Degree of Approximation of Function in the Generalized Hölder Metric by Product Summability Method of its Fourier Series

***Yamini Yadav*[1], *C. S. Rathore*[2 *]**

[1]*Research Scholar, Department of Mathematics, Atal Bihari Vajpayee Vishwavidyalaya, Bilaspur (C.G.) India*

[2*] *Department of Mathematics, Jajwalyadev Govt. Girls College, Janjgir (C.G.) India*

[1]*E-mail: yaminiyadav711@gmail.com*

[2*]*E-mail: rathoremaths20@gmail.com*

Abstract: *In this paper a new theorem established on the degree of approximation of function in the Generalized Hölder Metric by Matrix-Cesaro summability method of its Fourier series.*

Keywords: *Degree of Approximation, Hölder Metric, Generalized Hölder Metric, Matrix – Cesaro Summability Method, Fourier Series.*

Subject classification: *42B05, 42B08.*

Introduction

Alexits (1928) studied the degree of approximation of function of H_α $(0 < \alpha \leq 1)$ *and* $0 \leq \beta < \alpha$. *Prössdorf (1975) obtained an estimate for* $\|\sigma_n(f) - f\|_\beta$ *for* $f \in H_\alpha$, *where* $\sigma_n(f)$ *is the Fejer means of the Fourier series of* f *and he proved the following theorem-*

Theorem-A. *Let* $f \in H_\alpha$ $(0 < \alpha \leq 1)$ *and* $0 \leq \beta < \alpha$ *,*
then

$$\|\sigma_n(f) - f\|_\beta = (1) \begin{cases} O\left(n^{\beta-\alpha}\right) & (0 < \alpha < 1) \\ O\left[n^{\beta-1}(1+logn)^{1-\beta}\right] & (\alpha = 1). \end{cases}$$

The class β*=0 of theorem is due to Alexits.*

Mahapatra and Chandra (1982) studied for the Hölder continuous function f to obtain error bounds in Holder norm.

Theorem-B. *Let* $0 \leq \beta < \alpha \leq 1$ *and Let* $f \in H_\alpha$. *Then for* $n > 1$.

$$\left\|e_n^q(f) - f\right\|_\beta = O\left\{(n)2^{\frac{-(\propto-\beta)}{2}}(log\, n)^{\frac{\beta}{\alpha}}\right\}$$

*Chandra (1982) generalized Theorem **A** by replacing $\sigma_n(f)$ with $(\overline{N}, p_n)$ and Nörlund means of the Fourier series of f.*

In this way of generalization in 1983 R.N. Mohapatra & P. Chandra obtained new result using theorem A.

Again Prem Chandra (1988) generalized his results on Degree of approximation of functions in the Hölder metric.

Theorem-C. *Let $0 \leq \beta < \alpha \leq 1$ and Let $f \in H_\alpha$. Then*

$$\left\|e_n^q(f) - f\right\|_\beta = O\{n^{\beta-\alpha} \log n\}.$$

Later G. Das Tulika Ghosh and B.K. Roy (1995) studied the degree of approximation in the Hölder metric and proved the following theorem :

***Theorem-1**. Let $0 < \alpha \leq 1$ and $0 \leq \beta < \alpha$.*
Then

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \dfrac{\log n}{n^{\beta-\alpha}} & \left(0 < \alpha - < \beta \leq \dfrac{1}{2}\right) \\ \dfrac{1}{n^{\frac{1}{2}}} & \left(\dfrac{1}{2} < \alpha - < \beta \leq 1\right). \end{cases}$$

***Theorem-2**. Let $0 < \alpha \leq 1$ and $0 \leq \beta < \alpha$ and let $f \in H_\alpha$. Further, if*

$$\int_{2\pi/2n+1}^{\pi \log n/n^{1/2}} \frac{|\emptyset_x(t + 2\pi/2n + 1) - \emptyset_x(t)|}{t} exp\left(-\frac{nt^2}{4c}\right) dt = O\left(\frac{1}{n^\alpha}\right).$$

Then

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \dfrac{(\log n)^{\frac{\beta}{\alpha}}}{n^{\beta-\alpha}}, & \left(0 < \alpha - \beta \leq \dfrac{1}{2}\right) \\ \dfrac{1}{n^{\frac{1}{2}}}, & \left(\dfrac{1}{2} < \alpha - \beta \leq 1\right). \end{cases}$$

In 1996 G. Das Tulika Ghosh and B.K. Roy replaced Generalized Holder metric in place of Holder metric and proved some generalized theorem on degree of approximation of functions by their Fourier series .

