k-prime and k-semi-prime ideals in Ternary Gamma Semi rings

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Abstract

We study the properties of k-prime and k-semi-prime ideals of Ternary gamma semi rings, k-m-system and k-p-system. We make some properties and characterizations for k-prime and k-semi-prime ideals of ternary gamma semi rings in terms of k-m-system and k-p-system, respectively. We initiate the notion of prime ternary gamma semi-ring and semi-prime ternary gamma semi-ring to set up characterizations for these two ideals of a ternary gamma semi-ring. The concept of power ternary gamma semi-ring associated to a ternary gamma semi-group is also studied in this paper.

Key Words: Ternary gamma semi-ring, k-prime ideal, k-semi-prime ideal, prime ternary gamma semi-ring, semi-prime ternary gamma semi-ring, power ternary gamma semi-ring

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1. Introduction

In the literature of non-commutative algebra, the notion of prime (semi prime) ring is very important to characterize prime (semi-prime) ideals. In [3], Golan introduced the notion of prime (semi-prime) ideals in a semi-ring and he showed by examples that there are prime (semi-prime) ideals which are semi-ring theoretical but not ring theoretical. There are some remarkable research works [1, 5, 8] on prime ideals of Semi rings. The class of semi-ring theoretical prime (semi-prime) ideals which are equivalent to ring theoretical prime (semi-prime) ideals are called k-prime (k-semi-prime) ideals. To be precise, a prime ideal of a semi-ring is said to be a k-prime ideal if it is equal to its k-closure. There is not significant study of k-prime (k-semi-prime) ideals of Semi rings by the researchers till now. We establish here the concepts of prime and semi-prime semi-ring and using them we provide the characterizations for k-prime and k-semi-prime ideals. For references, refer the references. For a given semi-group S, power semi-ring is the algebraic structure $(P(S),\oplus,\odot)$,where P(S) is the set of all non-empty subsets of S; $\oplus$ and $\odot$are defined on P(S) as follows:
$A \oplus B = A \cup B$ and $A \odot B = \{ab: a \in A, b \in B\}$.It is a pillar result in structure theory of rings that an ideal is a semi-prime ideal if and only if it is the intersection of prime ideals. We generalize this result here for k-semi-prime ideals of a semi-ring. We prove that an ideal of a semi-ring is a k-semi-prime ideal if and only if it is the intersection of k-prime ideals.

2. Preliminaries

For the preliminaries, refer the references. A left ideal (respectively, right ideal, ideal) A of a ternary semi-ring T is said to be a left k-ideal (respectively, right k-ideal, k-ideal) of T if for any $x \in T$ and $y \in A$, $x + y \in A \Rightarrow x \in A$. Let A be an ideal of a ternary semi-ring T. Then the k-closure [6]

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of A, denoted by $\overline{A}$, is defined as $\overline{A}$ = {a ∈ T: a + b = c for some b, c ∈A}.

Lemma 2.1: Let T be a ternary gamma semi ring. Then for any ideals A, B, C of T, we have the following: (i) A ⊆ $\overline{A}$ (ii) A ⊆B ⇒ $\overline{A} \subseteq \overline{B}$ (iii) $\overline{\overline{A}}$ A = $\overline{A}$ (iv) $\overline{A\Gamma B\Gamma C}$= $\overline{A}\,\Gamma\,\overline{B}\,\Gamma\overline{C}$and(v) $\overline{A}$ is a k-ideal of T.
A non-zero element 'a' of a ternary gamma semi ring T is said to be a zero divisor if there exists b(≠ 0R) ∈T, α, β ∈ Γ such that aαbβc= 0R. A commutative ternary gamma semi ring T is said to be a semi-domain if for any a, b, c ∈T, α, β ∈ Γ, aαbβc= 0T ⟹ either a = 0T or b = 0T or 0T. A non-empty subset M of a ternary gamma semi ring T is said to be an m-system [3] if for any a, b ∈M, α, β ∈ Γ there exists z ∈T ∍ aαzβb ∈M.

3. Prime Ideals and k-Prime Ideals of Ternary gamma semi ring

In this section, we study the notion of k-prime ideal of Ternary gamma semi rings (TGSR) and construct some of its characterizations. We begin with the definition of prime and k-prime ideal of a ternary gamma semi ring.

Definition 3.1: A proper ideal P of a TGSR T is said to be a prime ideal (PI) of T if for any ideals A, B, C of T, AΓBΓΓC⊆P ⟹A ⊆P or B ⊆P or C⊆P.

Definition 3.2: A PI P of a TGSR T is said to be a k-prime ideal of T if $\overline{P}$ = P, where $\overline{P}$ denotes the k-closure of P.

