Some properties of fuzzy normed algebra

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Abstract:

This work, proposed new methodology to define the fuzzy normed algebra depending on the fuzzy setting Instead of the traditional definitions. In addition, to explain some application of fuzzy normed algebra.

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1- Introduction

The first definition of fuzzy sets demonstrated by Zadah in [1]1965 and then many authors applied it to different branches of pure and applied mathematics and then presented the Fuzzy Standard Space shortened by cheng, and Mordeson in [4] 1994, the definition of fuzzy metric space that was presented before kaleva and seikkala in [3] 1984.

Sedeqi and Amirpour In 2007 [1] defined fuzzy complete normed algebra and demonstrated a few of its findings. Some characteristics of intuitionistic fuzzy complete normed algebra were introduced by Dinda et al. in 2010 [2]. In 2016 Binzar et al. [5] established a need for the continuous product in a fuzzy normed algebra, and they also demonstrated several characteristics of the fuzzy full normed algebra.

In this study, we first defined fuzzy normed algebra and then established some of its fundamental characteristics. The definition of quotient normed algebra is then introduced, and the fundamental characteristics of this space are then demonstrated.

2-Fuzzy normed space

In this section explains the basic characteristics of fuzzy normed space.

[1]Definition (2.1) [6]

Assume that $\mathcal{X}$ be linear space over Ʀ A function $NM(\,.\,):\mathcal{X} \to Ʀ^*(I)$ is said fuzzy norm if

1. $NM(x) = \bar{0}\ iff\ x = 0$.
2. $NM(\lambda x) = |\lambda| \odot NM(x)$, $x \epsilon \mathcal{X}$, $\lambda \epsilon Ʀ$.
3. $NM(x+h) \le NM(x) \oplus NM(h)$.

[2]Definition (2.2) [6]

Assume that $\mathcal{X} \neq \phi$ over Ʀ. The Function $\bar{\mathcal{M}}: \mathcal{X} \times \mathcal{X} \to Ʀ^*(I)$ is said fuzzy metric if satisfies

1. $\bar{\mathcal{M}}(x\,,h) = \bar{0} \Leftrightarrow x = h$.

2.$\bar{\mathcal{M}}(x , h) = \bar{\mathcal{M}}(h , x)$
3. $\bar{\mathcal{M}}(x , h) \leq \bar{\mathcal{M}}(x , z) \oplus \bar{\mathcal{M}}(z , h), \forall x , h , z \epsilon \mathcal{X}$
Afuzzy metric space is $(\mathcal{X} , \bar{\mathcal{M}})$.
Where $\mathcal{X} \neq \phi$ and $\bar{\mathcal{M}}$ is a fuzzy metric function .
[3]**Example (2.3) [6]**
Assume that $\mathcal{X} \neq \phi$. the Function $\bar{\mathcal{M}}: \mathcal{X} \times \mathcal{X} \rightarrow \mathrm{R}^*(I)$ which shortened by
$\bar{\mathcal{M}}(x, h) = \begin{cases} \bar{0} , x = h \\ \bar{1} , x \neq h \end{cases}$
$\forall x , h \epsilon \mathcal{X}$. $(\mathcal{X} , \bar{\mathcal{M}})$ is afuzzymetric space.

[4]**Theorem (2.4) [6]**
Every fuzzynormed space is fuzzymetric space .
[5]**Remark (2.5) [6]**
The inverse of the previous theorem is not always true (see example **2.3**).
[6]**Definition (2.6) [6]**
Assume that $\mathcal{X}$ be a fuzzynormed space,$x_0 \epsilon \mathcal{X}$ and $\bar{r}$ a fuzzy number with $\bar{r} \neq \bar{0}$, the open ball $\beta_{\bar{r}}(x_0)$ in $\mathcal{X}$ of Radius $\bar{r}$ and center x_0 defined as

$$\beta_{\bar{r}}(x_0) = \{x \epsilon \mathcal{X}: NM(x - x_0) < \bar{r}\}$$

And the closed ball defined as

$$\overline{\beta_{\bar{r}}}(x_0) = \{x \epsilon \mathcal{X}: NM(x - x_0) \leq \bar{r}\}$$

[7]**Theorem (2.7) [6]**
Every open ball and closed ball are nonempty convex set .