In the present work we established a new theorem on degree of approximation of function in the generalized Hölder Metric by Matrix Cesaro - Summability method of its Fourier series .

eISSN 1303-5150

Definition and notations

Let f be a periodic function of period 2π and let $f \in L_p[0,2\pi]$ for $p \geq 1$. Let the Fourier series of f at $t = x$ be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.1}$$

In the case $0 < p < 1$, we can still regard (2.1) as the Fourier series of f by further assuming that $f(t)\cos nt$ and $f(t)\cos nt$ are integrable.

The space $L_p[0,2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0,2\pi]$. We write

$$\|f\|_c = \sup_{x\in(o,2\pi)}|f(t)| \quad (p = \infty)$$

$$\|f\|_\alpha = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p\,dt\right) & (p \geq 1) \\ \int_0^{2\pi}|f(t)|^p\,dt & (0 < p < 1). \end{cases}$$

We write

$$w(\delta) = w(\delta, f) = \sup_{0\leq h\leq\delta}\|f(x+h) - f(x)\|_c \tag{2.2}$$

$$w_p(\delta) = w_p(\delta, f) = \sup_{0\leq h\leq\delta}\|f(x+h) - f(x)\|_p \tag{2.3}$$

$${w_p}^{(2)}(\delta) = {w_p}^{(2)}(\delta, f) = \sup_{0\leq h\leq\delta}\|f(x+h) + f(x-h) - 2f(x)\|_p\,. \tag{2.4}$$

In the case $0 < \beta \leq 1$ and $w(\delta, f) = 0$

Let $C_{2\pi}$ denote the Banach Space of all 2π - periodic continuous function defined on $[\pi, -\pi]$ under sub-norm.

For $0 \leq \alpha \leq 1$ and some positive constant k the function space H_α is given by the following

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq k\,|x - y|^\alpha\}. \tag{2.5}$$

The space H_α is a Banach space with the norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y}[\Delta^\alpha f(x,y)], \tag{2.6}$$

where $\|f\|_c = \sup_{-\pi\leq x\leq\pi}|f(x)|,$

and $\Delta^\alpha f(x,y) = |f(x) - f(y)|/|x-y|^\alpha \quad x \neq y.$ *(2.7)*

We shall use the connection that $\Delta^0 f(x,y) = 0$.
The metric induced by norm in (2.5) on H_α is called the Hölder metric.
The set of functions f with $\|f\|_\alpha \leq K$ is a compact subset of C [0,1].

It can be seen that $\|f\|_\beta \le (2\pi)^{\alpha-\beta}\|f\|_\alpha$ *for* $0 \le \beta < \alpha \le 1$, *It follows that* $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$; *that is,Thus* $\{(H_\alpha, \|.\|_\alpha)\}$ *is a family of Banach spaces which decreases as* α *increases.*
Let $A = (a_{nk})(k, n = 0,1, \dots)$*be an infinite matrix of real numbers. We denote by* $T_n(f)$ *the A-transform of the Fourier series of* f *given by* $T_n(f;x) = \sum_{n=0}^{\infty} a_{nk}\, S_k(x) \qquad (n = 0,1, \dots)$,
Where $S_k(x)$ *is the n-th partial sum of the series (2.1). If* $A = (a_{nk})$ *is lower – triangular i.e.* $a_{nk} = 0$ *for* $K > n$, *we write*

$$t_n(f;x) = \sum_{n=0}^{\infty} a_{nk}\, S_k(x) \qquad (n = 0,1, \dots).$$

Since $\|f\|_\beta \le (2\pi)^{\alpha-\beta}\, \|f\|_\alpha, 0 \le \beta < \alpha \le 1, \quad \{H_\propto, ||.||_\propto\}$ *is a family of Banach space which decreases as* $\propto$ *increases.*
With a view to generalized Hölder metric, we proceed as follows. We define for $0 < \alpha \le 1$