Remark: It is clear from the definitions of prime and k-prime ideal of a ternary gamma semi ring that every k-prime ideal of a ternary gamma semi ring is also a prime ideal of that ternary gamma semi ring. But the converse is not true, in general. In the following example we can observe this.

Example 3.3: Consider the TGSR T= Γ = W of non-negative integers. Then pW are k-prime ideals of T, where p is a prime number. Also, I = W\{1} is a prime ideal of T but not a k-prime ideal of T.

Lemma 3.4: A proper k-ideal I of a TGSR T is a k-prime ideal of T⟺ for any k-ideals I1, I2, I3 of T, I1ΓI2ΓI3⊆I⟹I1⊆ I or I2⊆ I or I3⊆ I.

Proof: Let I, a k-prime ideal of a TGSR T. Then the condition follows easily. To prove the converse, let I a proper k-ideal of T, satisfying the given condition. Consider that I1, I2 and I3 are ideals of T such that I1ΓI2ΓI3⊆I. Accordingly, by Lemma 2.1, $\overline{I_1}\,\Gamma\overline{I_2}\Gamma\overline{I_3} \subseteq \overline{\overline{I_1}\,\Gamma\overline{I_2}\Gamma\overline{I_3}}$ = $\overline{I_1\Gamma I_2\Gamma I_3} \subseteq \overline{I}$= I, since I is a k-ideal. Then by our assumption, we have I1⊆I or I2⊆I or I3. Thus, we find that I1 ⊆I or I2⊆I or I3⊆I. Hence, I is a k-prime ideal of T. We are going to present a characterization theorem for a k-prime ideal of a TGSR.

Theorem 3.5: The following conditions are equivalent for an ideal P of a TGSR T:
(i) P is a prime ideal of T.
(ii) For any a, b, c ∈R, aΓTΓbΓTΓc⊆P ⟺ a ∈P or b ∈P or c ∈P

Theorem 3.6: For an ideal P of a TGSR T, the following statements are equivalent:
(i) P is a k-prime ideal of a TGSR T.
(ii) For any a, b, c∈ R, aΓRΓbΓRΓc ⊆P ⟺ a ∈ P or b ∈ P or c ∈ P.

Proof: (i) ⟹ (ii): Let P be a k-prime ideal of a TGSR T. Then P is a prime ideal of Rand P is a k-ideal of R, i.e., $\overline{P}$= P. So, the result follows by Theorem 3.5.
(ii) ⟹(i): Consider x ∈$\overline{P}$. Then x + p1 = p2 for some p1, p2∈P. Let y ∈ xΓTΓxΓTΓx. So, y = xαs1βx for some s1∈T⟹y + xαs1βp1 = xαs1β(x + p1). It follows that y + αxs1βp1 = xγs1δp2. Now, xαs1βp1, xγs1δp2 ∈P, since P is an ideal of T. Therefore, y ∈P. Accordingly, xΓTΓx ⊆P. Then by assumption, x ∈P. Consequently, $\overline{P}$⊆P and hence $\overline{P}$= P. This shows that P is a k-ideal of T. Again, P is a prime ideal of T, by Theorem 3.5. Consequently, P is a k-prime ideal of T. Next, we set up a very gripping characterization for k-prime ideals of a ternary gamma semi ring. For that we establish the notion of k-m-system.

Theorem 3.7: A proper ideal P of a TGSR T is a PI of T if and only if Pc (complement of P that is R\P or Pc) is an m-system.

Definition 3.8: A non-empty subset M of a TGSR T is said to be a k-m-system if (i) for any a, b ∈ M, α, β ∈ Γ there exists z ∈R such that aαzβb ∈M and (ii) a ∈M ⇒a ∉ $\overline{M_c}$

Example 3.9: Consider the TGSR T= (2n + 1)W, where n=1, 3, 5, …. of non-negative odd integers. Then J=T \ (lT), where l is a positive odd integer, is a k-m-system.

Theorem 3.10: A proper ideal P of a TGSR T is a k-prime ideal of T if and only if Pc is a k-m-system.

Proof: Let P be a proper ideal of a TGSR T. First suppose that Pc is a km-system. Let a, b ∈T be such that aΓTΓb $\subseteq\overline{P}$. If possible, let a ∉P, b ∉P and c ∉ P which shows that a ∈Pc, b ∈Pc and c ∈ P^c. Since Pc is a k-m-system, it follows that there exists z ∈T such that aαzβb ∈ Pc. Because Pc is a k-m-system, we find that aαzβb$\notin \overline{(P_c)}^c = \overline{P}$, which is a contradiction. Hence, a ∈ P or b ∈P or c ∈P. Therefore, P is a k-prime ideal of T, by Theorem 3.6. Conversely, suppose that P is a k-prime ideal of T. Then P is a k-ideal as well as a prime ideal of T. This implies that Pc is an m-system, by Theorem 3.7. Let a ∈Pc. Then a $\notin P = \overline{P}$, since P being a k-ideal. Accordingly, a $\notin \overline{(P^c)^c}$. Consequently, Pc is a k-m-system.