[8]**Definition (2.8) [6]**
Assume that $\mathcal{X}$ be a fuzzynormed space , A sequence $\{x_n\}$ in $\mathcal{X}$ is named:
1. converge to the point $x \epsilon \mathcal{X}$ if $\lim_{n \to \infty} NM(x_n - x) = \bar{0}$ 8847
If for each fuzzy number $\bar{\varepsilon} \neq \bar{0}$, $\exists k \epsilon Z^+$ such that
$NM(x_n - x) < \bar{\varepsilon}, \forall n \geq k$.
2 .Cauchy sequence if for each fuzzy number $\bar{\varepsilon} \neq \bar{0}$, $\exists k \epsilon Z^+$ such that
$NM(x_n - x_m) < \bar{\varepsilon}$, $\forall n , m \geq k$.
[9]**Remark (2.9)**
Each Convergent-Sequence is Cauchy-sequence But not Conversely.
[10]**Definition (2.10)**
A fuzzynormed space $\mathcal{X}$ is Complete if Each Cauchy-Sequence is converge to point $x \in \mathcal{X}$.

[11]**Definition (2.11) [6]**
Assume that $\mathcal{X}$ be a fuzzynormed space and $\mathcal{A}$ contained to $\mathcal{X}$:
1. the point $x \epsilon \mathcal{X}$ is alimit point to set A if $\forall \bar{r} > \bar{0}$, $\exists h \epsilon \mathcal{A}$ such that $x \neq h$ and if $NM(x - h) < \bar{r}$
Set all limit point of set A is named (Derived) of set $\mathcal{A}$and Shortened by $\acute{\mathcal{A}}$
$\acute{\mathcal{A}} = \{x \epsilon \mathcal{X}: \forall \bar{r} > \bar{0}, \exists h \in \mathcal{A} \ni h \neq x, NM(x - h) < \bar{r}\}$
The point $x \epsilon \mathcal{X}$ is a limitpoint to set $\mathcal{A}$ if G open set in $\mathcal{X}$ and if $x \epsilon G$ Then $\mathcal{A} \cap (G / \{x\}) \neq \emptyset$ Set all limitpoint of set $\mathcal{A}$ is named (Derived) of set $\mathcal{A}$ and Shortened by $\acute{\mathcal{A}}$.

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2. A point $x \epsilon \mathcal{X}$ is a closure point to set $\mathcal{A}$ if $\forall \bar{r} > \bar{0} \exists h \epsilon \mathcal{A}$ such that $NM(x-h) < \bar{r}$. the set whose elements all point closure of set $\mathcal{A}$ is named (closure) of set $\mathcal{A}$ and Shortened by $\bar{\mathcal{A}}$.
$\bar{\mathcal{A}} = \{x \epsilon \mathcal{X} : \forall \bar{r} > \bar{0}, \exists h \epsilon \mathcal{A} \ni NM(x-h) < \bar{r}\}$.

[12]**Theorem (2 .12) [6]**
Assume that $\mathcal{X}$ be afuzzy normed space and let $\mathcal{A} \subseteq \mathcal{X}$
1. $\acute{\mathcal{A}} \subset \bar{\mathcal{A}}$
2. $\bar{\mathcal{A}} = \mathcal{A} \cup \acute{\mathcal{A}}$

[13]**Theorem (2 .13) [6]**
Assume that $\mathcal{A}$ be a Subset in a fuzzynormed space $\mathcal{X}$. then $x \epsilon \bar{\mathcal{A}}$ iff exists sequence $\{x_n\}$ in $\mathcal{A}$ such that $\{x_n\}$ converge to x .