$$H(\alpha, p) = \{f \in L_p, 0 < p \le \infty : \|f(x+h) - f(x)\|_p \le k\, |h|^\alpha\}.$$

and introduce the following metric. For $\alpha > 0$

$$\|f\|_{(\alpha,p)} = \|f\|_p + sup\, \frac{\|f(x+h)-f(x)\|_p}{|h|^\alpha}.$$

$$\|f\|_{(0,p)} = \|f\|_p. \qquad (2.8)$$

It can be easily verified that (2.8) is a norm for $p \ge 1$ *and p-norm in the case* $0 < p < 1$. *Further it can be verified that* $H(\alpha, p)$ *is a Banach space for* $p \ge 1$ *and a complete p- normed space for* $0 < p < 1$.

Let $A = (a_{n,k})(k, n = 0,1, \dots)$ *be an infinite matrix of real numbers and let* $S_n(x)$ *be the nth partial sum of the series (2.1).We denote by* $T_n(f)$ *the A-transform of the Fourier series of* f *given by*

$$T_n(f;x) = \sum_{n=0}^{\infty} a_{nk}\, S_k(x).$$

The degree of approximation $E_n(f)$*of a function f: R→R by trigonometric polynomial* t_n *of degree n is defined by*

$$E_n(f) = \|t_n - f\|_\infty = sup\{|t_n(x) - f(x)| : x \in R\}. \quad \textit{Zygmund (1959).}$$

Let $\sum_{n=0}^{\infty} u_n$ *be the infinite series whose* n^{th} *partial sum is given by* $S_n = \sum_{k=0}^{\infty} u_k$ *.*

$$\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty. \qquad (2.9)$$

and $\sum_{k=0}^{\infty} a_{n,k} = 1$ *for each n=0, 1, 2 …* (2.10)

We write $A \in \tau$ *if conditions (2.9) and (2.10) hold. We use the following notations throughout:*

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x) \tag{2.11}$$

$$l_n(x) = T_n(f;x) - f(x) \tag{2.12}$$

$$K(n,t) = \sum_{k=0}^{\infty} a_{n,k} \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \tag{2.13}$$

$$\psi(n) = \sum_{k=0}^{\infty} \left| a_{n,k} - a_{n,k+1} \right| \tag{2.14}$$

$$\bar{a}_n(k) = \sum_{r=0}^{k} a_{n,r} \tag{2.15}$$

$$a'_{n,k} = \sum_{r=k}^{n} a_{n,r}. \tag{2.16}$$

Cesaro means (C,1) of sequence $\{S_n\}$ *is given by* $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$. *If* $\sigma_n \to S$, *as* $n \to \infty$ *then sequence* $\{S_n\}$ *or the infinite series* $\sum_{n=0}^{\infty} u_n$ *is said to be summable by Cesaro means (C,1) to S.*

Let $T = (a_{n,k})$ *be an infinite lower triangular matrix satisfying the conditions of regularity, i.e.* $\sum_{k=0}^{\infty} |a_{n,k}| \le M$, *a finite constant .*

Matrix – Cesaro means T(C₁) of the sequence $\{S_n\}$ *is given by*

$$t_n = \sum_{k=0}^{\infty} a_{n,n-k}\, \sigma_{n-k} = \sum_{k=0}^{\infty} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=0}^{n-k} S_r.$$

If $t_n \to S$ *as n* $\to \infty$ *, then the sequence* $\{S_n\}$ *or the infinite series* $\sum_{n=0}^{\infty} u_n$ *is said to be summable by Matrix Cesaro means T(C₁) method to S.*
We write

$$\emptyset(t) = f(x+t) + f(x-t) - f(x)$$

$$K(n,t) = a_{n,k} \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}.$$

Known Results

Das G, Ghosh Tulika and Ray B K (1995) studied Degree of approximation of functions in the Hölder metric by (e,c) means.