Now, we introduce the notion of prime ternary gamma semi ring which helps us to provide a characterization for k-prime ideal of a ternary gamma semi ring.

Definition 3.11: A TGSR T is said to be a prime ternary gamma semi ring if (0) is a prime ideal of T.

Remark: The notions of prime ternary gamma semi ring and k-prime ternary gamma semi ring are the same, since (0) is a k-prime ideal if and only if it is prime. In the following, we produce a characterization of a prime ternary gamma semi ring. The proof is immediate from Theorem 3.5. After that, we provide some properties of this class of Ternary gamma semi rings.

Theorem 3.12: A TGSR T is a prime ternary gamma semi ring if and only if for any a, b, c ∈ T, aΓbΓc= 0 ⟹ a = 0 or b = 0 or c = 0.

Example 3.13: Any semi-domain is a prime ternary gamma semi ring. Direct product of prime Ternary gamma semi rings may not be a prime ternary gamma semi ring. As for example, W is a prime semi ring but W×W×W is not a prime ternary gamma semi ring.

Lemma 3.14: If A, B and C are two ideals of a TGSR T then (i) Mn(AΓBΓC) =Mn(A)ΓMn(B)ΓMn(C) and (ii) A ⊆B or B ⊆C or C ⊆A ⇔ Mn(A) ⊆Mn(B) or Mn(B) ⊆Mn(C) or Mn(C) ⊆Mn(A) .

Proof: At first, let P ∈Mn(A)ΓMn(B)ΓMn(C). Then P = $\sum_{k=1}^{r} A_k \Gamma B_k \Gamma C_k$, where Ak∈Mn(A), Bk∈Mn(B) and Ck∈Mn(C) for each k = 1, . . ., r. Suppose that Ak= (a_{ij}^{k}) ∈Mn(A), Bk = (b) and Ck=(c_{ij}^{k}). Since A, B and C are ideals of T, we have $\sum_{j=1}^{n} a_{ij}^{k}\alpha b_{ij}^{k}\beta c_{ij}^{k}$ ∈AΓBΓC. This shows that AkΓBkΓCk ∈Mn(AΓBΓC), ∀k = 1, . . ., r. Consequently, P = $\sum_{k=1}^{r} A_k \Gamma B_k \Gamma C_k$∈Mn(AΓBΓC). To prove the reverse inclusion, let C ∈ Mn(ABC). Consider C = (cij) that is cij ∈AΓBΓC⟹cij= $\sum_{l=1}^{k_{ij}} a_l^{ij}\alpha b_l^{ij}\beta c_l^{ij}$, where a_l^{ij}∈A, b_l^{ij}∈Band c_l^{ij}∈C, ∀i, j = 1, . . ., n. Let A_k^{ij}be the matrix such that the (ij)th entry is $a_k^{ij}$and all otherentries are 0, B_k^{ij}be the matrix such that the (ij)th entry is $b_k^{ij}$and all otherentries are 0 and C_k^{ij}be the matrix such that the (ij)th entry is $c_k^{ij}$and all otherentries are 0 for all i, j = 1, . . ., n. Clearly, A_k^{ij}∈Mn(A), B_k^{ij}∈Mn(B) and C_k^{ij}∈Mn(C).
It demonstrates that C =$\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n} A_k^{ij}\Gamma B_k^{ij}\Gamma C_k^{ij}$∈Mn(A)ΓMn(B)ΓMn(C).
(ii) If A ⊆B then it is obvious that Mn(A) ⊆Mn(B). Conversely, let Mn(A) ⊆Mn(B). Consider x ∈A. Construct a matrix P such that P = (pij), where p11 = x and all other entries are 0. Then P ∈Mn(A) ⊆Mn(B)⟹x ∈B.

Lemma 3.15: A proper ideal I of a TGSR T with identity is a prime ideal of T⇔ Mn(I) is a prime ideal of Mn(T).