[14]**Definition (2.14) [6]**
Assume that $\mathcal{X}$ and Y be a fuzzynormed space. Afunction $f: \mathcal{X} \to Y$ is named
1. fuzzycontinuous at $x_0 \epsilon \mathcal{X}$ if for each fuzzy number $\bar{\varepsilon} \epsilon R^*(I)$, $\exists \bar{\delta} \neq \bar{0}$,
$\ni NM(f(x) - f(x_0)) < \bar{\varepsilon}$ When ever $NM(x - x_n) < \bar{\delta}$.
2. sequentiallycontinuous at $x_0 \epsilon \mathcal{X}$ if
$NM(f(x_n) - f(x_0)) \to \bar{0}$ in Y when ever $NM(x_n - x_0) \to \bar{0}$ in $\mathcal{X}$.

[15]**Theorem (2.15) [6]**
Let $\mathcal{X}$ and Y be a fuzzy normed space then the A function $f: \mathcal{X} \to Y$ is fuzzy continuous at $x_0 \epsilon \mathcal{X}$ if and only if for all fuzzy sequence $\{x_n\}$ is fuzzy convergent to $x_0 in\, \mathcal{X}$ then the sequence $\{f(x_n)\}$ is fuzzy convergent to $f(x_0)$ in Y 8848

3. Fuzzy normed algebra
This section deals with the concepts of fuzzy normed algebra ,and some of their properties.

Definition(3.1)
If the following axioms are true, a fuzzy normed algebra over a field F exists.:
1.$\mathcal{X}$ is an algebra over$\mathbb{F}$
2. NM is a fuzzy norm on $\mathcal{X}$
3. $NM(xh) \leq NM(x) \odot NM(h)$For all $x, \mathrm{h} \in \mathcal{X}$ and if $\mathcal{X}$ has an identity e , $\mathrm{NM(e)} = \bar{1}$.

Definition (3.2)
Let$(\mathcal{X}, NM)$ be a fuzzynormed algebra , A sequence $\{x_n\}$ in $\mathcal{X}$ is called
converge to the point $x \epsilon \mathcal{X}$ if $\lim_{n \to \infty} NM(x_n - x) = \bar{0}$
If for each fuzzy number $\bar{\varepsilon} \neq \bar{0}$, $\exists\, k \epsilon Z^+$such that $NM(x_n - x) < \bar{\varepsilon},\ \forall n \geq k$.

Definition (3.3)
Let$(\mathcal{X}, NM)$ be a fuzzynormed algebra, A sequence $\{x_n\}$ in $\mathcal{X}$ is called
Cauchy sequence if for each fuzzy number $\bar{\varepsilon} \neq \bar{0}$, $\exists k \epsilon Z^+$ such that
$NM(x_n - x_m) < \bar{\varepsilon}$, $\forall n, m \geq k$.

Definition (3.4)
A fuzzynormed algebra$(\mathcal{X}, NM)$ is Complete if Each Cauchy-Sequence is converge to point $x \in \mathcal{X}$.

Definition (3.5)

A complete fuzzy normed algebra is called fuzzy Banach algebra .

Definition (3.6)

If $(\mathcal{X}, \mathrm{NM})$ is a complete fuzzy normed space then the fuzzy normed algebra $(\mathcal{X}, \mathrm{NM})$ is called complete fuzzy normed algebra .

Theorem (3.7):

$(\mathcal{X}_e, \mathrm{NM})$ is a complete if and only if $(\mathcal{X}, \mathrm{NM})$ is complete.

Proof:

Suppose that $\mathcal{X}_e = \mathcal{X} \times \mathbb{C}$ is complete and let (x_n), (h_n) be two Cauchy sequence in $\mathcal{X}$ and $\mathbb{C}$ respectively that is for any fuzzy number $\bar{\varepsilon} \neq \bar{0}$,
$\exists K \in Z^+$, such that $\mathrm{NM}(x_n - x_m) < \bar{\varepsilon_1}, \forall n, m \geq K$
also ,$\mathrm{NM}_c(h_n - h_m) < \bar{\varepsilon_2}$ for all $n, m \geq K$.

$$NM_{\mathcal{X}_e}((x_n, h_n) - (x_m, h_m)) = NM_{\mathcal{X}_e}(x_n - x_m, h_n - h_m)$$
$$= NM_{\mathcal{X}}[x_n - x_m] \oplus \mathrm{NM}_c[h_n - h_m]$$
$$\leq \bar{\varepsilon_1} \oplus \bar{\varepsilon_2} < \bar{\varepsilon}$$