Theorem-1. $0 < \alpha \le 1$ *and* $0 \le \beta < \alpha$ *. Let* $f \in H_\alpha$*. Then*

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \frac{\log n}{n^{\beta-\alpha}} & \left(0 < \alpha - \beta \le \frac{1}{2}\right) \\ \frac{1}{n^{1/2}} & \left(\frac{1}{2} < \alpha - \beta \le 1\right). \end{cases}$$

Theorem-2. *Binod Prasad Dhakal (2010) determined the degree of approximation of certain function belonging to the* $Lip\ \alpha$ *class by Matrix Cesaro summability method.*

Main Theorem

In this paper we established a new theorem of Matrix- Cesaro product summability method in the Generalized Holder metric.

Theorem: *For* $p \geq 1$ *and* $f \in H(\alpha, p), 0 < \alpha \leq 1, 0 \leq \beta < \alpha$

$$\|l_n(x)\|_{(\beta,p)} = O(\psi(n)\big[(n+1)^{2+\beta-\alpha}\big],$$

where $l_n(x)$ *and* $\psi(n)$ *are respectively defined in (2.12) and (2.14).*

Lemmas

Lemma –I *Let* $0 < p \leq \infty$ *. Then*

(i) $w_p{}^{(2)}(\delta, f) \leq 2w_p(\delta, f)$ *(5.1)*

(ii) $\|\phi_x - \phi_{x+h}\|_p \leq 4K\|f(x) - f(x+h)\|_p,$ *(5.2)*

where K is some positive constant.

Proof: *For* $p \geq 1$ *and by Minkowski's inequality, we have*

$$\left(\int_0^{2\pi} |\phi_x(t)|^p dx\right) \leq \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^{2\pi} |f(x-t) - f(x)|^p dx\right)^{\frac{1}{p}},$$

and for $0 < p < 1$, *we have by the modified Minkowski type inequality*

$$\left(\int_0^{2\pi} |\phi_x(t)|^p dx\right) \leq \int_0^{2\pi} |f(x+t) - f(x)|^p dx + \int_0^{2\pi} |f(x-t) - f(x)|^p dx.$$

Now Lemma 1(i) follows from (3.3) and (3.4). For proving (ii) we first note that

$$\phi_x(t) - \phi_{x+h}(t) = \{f(x+t) - f(x+t+h)\} + \{f(x-t) - f(x+h-t)\} \\ -2\{f(x) - f(x+h)\},$$

and then apply Minkowski's inequality separately for $p \geq 1$ *and for* $0 < p < 1$.

Lemma –II *For* $0 < t < \frac{1}{n+1}$ *and fact that* $\frac{1}{\sin t} \leq \frac{\pi}{2t}$ *for* $0 < t \leq \frac{\pi}{2}$,

$$k(n,t) = O(n+1). \quad (5.3)$$

Lemma -III *For* $\frac{1}{n+1} < t < \pi$

$$k(n,t) = O\left(\frac{1}{(n+1)t^2}\right). \tag{5.4}$$

Proof of the main theorem

The n^{th} partial sum $S_n(x)$ of the Fourier series (2.1) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t)\frac{sin\left(n+\frac{1}{2}\right)}{sin\frac{t}{2}}dt \tag{6.1}$$

The (C, 1) transform i.e. σ_n of S_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}(S_k(x) - f(x)) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\emptyset(t)}{sin\frac{t}{2}}\sum_{k=0}^{n} sin\left(k+\frac{1}{2}\right)t\,dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\emptyset(t)\frac{sin^2(n+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt. \tag{6.2}$$

The matrix means of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_k(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(k+1)}\frac{sin^2(k+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt.$$

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_{n-k}(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(n-k+1)}\frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt.$$

$$t_n(x) - f(x) = \int_0^{\pi}\emptyset(t)K(n,t)d\,t. \tag{6.3}$$

Where $K(n,t)$ is defined in (2.13).The change of order of summation and integration is justified provided either side is convergent. We observe that by (2.9) the series for $K(n,t)$ is convergent and $K(n,t) = O(t^{-1})$ for all $0 < t \leq \pi$ and hence the integral given in (6.3) exists by (6.4) and by the fact that $f \in H(\alpha,p)$ in which case

$$\|f(x+t) - f(x)\|_p = O(|t|^{\alpha}).$$

And by Lemma 1(i)

$$w_p{}^{(2)}(t,f) = O(|t|^{\alpha}).$$

By generalized Minkowski's inequality for $p \geq 1$, we have

$$\|l_n(x) - l_n(x+y)\|_p \leq \frac{1}{\pi}\int_0^{\pi}\left\|\emptyset_x(t) - \emptyset_{x+y}(t)\right\|_p |K(n,t)|dt\,. \tag{6.4}$$