Proof: Let I, a prime ideal of a TGSR T. Now ideals of Mn(T) are of the form Mn(J), where J is an ideal of R, by [4, Proposition 5.13]. Consider two ideals Mn(A), Mn(B), Mn(C) of Mn(T) ∋ Mn(A)ΓMn(B)ΓMn(C)⊆Mn(I). Then by Lemma 3.14 it follows that Mn(AΓBΓC) = Mn(A)ΓMn(B)ΓMn(C)⊆Mn(I) ⟹AΓBΓC⊆I. It follows that A ⊆I or B ⊆ I or C⊆ I, since I is a prime ideal of T. Thus, we find that Mn(A) ⊆Mn(I) or Mn(B) ⊆Mn(I) or Mn(C)⊆ Mn(I) by Lemma 3.14. Hence Mn(I) is a prime ideal of Mn(T). Conversely, suppose that Mn(I) is a prime ideal of Mn(T). Consider AΓBΓC⊆I, for ideals A, B, C of I. Then Mn(A), Mn(B) and Mn(C) are ideals of Mn(T) and Mn(AΓBΓC)⊆Mn(I), by Lemma 3.14. This shows that Mn(A)ΓMn(B)ΓMn(C)⊆Mn(I), by Lemma 3.14.So, Mn(A) ⊆Mn(I) or Mn(B) ⊆Mn(I) or Mn(C)⊆Mn(I), since Mn(I) is a prime ideal of Mn(T). It demonstrates that A ⊆I or B ⊆I or C⊆ I, by Lemma 3.14. Consequently, I is a prime ideal T. By using Lemma 3.15, we have the following result.

Theorem 3.16: A TGSR T with identity is a prime ternary gamma semi-ring $\Leftrightarrow Mn(T)$ is a prime ternary gamma semi-ring.

Proof: T is a prime ternary gamma semi-ring $\Leftrightarrow (0)$ is a prime ideal of $T \Leftrightarrow Mn(0)$ is a prime ideal of $Mn(T) \Leftrightarrow Mn(T)$ is a prime ternary gamma semi-ring. It is immediate to check the following lemma in which we mention some properties of prime Ternary gamma semi rings.

Lemma 3.17: (i) Centre of a prime TGSR T is a semi-domain.
(ii) A TGSR T has no zero divisors if T is a prime TGSR.
Now, we state a characterization of a k-prime ideal of a semi-ring in terms of prime semi-ring. For this, we need the notion of quotient semi-ring, established by Golan in [3].

Definition 3.18: Let I be a proper ideal of a TGSR T. Then the congruence on T, denoted by ρI and defined by $s\rho Is$ if and only if $s + a1 = s1 + a2$ for some $a1, a2 \in I$, is called the Bourne congruence on T defined by the ideal I. The Bourne congruence class of I containing $r \in T$ is denoted by r/I and the set of all such congruence classes of T by T/I. It should be noted that for any $s \in T$ and for any proper ideal I of T, s/I is not necessarily equal to $s + I = \{s + a: a \in I\}$ but surely contains it. Now if we define the addition and multiplication on T/I by $a/I + b/I = (a + b)/I$ and $(a/I)\Gamma(b/I)\Gamma(c/I) = (a\alpha b\beta c)/I$ for all $a, b, c \in T$ and $\alpha, \beta \in \Gamma$ then with these two operations T/I forms a TGSR, called the Bourne factor TGSR or simply the FTGSR or a quotient TGSR.

Theorem 3.19: A k-ideal I of a TGSR T is a k-prime ideal of $T \Leftrightarrow T/I$ is a prime TGSR.

Remark: For commutative TGSRs, the notions of prime TGSR and semi-domain coincide. If S is a prime TGSR then T is a semi-domain because T is a prime TGSR $\Longrightarrow (0)$ is a prime ideal of T and hence a k-prime ideal of T. Therefore, $T/(0)$ is a semi-domain. Again, T is a semi-domain $\Longrightarrow T/(0)$ is a semi-domain. It follows that (0) is a k-prime ideal of T. So, T is a prime TGSR.
We conclude this section by building up a correspondence between prime ideals of a TGSG T and k-prime ideals of the associated power TGSR P(R). The following results form the base of this correspondence.

Lemma 3.20: Let T be a TGSG. Then (i) $P(I) \subseteq P(J) \Leftrightarrow I \subseteq J$ and(ii) $P(I)\Gamma P(J)\Gamma P(K) \subseteq P(A) \Leftrightarrow I\Gamma J\Gamma K \subseteq A$ for ideals I, J,A of T.