Then $NM_{\mathcal{X}_e}[(x_n, h_n) - (x_m, h_m)] < \bar{\varepsilon}$ for all $m, n \geq K$. Thus $[(x_n, h_n)]$ is Cauchy sequence in $\mathcal{X}_e$is complete so there is $(x, \lambda) \in \mathcal{X}_e$ but $\mathcal{X}_e$ such that $(x_n, h_n) \rightarrow (x, h)$ that is

$$1 = NM_{\mathcal{X}_e}[(x_n, h_n) - (x, h)]$$
$$= NM_{\mathcal{X}}(x_n - x) \oplus NM_c(h_n - h) \leq \bar{\varepsilon}$$

Therefore $NM_{\mathcal{X}}(x_n - x) \leq \bar{\varepsilon}$ and $NM_c(h_n - h) \leq \bar{\varepsilon}$.
Hence $\mathcal{X}$is complete.

Conversely, assume that $\mathcal{X}$ is complete and let $[(\mathcal{X}_n, h_n)]$ be a Cauchy sequence in $\mathcal{X}_e$ then for any given $\bar{\varepsilon} \neq \bar{0}$there is $K \in Z^+$ such that that $NM_{\mathcal{X}_e}[(x_n, h_n) - (x_m, h_m)] < \bar{\varepsilon}$ for all $m, n \geq K$ or
$NM_{\mathcal{X}}(x_n - x_m) \oplus NM_c(h_n - h_m) < \bar{\varepsilon}$, Hence
$NM_{\mathcal{X}}(x_n - x_m) < \bar{\varepsilon}_1$ and $NM_c(h_n - h_m) < \bar{\varepsilon}_2$ for all $m, n \geq K$. for some $\bar{\varepsilon}_1 < \bar{\varepsilon}$ *and* $\bar{\varepsilon}_2 < \bar{\varepsilon}$, This implies (x_n), (h_n) are Cauchy sequence in $\mathcal{X}$ and $\mathbb{C}$ respectively, but $\mathcal{X}$ and $\mathbb{C}$ are complete so there is $x \in \mathcal{X}$ and $h \in \mathbb{C}$ such that $NM_{\mathcal{X}}(x_n - x) \leq \bar{\varepsilon}$ and $NM_c(h_n - h) \leq \bar{\varepsilon}$, Now

$$NM_{\mathcal{X}_e}[(x_n, h_n) - (x, h)]$$
$$= NM_{\mathcal{X}}(x_n - x) \oplus NM_c(h_n - h) = \bar{\varepsilon}_1 \oplus \bar{\varepsilon}_2 \leq \bar{\varepsilon}.$$

Hence $(x_n, h_n) \rightarrow (x, h)$ thus$\mathcal{X}_e$ is complete.

Proposition (3.8)

Let $(\mathcal{X}, NM)$be a fuzzy normed space and M be a closed subspace of $\mathcal{X}$. Then the quotient space $\frac{\mathcal{X}}{M}$ is fuzzy normed space with respect to the quotient fuzzy norm defined by: $\mathrm{QM}[x / M] = \sup\{NM(x / a) : a \in M\} = \frac{\mathcal{X}}{M} = \{x / M : x \in \mathcal{X}\}$.

Theorem (3.9)

Let $\mathcal{X}$ be a complete fuzzy normed space and M be a closed subspace of $\mathcal{X}$ then $(\frac{\mathcal{X}}{M})$ is a complete fuzzy normed space.

Proof:

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Let (x_n / M) be a fuzzy Cauchy sequence in $\left(\frac{\mathcal{X}}{\mathrm{M}}\right)$.
hence for every $\bar{\varepsilon} > \bar{0}, NM((x_m - x_n)/M) < \bar{\varepsilon}$
but $NM(x_m - x_n) < NM((x_m - x_n)/M) < \bar{\varepsilon}$
thus $NM(x_m - x_n) < \bar{\varepsilon}$. That is x_n is a fuzzy Cauchy sequence in $\mathcal{X}$
but $\mathcal{X}$ is fuzzy complete, so $x_n \longrightarrow x \in \mathcal{X}$.
thus $\frac{x_n}{\mathrm{M}} \longrightarrow \frac{x}{\mathrm{M}} \in \frac{\mathcal{X}}{\mathrm{M}}$
since NM is fuzzy continuous
hence $\frac{\mathcal{X}}{\mathrm{M}}$ is fuzzy complete.