We split the integral in (6.3) as I_1 and I_2 with limits of integration from 0 to $\frac{1}{n+1}$ and from $\frac{1}{n+1}$ to π.

$$\|l_n(x) - l_n(x+y)\|_p \leq \frac{1}{\pi}\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right]\left\|\emptyset_x(t) - \emptyset_{x+y}(t)\right\|_p |K(n,t)|dt$$

$$l_n(x) = \frac{1}{\pi}\int_0^{\frac{1}{n+1}}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p K(n,t)\,dt + \frac{1}{\pi}\int_{\frac{1}{n+1}}^{\pi}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p K(n,t)\,dt$$

$$= I_1 + I_2,\ say \qquad (6.5)$$

We use $\left|\sum_{k=0}^{\infty} a_{n,k}\ \sin\left(k+\frac{1}{2}\right)t\right| \le \|A\| < \infty.$

Again $I_1 = \frac{1}{\pi}\int_0^{\frac{1}{n+1}}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p |K(n,t)|\,dt\,.$

by Lemma 1 (i), we get

$$I_1 = O(1)\int_0^{\frac{1}{n+1}} t^{\alpha}|K(n,t)|\,dt$$

$$= O(1)\int_0^{\frac{1}{n+1}} t^{\alpha-2}\,dt$$

$$= O(1)\left[t^{\alpha-1}\right]_0^{\frac{1}{n+1}}$$

$$= O(1)\frac{1}{(n+1)^{\alpha-1}}. \qquad (6.6)$$

We make use of the fact that, by Abel's transformation

$$\sum_{k=0}^{\infty} a_{n,k}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} = O(t^{-2})\sum_{k=0}^{\infty}\left|a_{n,k}-a_{n,k+1}\right|. \qquad (6.7)$$

again by using Lemma 1 (i), we obtain

$$I_2 = \frac{1}{\pi}\int_{\frac{1}{n+1}}^{\pi}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p |K(n,t)|\,dt$$

$$= O(1)\int_{\frac{1}{n+1}}^{\pi} t^{\alpha}|K(n,t)|\,dt$$

$$= O\left(\psi(n)\frac{1}{(n+1)^{\alpha-2}}\right), 0<\alpha\le 1. \qquad (6.8)$$

Again $I_1 = \int_0^{\frac{1}{n+1}}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p |K(n,t)|\,dt$

$$= O((|y|^{\alpha})(n+1)) \qquad (6.9)$$

$$I_2 = \int_{\frac{1}{n+1}}^{\pi}\left\|\emptyset_x(t)-\emptyset_{x+y}(t)\right\|_p |K(n,t)|\,dt$$

$$= O\left(|y|^{\alpha}\psi(n)\right)\int_{\frac{1}{n+1}}^{\pi}|K(n,t)|\,dt$$

$$= O\left(|y|^{\alpha}\psi(n)\right)\int_{\frac{1}{n+1}}^{\pi} t^{-3}\,dt$$

$$= O\left(|y|^{\alpha}\psi(n)\right)\left[t^{-2}\right]_{\frac{1}{n+1}}^{\pi}$$

$$= O\left(|y|^{\alpha}\psi(n)\right)(n+1)^2. \qquad (6.10)$$

Now $I_r = I_r^{\,1-\frac{\beta}{\alpha}}\ I_r^{\,\frac{\beta}{\alpha}}$ *where* $r = 1,2,3\ldots$

From (6.6) and (6.9) we get

eISSN 1303-5150

$$I_1 = O\left[\left\{\frac{1}{(n+1)^{\alpha-1}}\right\}^{1-\frac{\beta}{\alpha}} \{(|y|^{\alpha})(n+1)\}^{\frac{\beta}{\alpha}}\right]$$
$$= O\left[(n+1)^{1+\beta-\alpha-\frac{\beta}{\alpha}}\left\{|y|^{\beta}(n+1)^{\frac{\beta}{\alpha}}\right\}\right]$$
$$= O\left[(n+1)^{1+\beta-\alpha}\,|y|^{\beta}\right]. \qquad (6.11)$$