Proof: (i) It is obvious that $I \subseteq J \Longrightarrow P(I) \subseteq P(J)$. Conversely, let $P(I) \subseteq P(J)$. Consider $a \in I$. Then $\{a\} \in P(I) \subseteq P(J)$. This shows that $a \in J$. (ii) At first, suppose that $I\Gamma J\Gamma K \subseteq A$. Consider $L \in P(I)\Gamma P(J)\Gamma P(K)$. Then $L = B1\Gamma C1\Gamma D1 \cup B2\Gamma C2\Gamma D2 \cup \cdot\cdot\cdot \cup Bn\Gamma Cn\Gamma Dn$ for some $Bi \in P(I)$, $Ci \in P(J)$ and $Di \in P(K)$, $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, we have $Bi\Gamma Ci\Gamma Di \subseteq I\Gamma J\Gamma K$. Accordingly, $L \subseteq I\Gamma J\Gamma K \subseteq A$ that is $L \in P(A)$. Conversely, let $P(I)\Gamma P(J)\Gamma P(K) \subseteq P(A)$. Consider $x \in I\Gamma J\Gamma K \Longrightarrow x = a\alpha b\beta c$ for some $a \in I$, $b \in J$ and $c \in K$, $\alpha, \beta \in \Gamma$. That being the case, $\{a\} \in P(I)$, $b \in P(J)$ and $\{c\} \in P(K) \Longrightarrow \{x\} = \{a\alpha b\beta c\} = \{a\}\Gamma\{b\}\Gamma\{c\} \in P(I)\Gamma P(J)\Gamma P(K) \subseteq P(A)$. Hence, $x \in A$.

Theorem 3.21: Let T be a ternary gamma semi-group (TGSG). Then J is a k-ideal of the power semi-ring $P(T) \Leftrightarrow J = P(I)$ for some ideal I of T.

Theorem 3.22: Let T be a TGSG. Then a proper k-ideal J of P(T) is a k-prime ideal of $P(T) \Leftrightarrow J = P(I)$ for some prime ideal I of T.

Proof: Let J be a proper k-ideal of T and $J = P(I)$ for some prime ideal of T. Consider I1, I2, I3 be k-ideals of P(R) such that $I1\Gamma I2\Gamma I3 \subseteq J$. Then $I1 = P(A)$, $I2 = P(B)$ and $I3 = P(C)$ for some ideals A, B and C of the TGSG T, by Theorem 3.21. Thus, we find that $P(A)\Gamma P(B)\Gamma P(C) \subseteq J = P(I)$. By Lemma 3.20, it follows that $A\Gamma B\Gamma C \subseteq I$. It demonstrates that $A \subseteq I$ or $B \subseteq I$ or $C \subseteq I$, since I is a PI of $T \Longrightarrow P(A) \subseteq P(I)$ or $P(B) \subseteq P(I)$ or $P(C) \subseteq P(I)$, by Lemma 3.20. Therefore, $I1 \subseteq J$ or $I2 \subseteq J$ or $I3 \subseteq J$. Consequently, J is a k-prime ideal of P(T), by Lemma 3.4.Conversely, let J be a k-prime ideal of P(T). So, $J = P(I)$ for some ideal I of T, by Theorem 3.21. Consider two ideals J1, J2, J3 of T such that $J1\Gamma J2\Gamma J3 \subseteq I$. This shows that $P(J1)\Gamma P(J2)\Gamma P(J3) \subseteq P(I) = J$, by Lemma 3.20. Then we find that $P(J1) \subseteq P(I)$ or $P(J2) \subseteq P(I)$ or $P(J3) \subseteq P(I)$, since J is a k-prime ideal of P(T). Accordingly, $J1 \subseteq I$ or $J2 \subseteq I$ or $J3 \subseteq I$, by Lemma 3.20. Hence, I is a prime ideal of T

4. k-Semi-prime Ideals of Ternary gamma semi rings

In this section, we study the notion of k-semi prime ideal of a TGSR. We exhibit some of its captivating properties and characterizations. First, we begin with the following definitions.

Definition 4.1: A proper ideal I of a TGSR T is said to be a semi-prime ideal of T if for any ideal H of T, $H\Gamma H\Gamma H \subseteq I \Longrightarrow H \subseteq I$.

Definition 4.2: A semi-prime ideal I of a TGSR T is said to be a k-semi-prime ideal of T if $\bar{I} = I$.

Remark: From previous definitions, it follows that every k-prime ideal of a TGSR T is a k-semi-prime ideal of T and every k-semi-prime ideal of T is a semi-prime ideal of T. But the converse is not true. This follows from the following example.

Example 4.3: Consider the TGSR T= W of non-negative integers. Then qW is a k-semi-PI of T, where q is a square free positive integer. These are k-semi-prime ideals of T which are not k-prime ideals of T. Also, in this example, I = W\{1}is a semi-prime ideal which is not a k-semi-prime ideal of T. In the following, we produce a characterization theorem of k-semi-prime ideal of a TGSR. The proof is similar to the proof of Theorem 3.6. So we omit the proof.

Theorem 4.4: The following statements are equivalent for an ideal I of a TGSR R.
(i) I is a k-semi-prime ideal of a TGSR T.
(ii) For any $a \in R$, $a\Gamma T\Gamma a\Gamma T\Gamma a \subseteq \bar{I} \Leftrightarrow a \in I$.
Now, similar to k-m-system, we can define the analogous version k-p-system which helps us to provide another characterization for k-semi-prime ideal of a TGSR.