Theorem (3.10)

Let $(\mathcal{X}, NM)$ be a complete fuzzy normed algebra and suppose M is a closed ideal in $\mathcal{X}$. Then $\left(\frac{\mathcal{X}}{\mathrm{M}}, QM\right)$ is a complete fuzzy normed algebra. If $\mathcal{X}$ has identity then $\frac{\mathcal{X}}{\mathrm{M}}$ has identity. Moreover the identity of $\frac{\mathcal{X}}{\mathrm{M}}$ has fuzzy norm equa
to $\bar{1}$.

Proof:

We know that $\frac{\mathcal{X}}{\mathrm{M}}$ is complete fuzzy normed space by theorem 3.1.8. Since M is an ideal it is easy to see that $\frac{\mathcal{X}}{\mathrm{M}}$ is an algebra with multiplication given by
$(x / M) \cdot (h / M) = (xh) / M.$ *Now*

$$QM[(x / M)(h / M)] = QM[(xh / M)]$$
$$= sup_{a \in M} NM(xh / a)$$
$$\leq sup_{a \in M} NM[(x / a)(h / b)]$$
$$\leq sup_{a \in M} NM(x / a) \odot sup_{b \in M} NM(h / b)$$
$$= QM(x / M) \odot QM(h / M).$$

Thus $(\frac{\mathcal{X}}{\mathrm{M}}, \mathrm{QM})$ is a complete fuzzy normed algebra. Furthermore if e is the identity of $\mathcal{X}$ with $\mathrm{NM(e)} = \bar{1}$ then e / M identity of $\frac{\mathcal{X}}{\mathrm{M}}$. Also
$\mathrm{Q}M = \sup_{a \in M} NM[\mathrm{e} / \mathrm{a}] \leq \mathrm{NM(e)} = \bar{1}$ where $[\mathrm{a} = 0]$ so $\mathrm{QM[e / M]} = \bar{1}$.

Definition (3.11):

let $(\mathcal{X}, \mathrm{NM})$and $(\mathrm{Y}, \mathrm{NM})$be two fuzzy normed algebra. Afunction $f: \mathcal{X} \to Y$ is named
1. fuzzy continuous at $x_0 \epsilon \mathcal{X}$ if for each fuzzy number $\bar{\varepsilon}\, \epsilon \mathrm{R}^*(I)$, $\exists \bar{\delta} \neq \bar{0}$,
$\ni NM(f(x) - f(x_0)) < \bar{\varepsilon}$ When ever $NM(x - x_0) < \bar{\delta}$.
2. sequentially continuous at $x_0 \epsilon \mathcal{X}$ if
$NM(f(x_n) - f(x_0)) \to \bar{0}$ in Y when ever $NM(x_n - x_0) \to \bar{0}$ in $\mathcal{X}$.

Lemma (3.12)

In a fuzzy normed algebra $(\mathcal{X}, NM)$multiplication is fuzzy continuous function.

Proof:

Suppose that $(\mathrm{a_n})$ and $(\mathrm{b_n})$ be two sequences in $\mathcal{X}$. If $\mathrm{a_n} \to \mathrm{a}$, $\mathrm{b_n} \to \mathrm{b}$ as
$\mathrm{n} \to \infty$ then for each fuzzy number $\bar{\varepsilon} \neq \bar{0}, \exists\ k\ \epsilon Z^+$ such that
$NM(a_n - a) < \bar{\varepsilon}_1,\ \forall n \geq k$
Also $NM(b_n - b) < \bar{\varepsilon}_2,\ \forall n \geq k$

