasticity based on electromagnetic thermoelastic fluid with two-temperature model.

Stress distribution along radial direction in case of thick hollow cylinder with internal heat under fractional derivatives was analyzed by Thakare and Warbhe [19]. Deflection in vibrating plate was successfully estimated under fractional derivatives by Mungle et al. [20]. Physical convective heat exchange boundaries impact under fractional order parameters are obtained by Kumar and Kamdi [21]. Also, investigation in nonhomogeneous cylindrical region is done by Thakare et al. [22] and obtained its thermal behaviour within the context of fractional calculus. Recently, Lamba [23] investigated Functionally Graded Cylinder

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response under fractional derivatives and successfully obtained analytical results of temperature and stresses.

The present article based on the thermoelastic analysis of finite length hollow cylinder with convective heat exchange boundaries on curved surfaces and subjected to jump heating on upper face, while keeping lower face at zero temperature. Method of integral transformation is used to evaluate the solution of problem analytically in terms of temperature, displacement and stresses. All the obtained results are plotted graphically and illustrated numerically by considering properties of pure aluminum metal.

2. Modeling of the problem:

Assume a finite length hollow cylinder having length *2h* and occupying the dimension D: $a \le r \le b$, $-h \le z \le h$ in the cylindrical coordinate system as shown in Fig. 1.

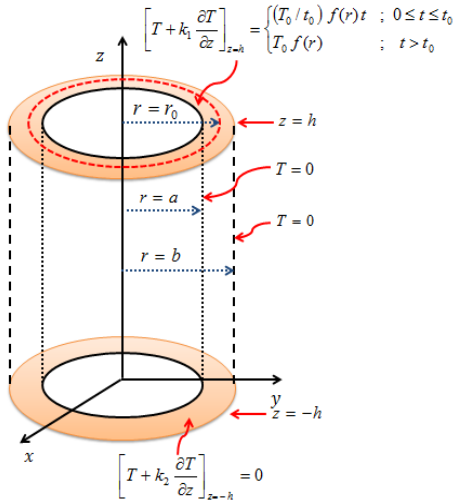

Fig. 1. Geometry of the problem

2.1 Transient heat conduction formulation

The expression of heat equation for a finite hollow cylinder is as below

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial^\alpha T}{\partial t^\alpha}; \qquad a\le r\le b\ ,\ -h\le z\le h \tag{2}$$

Where,

T - Temperature function at a point (r,z) at a time t,

$k=\kappa/c_v\,\rho$ - Coefficient of thermal diffusivity,

c_v - Calorific capacity,

ρ - Density,

κ - Coefficient of thermal conductivity,

and $\partial^\alpha T/\partial t^\alpha$ is the Caputo fractional derivative, respectively.

Caputo type nonlocal time fractional derivative of temperature function T w.r.t. t of order α i.e. $\partial^\alpha T/\partial t^\alpha$ defined as [1, 2 and 3]

$$\frac{d^\alpha F(t)}{dt^\alpha}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n F(\tau)}{d\tau^n}d\tau, & n-1<\alpha<n,\\ \frac{d^n F(\tau)}{dt^n}, & \alpha=n\end{cases} \tag{3}$$

subject to the initial conditions

$$T\{t=0\}=0 \quad \text{for } 0<\alpha\le 2, \tag{4a}$$

$$\frac{\partial T}{\partial t}\{t=0\}=0 \quad \text{for } 1<\alpha\le 2, \tag{4b}$$

the boundary conditions

$$T\{r=a,z,t\}=T\{r=b,z,t\}=0 \tag{4c}$$

$$\left[T+k_1\frac{\partial T}{\partial z}\right]_{z=h}=\begin{cases}(T_0/t_0)\,f(r)t & ;\ 0\le t\le t_0\\ T_0\,f(r) & ;\ t>t_0\end{cases} \tag{4d}$$

$$\left[T+k_2\frac{\partial T}{\partial z}\right]_{z=-h}=0$$

in which

$$f(r)=\begin{cases}1 & ;\ r_0\le t\le b\\ 0 & ;\ a\le t\le r_0\end{cases} \tag{4e}$$

Where,

k_1 and k_2 - radiation constants on the two plane surfaces,

k - thermal diffusivity of the material of the cylinder.