Definition 4.5: A non-empty subset I of a TGSR S is said to be a p-system if for any $a \in I$, $\alpha, \beta \in \Gamma$ $\exists z \in R \ni a\alpha z\beta a \in I$. A p-system is said to be a k-p-system if a $\in I \Longrightarrow a \notin Ic$.

Example 4.6: Consider the TGSRR= W of non-negative integers. Then J = T\ nT, where n is a square free positive integer is a k-p-system. Note that every k-m-system is also a k-p-system but not conversely. As for example, in this TGSR T, I = T\ kT, where k is a positive square free integer but not a prime integer is a k-p-system but not a k-m-system. Similar to Theorem 3.10, we have the following interesting characterization fork-semi-prime ideals of a TGSR in terms of k-p-system.

Theorem 4.7: An ideal I of a TGSR T is a k-semi-prime ideal ⟺ Ic is a k-p-system.
Now, we initiate the concept of semi-prime TGSR which helps us to produce an interesting characterization for k-semi-prime ideal of a TGSR.

Definition 4.8: A TGSR T is said to be a semi-prime TGSR if (0) is a semi-prime ideal of T.

Remark: The notions of semi-prime TGSR and k-semi-prime TGSR are the same because (0) is a k-semi-prime ideal ⟺ it is semiprime.

Example 4.9: Any prime TGSR is a semi-prime TGSR. But the converse need not be true, in general. As for example, W× W × W is a semiprime TGSR but not a prime TGSR.
Now similar to Theorem 3.12, we produce a characterization of a semi-prime TGSR.

Theorem 4.10: A TGSR T is a semi-prime TGSR ⟺ for any $a \in T$, $a\alpha T\beta a\gamma T\delta a = 0 \Longrightarrow a = 0$. Again similar to Theorem 3.16, we have the following result.

Theorem 4.11: A TGSR T with identity is a semi-prime TGSR⟺ Mn(R) is a semi-prime TGSR.

Remark: A TGSR T is said to be reduced if it has no non-zero nilpotent element. It can be shown that any reduced TGSR is a semi-prime TGSR and in case of commutative TGSR, the notions of reduced TGSR and semi-prime TGSR coincide.

Theorem 4.12: A k-ideal I of a TGSR T is a k-semi-prime ideal of T⟺ T/I is a semi-prime TGSR.

Proof: If T/I is a semi-prime TGSR then I = 0/I is a semi-prime ideal of T/I. Also 0/I = I because I is a k-ideal of T. Thus, we find that I is a k-semi-prime ideal of T. Conversely, let I be a k-semi-prime ideal of T. Then 0+I is a k-semi-prime ideal of T/I whence 0/I is a k-semi-PI of T/I. Consequently, T/I is semi-prime TGSR. It is one of the pillar results in the structure theory of rings that an ideal of a ring is a semi-prime ideal ⟺ it is the intersection of some prime ideals. Weprove a similar result here for k-semi-prime ideals of TGSRs. For that we establish the following results.
At first, we mention the following lemma. A similar result was proved for ring in [8, Proposition 10.5]. So we omit the proof.

Lemma 4.13: Let M be an m-system of a TGSR T and I be a maximal ideal, maximal with respect to condition that $M \cap I = \emptyset$. Then I, a PI of T.

Definition 4.14: For any TGSR T and any k-ideal I of

T, we define β(I) ={s ∈ T: M ∩ I ≠∅ for any k-m-system M containing s}.

Theorem 4.15: For any TGSR T and any k-ideal I of T, β(I) =$\bigcap_{I\subseteq P, P \text{ is a K-prime ideal}} P$.

Proof: Let x ∈$\bigcap_{I\subseteq P, P \text{ is a K-prime ideal}} P$. So, x ∈P ∀P which contains I. If possible, let x ∉β(I). So, ∃ k-m-system M ∋x ∈M and M ∩ I = ∅. By Zorn's Lemma, ∃ a maximal k-ideal J of T such that M ∩ J = ∅. By Lemma 4.13, J is a k-prime ideal. Now x ∈M and M ∩ J = ∅ imply that x ∉ J. Again, I ⊆J and x ∉J ⟹x ∉ $\bigcap_{I\subseteq P, P \text{ is a K-prime ideal}} P$, which is a contradiction. Thus x ∈ β(I), whence $\bigcap_{I\subseteq P, P \text{ is a K-prime ideal}} P$⊆β(I). To prove the reverse inclusion, consider x ∈ β(I). Let P be a k-prime ideal of T ∋ I ⊆ P. If possible, suppose that x ∉P that is x ∈ Pc. Now Pc is a k-m-system, by Theorem 3.10. Since x ∈ β(I), it follows that Pc ∩ I ≠∅. This contradicts the fact that I ⊆P. Consequently, x ∈P for all k-prime ideals P such that I ⊆P. So, x ∈$\bigcap_{I\subseteq P, P \text{ is a K-prime ideal}} P$.