$$NM(a_nb_n - ab) = NM(a_nb_n - a_nb + a_nb - ab)$$
$$\leq NM(a_nb_n - a_nb) \oplus NM(a_nb - ab)$$
$$\leq NM\big(a_n(b_n - b)\big) \oplus NM\big(b(a_n - a)\big)$$
$$\leq NM(a_n) \odot NM(b_n - b) \oplus NM(b) \odot NM(a_n - a)$$
$$\leq \bar{r}_1 \odot \bar{\varepsilon}_1 \oplus \bar{r}_2 \odot \bar{\varepsilon}_2 \leq \bar{\varepsilon}$$

then multiplication is continuous function.
Now using theorem 2.5.9 we have

Definition (3.13)
let $(\mathcal{X}, \mathrm{NM})$ be a fuzzy normed algebra and $T: \mathcal{X} \rightarrow \mathcal{X}$ be an linear operator , we say that T is fuzzy bounded if then exists $\bar{\varepsilon}$ such that $NM(T(x)) \leq \bar{\varepsilon} \odot NM(x)$
If T is fuzzy bounded then $NM(T) = sup_{x\in\mathcal{X}} NM\big(T(x)\big), NM(x) = \bar{1}$.

Theorem (3.14)
Every fuzzy normed algebra can be embedded as a closed subalgebra of $\mathrm{B(X)}$**.**

Proof:
Define $\mathrm{N_x: X \rightarrow X}$ by $\mathrm{N_x(a) = xa}$ for all $\mathrm{a} \in \mathrm{X}$ .Then $\mathrm{N_x} \in \mathrm{B(X)}$ since $\mathrm{N_x(a_1 + a_2) = x(a_1 + a_2) = xa_1 + xa_2 = N_x(a_1) + N_x(a_2)}$ and
$\mathrm{N_x(\alpha a) = x(\alpha a) = \alpha(xa) = \alpha N_x(a)}$.Also

$$\mathrm{NM\big(N_x(a)\big) = NM(xa) \leq NM(x) \odot NM(a)}$$

so that $\mathrm{NM(N_x) \leq NM(x)}$.
Now we show that $\mathrm{N_{a+b} = N_a + N_b, N_{ab} = N_a \cdot N_b, N_{\alpha a} = \alpha N_a, N_e = I}$
$\mathrm{N_{a+b}(x) = (a + b)x = ax + bx = N_a(x) + N_b(x)}$
$\mathrm{N_{\alpha a}(x) = (\alpha a)x = \alpha(ax) = \alpha N_a(x)}$
$\mathrm{N_{ab}(x) = (ab)x = a(bx) = N_a \cdot N_a(x)}$
$\mathrm{N_e(x) = e \cdot x = x = I_x(x)}$.
Since $\mathrm{NM\big(N_x(y)\big) = NM(xy) \leq NM}(x)\mathrm{\odot NM(y)}$.
Put $\mathrm{NM(x) = \varepsilon}$ for some $0 < \varepsilon < 1$. That is
$\mathrm{NM\big(N_x(y)\big) \leq \varepsilon \odot NM(y)}$. Therefore $\mathrm{N_x}$ is bounded. Put 8851
N = $\{\mathrm{N_x: x \in X}\}$ so N is a subalgebra of $\mathrm{B(X)}$ and the function
$\mathrm{F: X \rightarrow B(X)}$ defined by $\mathrm{F(a) = N_a}$ is an isometric so it is injective.
Moreover, the image of the function F i.e F(X) is a closed subalgebra of
$\mathrm{B(X)}$.Now suppose that $(\mathrm{N_{a_n}})$ be a sequence in $\mathrm{B(X)}$ such that
$\mathrm{N_{a_n}}$ →S in $\mathrm{B(X)}$ then $\mathrm{N_{a_n}(x) = xa_n = N_e(a_n)x}$ and so as $\mathrm{n \rightarrow \infty}$

Proposition (3.15)
If $(\mathcal{X}, NM)$ is a complete fuzzy normed algebra then the inverse operator
$x \rightarrow x^{-1}$ is fuzzy continuous mapping.