2.2 Displacement and stress field

The relationship between thermoelastic displacement function Poisson's ratio and temperature function as in [24] is

$$\nabla^2\phi=\frac{(1+\nu)}{(1-\nu)}a_t T \tag{5}$$

$$\text{with } \phi\{r=a;\ r=b\}=0 \tag{6}$$

where,

∇^2 -Laplacian operator,

ν - Poisson's ratio and a_t - Poisson's ratio.

The radial and axial displacement U and W satisfying the uncoupled thermoelastic equations

as in [25]

$$\nabla^2 U - \frac{U}{r^2} + (1-2\nu)^{-1}\frac{\partial e}{\partial r} = 2\frac{(1+\nu)}{(1-2\nu)}a_t\frac{\partial T}{\partial r} \quad (7)$$

$$\nabla^2 W + (1+2\nu)^{-1}\frac{\partial e}{\partial z} = 2\frac{(1+\nu)}{(1-2\nu)}\frac{\partial T}{\partial z} \quad (8)$$

where $e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z}$ is the volume dilation and

$$U = \frac{\partial \phi}{\partial r} \quad (9)$$

$$W = \frac{\partial \phi}{\partial z} \quad (10)$$

The stress functions are given by

$$\tau_{rz}(a,z,t) = 0 \qquad \tau_{rz}(b,z,t) = 0 \qquad \tau_{rz}(r,0,t) = 0 \quad (11)$$

and

$$\sigma_r(a,z,t) = p_1, \qquad \sigma_r(b,z,t) = -p_0, \qquad \sigma_z(r,h,t) = 0 \quad (12)$$

Where,

p_1 and p_0 – uniform surface pressures over the boundaries of the cylinder.

The stress in terms of displacement components using (11) and (12) are as:

$$\sigma_r = (\lambda + 2G)\frac{\partial U}{\partial r} + \lambda\left[\frac{U}{r} + \frac{\partial w}{\partial z}\right] \quad (13)$$

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z} + \lambda\left[\frac{\partial U}{\partial r} + \frac{U}{r}\right] \quad (14)$$

$$\sigma_\theta = (\lambda + 2G)\frac{U}{r} + \lambda\left[\frac{\partial U}{\partial r} + \frac{\partial w}{\partial z}\right] \quad (15)$$

$$\tau_{rz} = G\left[\frac{\partial w}{\partial r} + \frac{\partial U}{\partial z}\right] \quad (16)$$

Where,

$\lambda = 2G\nu/1-2\nu$ - lame's constant,

G - shear modulus

U and W - displacement components.

Above equations (1) to (16) constitutes the formulation of the problem mathematically.

3. Solution for the Problem:

3.1 Transient heat conduction analysis:

In order to solve equation (1), we firstly define the Laplace transform of time fractional order derivative of order α over the variable t as [1, 2 and 3]

$$L\left\{\frac{d^\alpha F(r,z,t)}{dt^\alpha}\right\} = s^\alpha F^*(r,z,s) - \sum_{k=0}^{n-1} F^{(k)}(0^+)s^{\alpha-1-k}, \quad n-1<\alpha<n \quad (17)$$

$$L\{T(r,z,s)\} = T^*(r,z,s)$$

Secondly, we define Hankel transform and its inversion of order m over the variable r as

$$\bar{\theta}^*(\beta_m,z,s) = \int_{r'=a}^{b} r' K_0(\beta_m,r')\theta^*(r',z,s)dr' \quad (18)$$

$$\theta^*(r,z,t) = \sum_{m=1}^{\infty} K_0(\beta_m,r)\,\bar{\theta}^*(\beta_m,z,s) \quad (19)$$

where,

$$K_0(\beta_m,r) = \frac{R_0(\beta_m,r)}{\sqrt{N}} \quad (20)$$

$$R_0(\beta_m,r) = \frac{J_0(\beta_m,r)}{k_2\beta_m J_0'(\beta_m,b) + J_0(\beta_m,b)} - \frac{Y_0(\beta_m,r)}{k_2\beta_m Y_0'(\beta_m,b) + Y_0(\beta_m,b)} \quad (21)$$

$$N = \frac{b^2}{2}\left[\frac{h_2^2}{k_2^2\beta_m^2} + 1\right]R_0^2(\beta_m,b) - \frac{a^2}{2}\left[\frac{h_1^2}{k_1^2\beta_m^2} + 1\right]R_0^2(\beta_m,a) \quad (22)$$