Lemma 4.16: Let N be a k-p-system of a TGSR T and a ∈N. Then there is a k-m-system M of T ∋ M ⊆N and a ∈ M.

Proof: Let N be a k-p-system and a ∈N. Construct M∗= {a1, a2, a3, . . .}, where a1 = a. Now a1∈N ⟹ there exists some s1∈T such that a1αs1βa1∈N. Take a2 = a1αs1βa1. Continuing this way, we construct M∗. Clearly, a ∈M∗ and M∗⊆N. Now we prove that M∗ is an m-system. Consider x, y ∈M∗.. Then x = ai and y = aj for some odd positive integers i and j. If i > j, choose s∗= $\mathrm{si}\alpha(\prod_{k=1}^{i-j}\times\ a_{i-k}\ \beta s_{i-k})$.
Then aiαs∗βaj = aiαsiβ(
$a_{i-1}\alpha_1\ s_{i-1}\ \alpha_2\ a_{i-2}\ \alpha_2\ s_{i-2}\ \alpha_3\ \ldots\ldots\ldots\alpha_{j-1}\ a_j\alpha_j s_j)\gamma$
aj = aiαsiβai = ai+1∈M∗.
If j>i, choose s∗= $(\prod_{k=0}^{j-i-1} s_{i+k}\alpha a_{i+k})$sj. In this case, aiαs∗βaj=aiα(
$s_i\ \beta_1\ a_i\ \beta_2 s_{i+1}\beta_3 a_{i+1}\ \beta_4\ \ldots\ldots\ldots.\ \beta_{j-1}s_{j-1}\beta_j a_{j-1})$γsjδaj = ajαsjβaj= aj+1∈M∗. If i = j then considering s∗= si or sj we can show that xαs∗βy ∈M∗. Hence, M∗ is an m-system. Construct M = M∗+ Nc. Since N is a k-p-system, Nc is a k-semi-prime ideal of R, by theorem 4.7. Thus, a ∈M, since 0 ∈Nc and a ∈M∗. Consider x ∈M. This shows that x = m + m1 for some m ∈M∗ and m1∈Nc. Now, m ∈M∗ ⟹m ∈N, because M∗⊆N. Thus m ∉Nc, since N is a k-p-system. Therefore, m + k ∉Nc for any k ∈Nc i.e. m + k ∈N for any k ∈Nc. As a result, m + m1∈N. This concludes that M ⊆N. Now we prove that M is an m-system. Consider that x1, y1∈M. At first suppose that x1 = a and y1= a. Then y = ai + m2 for some ai∈M∗ and m2∈Nc. Choose s∗= $(\prod_{k=1}^{i-1} s_k\alpha a_k)$βsi. Then x1αs∗βy1= $a1\alpha(s_1\alpha_1 a_1\alpha_2 s_2\ \ \alpha_3\ \ldots\ldots\ \alpha_{i-1}s_{i-1}\alpha_i a_i)$βsi γ(ai+m2) = aiαsiβ(ai+m2) = ai+1+aiαsiβm2∈M because ai+1∈M∗ and aiαsiβm2∈Nc, Nc being a k-semi-prime ideal of R. If x1≠a and y1=a, we can deal in a similar way. Let x1≠a and y1≠a. Consider x1 = ai+ m1 and y1 = aj+ m2 for some ai, aj ∈M∗ and m1, m2∈Nc. If i > j, choose s∗ = $\mathrm{si}\alpha(\prod_{k=1}^{i-j} a_{i-k}\ \beta s_{i-k})$. That being the case, x1αs∗βy1 = (ai+m1)αsi $\beta(a_{i-1}\alpha_1 s_{i-2}\alpha_2 a_{i-2}\alpha_3\ \ldots\ldots\alpha_{j-1}a_j\alpha_j s_j)$(aj+ m2) = ai+1 + $ai\alpha_1$s∗β1m2 + m1α2s∗ β2 aj+m1α3s∗ β3m2∈M because aj+1∈M∗ and $ai\alpha_1$s∗β1m2 + m1α2s∗ β2 aj+m1α3s∗ β3m2∈Nc, Nc being a k-semi-prime ideal of R. If j > i, choose s∗= $(\prod_{k=0}^{j-i-1} s_{k+i}\alpha a_{k+i})$βsj. In this case, x1αs∗βy1 = (ai+ m1)$(a_i\alpha_i s_{i+1}\beta_i a_{i+1}\alpha_{i+1}\ \ldots\ldots\ldots\ \alpha_{j-2}s_{j-1}\beta_{j-1}a_{j-1})\beta_j$sj α_j(aj+m2)=aj+1+$ai\alpha_1 s*\beta_1$m2+m1α2s*β2m2∈M as aj+1∈M∗ and $ai\alpha_1 s*\beta_1$m2+m1α2s*β2m2∈Nc, since Nc is a k-semi-prime ideal of T. Accordingly, M is an m-system. Consider t ∈M. If possible, let t ∈Mc. Then t + n ∈M c for some n ∈M c. Suppose t = m∗+ n∗ for some m∗∈M∗ and n∗∈Nc. This shows that m∗+ n∗+ n ∈Mc, where m∗∈M∗, n∗∈Nc and n ∈Mc. Now M∗⊆M∗ + Nc= M implies that Mc⊆(M∗)c. Since N is a k-p-system and m∗∈M∗⊆N, it follows that m∗∉Nc which contradicts the fact that t + n ∈Mc⊆(M∗)c. Consequently, t ∉Mc. Hence, Mc is a k-m-system.