Proof
First we show that the inverse map is fuzzy continuous at e, if $\bar{\varepsilon} \neq \bar{0}$ be given and for all $\bar{\delta} \neq \bar{0}$ such that$\mathrm{NM(a - e)} < \bar{\delta}$ implies $NM\big(\mathrm{a^{-1} - e}\big) < \bar{\varepsilon}$. Now since $NM(a - e) < \bar{1}$ *implies* $a^{-1} = \sum_{n=0}^{\infty}(e - a)^n$ *therefore*

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$$NM(a^{-1} - e) = NM\left(\sum_{n=0}^{\infty}(e-a)^n - e\right)$$
$$= NM\left(\sum_{n=1}^{\infty}(e-a)^n\right)$$
$\leq \bar{\delta} \oplus \bar{\delta} \odot \bar{\delta} \oplus \bar{\delta} \ldots$
$$\leq \bar{\varepsilon}$$
we get $NM(a^{-1} - e) < \bar{\varepsilon}$. *Now* $x_n \to x$ *implies*
$x_n x^{-1} \to x x^{-1} = e$ *implies* $(x_n x^{-1})^{-1} \to e$ *implies* $x {x_n}^{-1} \to e$
implies $x_n^{-1} \to x^{-1}$.

Theorem (3.16)

A fuzzy normed algebra$(\mathcal{X}, \mathrm{NM})$without identity can be embedded into fuzzy normed algebra $\mathcal{X}_{\mathrm{e}}$with identity e and $\mathcal{X}$ is an ideal in $\mathcal{X}_{\mathrm{e}}$.

Proof:

Put $\mathcal{X}_{\mathrm{e}} = \mathcal{X} \times \mathbb{C}$ and define multiplication in $\mathcal{X}_{\mathrm{e}}$by
$(\mathrm{a}, \alpha) \cdot (\mathrm{b}, \beta) = (\mathrm{ab} + \beta\mathrm{a} + \alpha\mathrm{b}, \alpha\beta)$ then $\mathcal{X}_{\mathrm{e}}$ is an algebra with
identity$\mathrm{e} = (0,1)(\mathrm{a}, \alpha) = (\mathrm{a}, \alpha)$
$$\mathrm{e} = (\mathrm{a}, \alpha)(0,1) = (\mathrm{a}, \alpha)$$
also $\mathcal{X}_{\mathrm{e}}$ is a fuzzy normed space 8852
with fuzzy norm :: $\mathrm{NM}_{\mathcal{X}_{\mathrm{e}}} : \mathcal{X}_{\mathrm{e}} \to \mathrm{R}^*(I)$ define by
$\mathrm{NM}_{\mathcal{X}_{\mathrm{e}}}(\mathrm{a}, \alpha) = NM_{\mathcal{X}}(\mathrm{a}) \odot |\alpha|$,where $\mathrm{NM}_c(\alpha) = |\alpha|$
$NM_{\mathcal{X}_{\mathrm{e}}}((\mathrm{a}, \alpha) \cdot (\mathrm{b}, \beta)) = \mathrm{NM}_{\mathcal{X}_{\mathrm{e}}}((\mathrm{ab} + \beta\mathrm{a} + \alpha\mathrm{b}, \alpha\beta))$
$$\leq NM_{\mathcal{X}}(\mathrm{ab} + \beta\mathrm{a} + \alpha\mathrm{b}) \oplus \mathrm{NM}_c(\alpha\beta)$$
$$\leq NM_{\mathcal{X}}(\mathrm{ab}) \oplus \mathrm{NM}_c(\alpha\beta)$$
$\leq NM_{\mathcal{X}}(\mathrm{a}) \odot \mathrm{NM}_{\mathcal{X}}(\mathrm{b}) \oplus |\alpha| \odot |\beta|$, where $\mathrm{NM}_c(\beta) = |\beta|$
$$\leq \mathrm{NM}_{\mathcal{X}}(\mathrm{a}) \oplus |\alpha| \odot [NM_{\mathcal{X}}(\mathrm{b}) \oplus |\beta|]$$
$$= NM_{\mathcal{X}_{\mathrm{e}}}(\mathrm{a}, \alpha) \odot NM_{\mathcal{X}_{\mathrm{e}}}(\mathrm{b}, \beta)$$

Theorem (3.17)

Let $(\mathcal{X}, \mathrm{NM})$ be a fuzzy normed algebra with idendtity e. then there is a fuzzy norm H on A such that $\mathrm{NM}_{\mathcal{X}}$ is equivalent to H such that $(\mathcal{X}, \mathrm{H})$ is a fuzzy normed algebra with $\mathrm{H(e)} = \bar{1}$ and $\mathrm{a} \in \mathcal{X}$.