β_m are the positive roots of

$$\frac{-\beta k_1 J_0'(\beta,a) + J_0(\beta,a)}{k_2\beta J_0'(\beta,b) + J_0(\beta,b)} - \frac{-\beta k_1 Y_0'(\beta,a) + Y_0(\beta,a)}{k_2\beta Y_0'(\beta,b) + Y_0(\beta,b)} = 0 \quad (23)$$

Thirdly, the finite Marchi-Fasulo integral transform for $f(z)$ in range $-h < z < h$ defined below with its inversion of order m over the variable r as

$$\bar{G}(n) = \int_{-h}^{h} g(z)P_n(z)dz \quad (24)$$

then at each point of (-h, h) at which f(z) is continuous,

$$g(z) = \sum_{n=1}^{\infty}\frac{\bar{G}(n)}{\lambda_n}P_n(z) \quad (25)$$

where

$$P_n(z) = Q_n\cos(a_n z) - W_n\sin(a_n z)$$

$$Q_n = a_n(\alpha_1+\alpha_2)\cos(a_n h) + (\beta_1-\beta_2)\sin(a_n h)$$

$$W_n = (\beta_1+\beta_2)\cos(a_n h) + (\alpha_2-\alpha_1)a_n\sin(a_n h)$$

$$\lambda_n = \int_{-h}^{h} P_n^{\,2}(z)\,dz = h\left[Q_n^{\,2} + W_n^{\,2}\right] + \frac{\sin(2a_n h)}{2a_n}\left[Q_n^{\,2} - W_n^{\,2}\right]$$

The eigen values a_n are the solutions of the equation

$$\left[\alpha_1 a\cos(ah) + \beta_1 \sin(ah)\right]\times\left[\beta_2\cos(ah) + \alpha_2 a\sin(ah)\right] = \left[\alpha_2 a\cos(ah) - \beta_2\sin(ah)\right]\times\left[\beta_1\cos(ah) - \alpha_1 a\sin(ah)\right] \quad (26)$$

α_1, α_2, β_1 and β_2 are constants.

Moreover, satisfying the following property:

$$\int_{-h}^{h}\frac{\partial^2 f(z)}{\partial z^2}P_n(z)dz = \frac{P_n(h)}{\alpha_1}\left[\beta_1 f(z) + \alpha_1\frac{\partial f(z)}{\partial z}\right]_{z=h} - \frac{P_n(-h)}{\alpha_2}\left[\beta_2 f(z) + \alpha_2\frac{\partial f(z)}{\partial z}\right]_{z=-h} - a_n^{\,2}\overline{F}(n) \quad (27)$$

Applying Laplace transform, Hankel transform and finite Marchi-Fasulo integral transform and their corresponding inversion formula stated in (17) to (27) under the assumed boundary conditions the temperature function is obtained as

$$T(r,z,t) = \frac{r_0 k T_0}{k_3\, t_0}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau \quad (28)$$

where,

$$t^{\alpha-1}E_{\alpha,\alpha}\left(-k(\lambda_m^{\,2} + a_n^{\,2})\,t^{\alpha}\right) = L^{-1}\left\{\frac{1}{s^{\alpha} + k(\lambda_m^{\,2} + a_n^{\,2})}\right\}$$

Where,

m, n -positive integers,

λ_m - positive roots of the transcendental.

and

$$J_0(\lambda_m a)\,Y_0(\lambda_m b) - J_0(\lambda_m b)\,Y_0(\lambda_m a) = 0$$

$$\overline{f}(n) = \int_{-h}^{h} f(z)P_n(z)\,dz, \quad \lambda_n = \int_{-h}^{h} P_n^{\,2}(z)\,dz$$

$$P_n(z) = Q_n\cos(a_n z) - W_n\sin(a_n z)$$

$$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h)$$

$$W_n = (\beta_1 + \beta_2)\cos(a_n h) + (\alpha_2 - \alpha_1)a_n\sin(a_n h)$$

and the eigen values a_n are the solutions of the equation

$$\left[\alpha_1 a\cos(ah) + \beta_1\sin(ah)\right]\times\left[\beta_2\cos(ah) + \alpha_2 a\sin(ah)\right] = \left[\alpha_2 a\cos(ah) - \beta_2\sin(ah)\right]\times\left[\beta_1\cos(ah) - \alpha_1 a\sin(ah)\right]$$