From (6.8) and (6.10) we get

$$I_2 = O\left[\left\{\psi(n)\frac{1}{(n+1)^{\alpha-2}}\right\}^{1-\frac{\beta}{\alpha}} \{(|y|^{\alpha}\psi(n))(n+1)^2\}^{\frac{\beta}{\alpha}}\right]$$
$$= O(\psi(n))\left[(n+1)^{2+\beta-\alpha-\frac{2\beta}{\alpha}}\,|y|^{\beta}(n+1)^{\frac{2\beta}{\alpha}}\right]$$
$$= O(\psi(n))\,|y|^{\beta}(n+1)^{2+\beta-\alpha}. \qquad (6.12)$$

Now form (6.11) and (6.12) we get

$$\|l_n(x+y)-l_n(x)\|_p = O\left[(n+1)^{1+\beta-\alpha}\,|y|^{\beta}\right] + O(\psi(n))\,|y|^{\beta}(n+1)^{2+\beta-\alpha}$$
$$= O((\psi(n))\,|y|^{\beta})\left[(n+1)^{1+\beta-\alpha}+(n+1)^{2+\beta-\alpha}\right]. \qquad (6.13)$$

Hence $$\sup_{y\neq 0}\frac{\|l_n(x+y)-l_n(x)\|_p}{|y|^{\beta}} = O(\psi(n))\left[(n+1)^{2+\beta-\alpha}\right]. \qquad (6.14)$$

Now

$$\|l_n(x)\|_p = O(\psi(n))[(n+1)^{2-\alpha}]. \qquad (6.15)$$

Now we combine (6.14) and (6.15) to obtain the degree of approximation for $\|l_n(x)\|_{(\beta,p)}$

$$\|l_n(x)\|_{(\beta,p)} = O(\psi(n))\left[(n+1)^{2+\beta-\alpha}\right].$$

Thus the theorem is completely proved.

References

Alexits, G. Über die Annäherung einer stetigen function durch die Cesaroschen Mittel inhrer Fourier reihe. Math. Annalen 1928; 100: 264 - 277.

Chandra, P. On the degree of approximation of functions belonging to Lipschitz class. Nanta Math. 1978; 8: 88 - 91.

Chandra, P. On the generalized Fejer means in the metric of the Hölder space. Math. Nachr.Stud. 1984; 52: 121 - 125.

Chandra, P. Degree of approximation of functions in the Hölder metric. J. Indian Math. Soc. 1988; 53: 99 - 114.

Das, G. Ghosh, T. and Ray, B. K. Degree of approximation of functions in the Hölder metric by (e, c) means. Proc. Indian Acad. Sci. (Math. Sci.). 1995; 105 (3): 315 - 327.

Das, G. Ghosh, T. and Ray, B.K. Degree of approximation of functions by their Fourier series in the generalized Hölder metric, Proc. Indian Acad. Sci. (Math. Sci.). 1996; 106(2): 139 - 153.

Dhakal, B.P. Approximation of functions belonging to the $Lip\ \alpha$ class by Matrix Cesaro Summability method. International Mathematical forum. 2010; 5(35): 1729 - 1735.

Hardy, G. H. and Littlewood, J. E. a Convergence criterion for Fourier series. Math. Z. 1928; 28: 612 - 634.

Hardy G H, Divergent series, 1949: Oxford.

Mohapatra, R.N. and Chandra, P. Continuous function and their Euler, Borel and Taylor mean. Math. Chonical 1982; 11: 81 - 96.

Mohapatra, R.N. and Chandra, P. Degree of approximation of functions in Hölder metric, Acta Mathematica Hungaria 1983; 41(1-2): 67 - 76.

Prösdroff s, Zur Konvergenz der Fourier reihen Holder Stetiger Funktionen. Math.Nachr 1975; 69: 7 - 14.

Yadav, Y. and Rathore, C.S. Approximation of Functions belonging to the weighted $W(L^p\xi(t))$ class by Matrix – Cesaro summability method, Universe of emerging technologies and science 2015: Volume II Issue VIII-August.

Zygmund, A, Trignometric. Series, Cambridge University Press, New York, 1959: Vol.1.