Proof:

For each $x \in \mathcal{X}$ let N_x be a linear operator defined by $\mathrm{N}_x(\mathrm{h}) = \mathrm{xh}$for all $\mathrm{h} \in \mathcal{X}$. Now if $\mathrm{N}_x = \mathrm{N}_{\mathrm{h}}$ it follows that $\mathrm{N}_x(\mathrm{e}) = \mathrm{N}_{\mathrm{h}}(\mathrm{e})$. And so $x = \mathrm{h}$ hence $x \longmapsto \mathrm{N}_x$ is an injective map from $\mathcal{X}$ into the set of all linear operator on $\mathcal{X}$.

$NM_{\mathcal{X}}(\mathrm{N}_x(\mathrm{h})) = NM_{\mathcal{X}}(x\mathrm{h}) \leq NM_{\mathcal{X}}(x) \odot NM_{\mathcal{X}}(\mathrm{h})$ for $\mathrm{h} \in \mathcal{X}$ which implies that N_x is fuzzy bounded and $NM_{\mathcal{X}}(\mathrm{N}_x) \leq \bar{\varepsilon} \odot NM_{\mathcal{X}}(\mathcal{X})$ put $\mathrm{H(a)} = NM_{\mathcal{X}}(\mathrm{N}_{\mathrm{a},})$ so $\mathrm{H}(x) \leq \bar{\varepsilon} \odot NM_{\mathcal{X}}(\mathcal{X})$ ...(1), for some $\bar{\varepsilon}$, $\bar{0} \leq \bar{\varepsilon} \leq \bar{1}$.

On the other hand
$\mathrm{H(a)} = NM_{\mathcal{X}}(\mathrm{N}_a) = \sup_{\mathrm{b} \in \mathrm{D}(\mathrm{N_a})} NM_{\mathcal{X}}(\mathrm{N_a(b)})$
$$= \sup_{\mathrm{b} \in \mathrm{D}(\mathrm{N_a})} NM_{\mathcal{X}}(\mathrm{ab})$$

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$$\leq NM_{\mathcal{X}}(\mathrm{ae}) = NM_{\mathcal{X}}(\mathrm{a}) \odot NM_{\mathcal{X}}(\mathrm{e}) = \frac{1}{\bar{\varepsilon}} \odot NM_{\mathcal{X}}(\mathrm{a})$$

so $\mathrm{H(a)} \leq \frac{1}{\bar{\varepsilon}} \odot NM_{\mathcal{X}}(\mathrm{a}) \ldots (2)$.

Thus $\bar{\varepsilon} \odot NM_{\mathcal{X}}(\mathrm{a}) \leq \mathrm{H(a)} \leq \frac{1}{\bar{\varepsilon}} \odot NM_{\mathcal{X}}(\mathrm{a})$

for all a $\in \mathcal{X}$

Hence $NM_{\mathcal{X}}(\mathrm{a})$ is equivalent to $\mathrm{H(a)}$. Now

$$\mathrm{H(ab)} = NM_{\mathcal{X}}(\mathrm{N_{ab}}) = NM_{\mathcal{X}}(\mathrm{N_a}.\mathrm{N_b})$$
$$\leq NM_{\mathcal{X}}(\mathrm{N_a}) \odot NM_{\mathcal{X}}(\mathrm{N_b}) = \mathrm{H(a)} \odot \mathrm{H(b)}$$

Therefore $(\mathcal{X}, \mathrm{H})$ is a fuzzy normed algebra. We have now

$\mathrm{H(e)} = NM_{\mathcal{X}}(\mathrm{N_e}) = \bar{1}$.

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