3.2 Determination of Thermoelastic Displacement Function:

Displacement potential $\phi(r,z,t)$ obtained by using $T(r,z,t)$ from equation (28) in equation (5) as below

$$\phi(r,z,t) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3\, t_0}\sum_{n=1}^{\infty}\frac{r^2 P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau \quad (29)$$

Using (29) in (9) and (10) one obtains the radial and axial displacement U and W as

$$U = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3\, t_0}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[2r\left(J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right) + \lambda_m r^2\left(J_0(\lambda_m a)\,Y_0'(\lambda_m r) - J_0'(\lambda_m r)\,Y_0(\lambda_m a)\right)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau \quad (30)$$

$$W = \left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3\, t_0}\sum_{n=1}^{\infty}\frac{r^2 P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau \quad (31)$$

3.3 Determination of Thermal Stresses:

Using (30) and (31) in (13), (14), (15) and (16) the stress functions are obtained as

$$\sigma_{rr} = (\lambda + 2G)\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3\, t_0}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[2\left(J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right) + 4\lambda_m r\left(J_0(\lambda_m a)\,Y_0'(\lambda_m r) - J_0'(\lambda_m r)\,Y_0(\lambda_m a)\right) + \lambda_m^2 r^2\left(J_0(\lambda_m a)\,Y_0''(\lambda_m r) - J_0''(\lambda_m r)\,Y_0(\lambda_m a)\right)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau + \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left[2\left(J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right) + \lambda_m r\left(J_0(\lambda_m a)\,Y_0'(\lambda_m r) - J_0'(\lambda_m r)\,Y_0(\lambda_m a)\right)\right] \times\int_0^t(\tau - S(\tau))\left\{(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-k\,(\lambda_m^{\,2} + a_n^{\,2})\,(t-\tau)^{\alpha}\right)\right\}d\tau + \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{r^2 P_n''(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a)J_0(\lambda_m h)}{\left(\lambda_m^2 + a_n^2\right)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)} \times\left(J_0(\lambda_m a)\,Y_0(\lambda_m r) - J_0(\lambda_m r)\,Y_0(\lambda_m a)\right)$$

$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau \quad (32)$$

$$\sigma_{zz} = (\lambda + 2G)\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3 t_0}\sum_{n=1}^{\infty}\frac{r^2 P_n''(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau$$
$$+ \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left[2\left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + 4\lambda_m r\left(J_0(\lambda_m a)\, Y_0'(\lambda_m r) - J_0'(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + \lambda_m^2 r^2\left(J_0(\lambda_m a)\, Y_0''(\lambda_m r) - J_0''(\lambda_m r)\, Y_0(\lambda_m a)\right)\right]$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau$$
$$+ \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left[2\left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + \lambda_m r\left(J_0(\lambda_m a)\, Y_0'(\lambda_m r) - J_0'(\lambda_m r)\, Y_0(\lambda_m a)\right)\right]$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau \quad (33)$$

$$\sigma_{\theta\theta} = (\lambda + 2G)\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3 t_0}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left[2\left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + \lambda_m r\left(J_0(\lambda_m a)\, Y_0'(\lambda_m r) - J_0'(\lambda_m r)\, Y_0(\lambda_m a)\right)\right]$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau$$
$$+ \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left[2\left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + 4\lambda_m r\left(J_0(\lambda_m a)\, Y_0'(\lambda_m r) - J_0'(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + \lambda_m^2 r^2\left(J_0(\lambda_m a)\, Y_0''(\lambda_m r) - J_0''(\lambda_m r)\, Y_0(\lambda_m a)\right)\right]$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau$$
$$+ \lambda\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\sum_{n=1}^{\infty}\frac{r^2 P_n''(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau \quad (34)$$

$$\tau_{rz} = G\left(\frac{1+\nu}{1-\nu}\right)\frac{\alpha_t}{4}\,\frac{r_0 k T_0}{k_3 t_0}\sum_{n=1}^{\infty}\frac{P_n'(z)}{\lambda_n}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_0(\lambda_m a) J_0(\lambda_m h)}{(\lambda_m^2 + a_n^2)\left(J_0^2(\lambda_m a) - J_0^2(\lambda_m h)\right)}$$
$$\times \left[2r\left(J_0(\lambda_m a)\, Y_0(\lambda_m r) - J_0(\lambda_m r)\, Y_0(\lambda_m a)\right)\right.$$
$$\left. + \lambda_m r^2\left(J_0(\lambda_m a)\, Y_0'(\lambda_m r) - J_0'(\lambda_m r)\, Y_0(\lambda_m a)\right)\right]$$
$$\times \int_0^t (\tau - S(\tau))\left\{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-k\,(\lambda_m{}^2 + a_n{}^2)\,(t-\tau)^{\alpha}\right)\right\} d\tau \quad (35)$$

4. Numerical results and discussion

For the purpose of numerical computation the following dimensionless values are introduce as

$$\bar{T} = \frac{T}{T_R},\ \bar{r} = \frac{r}{a},\ \bar{z} = \frac{z - h_1}{a},\ \bar{t} = \frac{\kappa t}{a^2},\ \bar{h} = \frac{h}{a},\ \bar{U} = \frac{U}{\alpha_0 T_R a},\ \bar{W} = \frac{W}{\alpha_0 T_R a}, \quad (46)$$
$$\bar{\sigma}_{rr} = \frac{\sigma_{rr}}{\alpha_0 E T_R},\ \bar{\sigma}_{\theta\theta} = \frac{\sigma_{\theta\theta}}{\alpha_0 E T_R}, \bar{\sigma}_{zz} = \frac{\sigma_{zz}}{\alpha_0 E T_R},\ \bar{\tau}_{rz} = \frac{\sigma_{rz}}{\alpha_0 E T_R},$$

The thermo-mechanical properties of Aluminum (pure) cylinder are as:

Modulus of elasticity	E = 70 *GPa,*
Poisson's ratio	υ = 0.35,
Thermal expansion coefficient	α = 23×10-6/^{0}C,
Thermal diffusivity	κ = 84.18×10-6 m^2s^{-1}
Thermal conductivity	λ = 204.2 $Wm^{-1}K^{-1}$

The physical parameter is as: $a = 1m, b = 2m$, $h = 0.05m$, $t_0 = 0.05$ and $T_0 = 120^0 C$

Mathematica software is used to analyze the jump heating response on the upper plane face while keeping zero temperature at lower face of cylinder with assumed boundary conditions under time fractional order derivative.

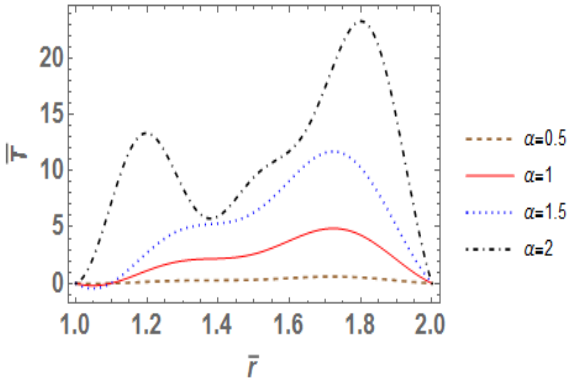

Figure 2. Temperature distribution

Figure 2 shows the variation of temperature function $T(r,z,t)$ against radius for different fractional parameters $\alpha = 0.5, 1, 1.5, 2$. It is observed that temperature varies directly proportional to fractional parameters. Temperature is zero at both inner and outer surfaces which matches with assumed mathematical condition (4c). Also high

temperature distribution noted at upper plane surface due to the effect of prescribed jump heating on the upper face.

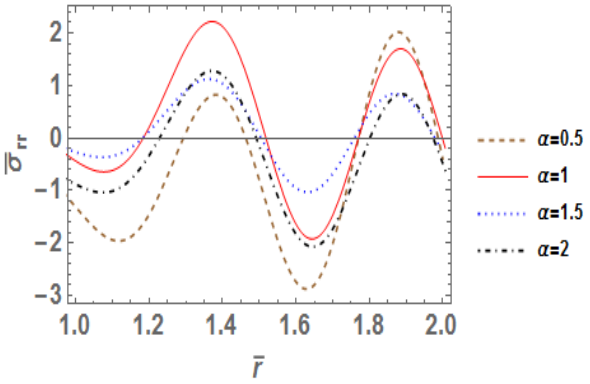

Figure 3. Radial stress distribution

Figure 3 represents the distribution of radial stress function σ_{rr} against radius for different fractional parameters $\alpha = 0.5, 1, 1.5, 2$ Sinusoidal behaviour of radial stress obtained graphically as well as for high value of fractional parameter large stress variation is noted. Also effect of jump heating is observed at upper plane surface.

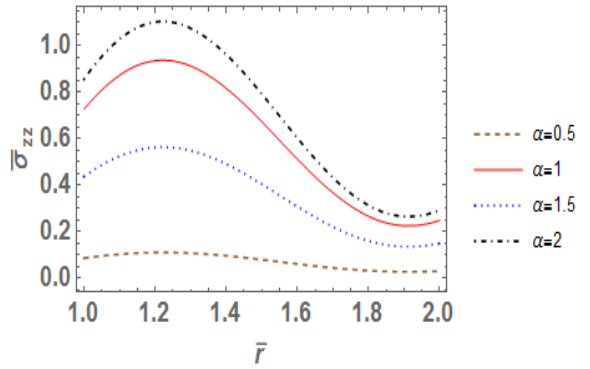

Figure 4. Axial stress distribution

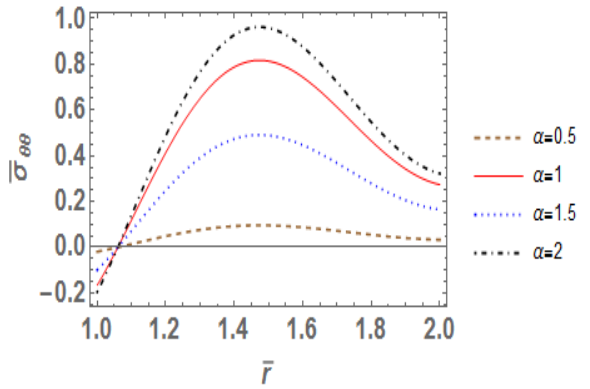

Figure 5. Tangential stress distribution

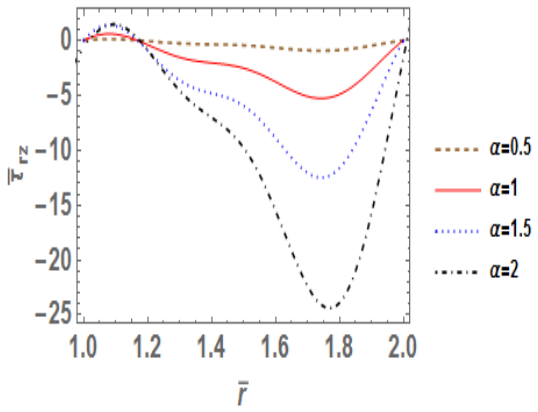

Figure 6. Shear stress distribution

Figure 4, 5 and 6 represents the distribution of axial stress, tangential stress and shear stress σ_{zz}, $\sigma_{\theta\theta}$ and τ_{rz} against radius for different fractional parameters $\alpha = 0.5, 1, 1.5, 2$. Axial stress distribution increases initial along radial direction attain maximum value at $r = 1.2$ and then goes on decreasing towards outer radii. Tangential stress increases initially and reaches at peak at the mid of radius and decreases outward, while shear stress distribution is zero at inner and outer curved surface which matches the boundary condition (11). From the above plotting it noted that varies of stresses found directly proportional to fractional parameters α.

Conclusion

In this problem, the Caputo fractional model is discussed for analyzing the thermoelastic response of a finite length hollow cylinder. The upper face of hollow cylinder is considered with impact of jump heating and lower face at zero temperature while boundaries of curved surfaces are thermally insulated. The theory of integral transformation is used to obtain the analytical solution for the Caputo fractional time derivative model. From the numerical investigations for the properties of pure aluminum metal the following results has been noted down:

1. The temperature distribution and thermal stresses response are dependent on the values of fractional-order parameter along radial directions.
2. Temperature variation is noted between the curved surfaces and at the inner and outer surfaces its zero.
3. The maximum radial stress distribution is

noted at the mid of the cylinder along the radii which may be due to the supply of energy in terms of heating.

4. The plotting of axial stress indicates that compressive forces are high in the middle part due to the availibity of jump heating.
5. Tangential stress is initially tensile and then becomes compressive throughout the radial direction.
6. The value of shear stress is zero at inner and outer curved surfaces and attains maximum value in the middle due to accumulation of heat energy.

All the obtained results can be applied to the design of useful structures or machines and applicable in engineering applications. Also, such type of study is useful for researcher working in field of material sciences using fractional calculus.

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