Theorem 4.17: For any ideal I of a TGSR T, the following statements are equivalent:
(i) I is a k-semi-prime ideal of T.
(ii) I is the intersection of k-prime ideals of T.
(iii) I = β(I).

Proof: (iii) ⇒ (ii) It follows from Theorem 4.15.
(ii) ⇒(i) It can be proved from the definitions of k-prime ideals and k-semi-prime ideals only.
(i) ⇒(iii) From definition of β(I), it is clear that I ⊆β(I). Consider that x∉ I. Then x ∈Ic. Since I is a k-semi-prime ideal, Ic is a k-p-system, by Theorem 4.7. Accordingly, there exists a k-m-system M such that x ∈M and M ⊆Ic, by Lemma 4.16. Thus x ∈M and M ∩ I = ∅. It follows that x ∉ β(I). Consequently, β(I) ⊆I. Hence, I = β(I). We conclude this section by stating the correspondence of semi-prime ideals of a semi-group Rand k-semi-prime ideals of associated power ternary gamma semi ring P(T). Proof of this theorem is similar to the proof of Theorem 3.22.

Theorem 4.18: Let T be a TGSG. Then a proper k-ideal J of P(T) is a k semi-prime ideal of P(R) ⟺ J = P(I) for some semi-prime ideal I of T.

Conclusion

In this paper, we have produced a significant study of k-prime and k-semi-prime ideals of a ternary gamma semi ring. There is enough scope for the researchers to work with the very much new notion of k-m-system, established in this paper. Power ternary gamma semi ring is another broad area in which researchers may be interested. It is a very much fascinating research topic to study the correspondence between various classes of semi-groups and the analogous classes of the power ternary gamma semi ring.

References

M. K. Dubey, Prime and weakly prime ideals in Ternary gamma semi rings, Quasi groups Related Systems20 (2012) 197–202.

S. Ghosh, Fuzzy k-ideals of Ternary gamma semi rings, Fuzzy Sets Systems 95 (1998) 103–108.

J. S. Golan, Ternary gamma semi rings and Their Applications (Kluwer Academic Publishers, Netherlands, Springer, 1999).

J. S. Golan, Ternary gamma semi rings and Affine Equations Over Them: Theory and Applications (Kluwer Academic Publishers, Netherlands, Springer, 2003).

V. Gupta and J. N. Chaudhari, Prime ideals in Ternary gamma semi rings, Bull. Malays. Math. Sci.Soc. 34 (2011) 417–421.

S. Kar and S. Purkait, Characterization of some k-regularities of Ternary gamma semi Rings in termsof fuzzy ideals of Ternary gamma semi rings, J. Intell. Fuzzy Syst. 27 (2014) 3089–3101.

P. Lescot, Prime and primary ideals in Ternary gamma semi rings, Osaka J. Math. 52 (2015) 721–736.

M. K. Sen and A. K. Bhuniya, On Ternary gamma semi rings whose additive reduct is a semilattice, Semigroup Forum 82 (2011) 131–140.

G. Srinivasa Rao, D. Madhusudhanarao and P. Siva Prasad, Simple Ternary Semi-rings, The Global Journal of Mathematics & Mathematical Sciences, 9(2) (2016), 185-196.

D. Madhusudhana Rao, G. Srinivasa Rao, Special Elements in ternary semi rings, International Journal of Engineering Research and Applications, 4(11) (2014), 123-130.

G. Srinivasa Rao, D. Madhusudhana Rao, Structure of certain ideals in ternary semi rings, Int. J. of Innovative Science and Modern Engg., 3(3) (2015), 49-56.

G. Srinivasa Rao, D. Madhusudhana Rao, A Study on Ternary Semi rings, Int. J. of Math. Archive, 5(12) (2014), 24-30.

G. Srinivasa Rao, D. Madhusudhana Rao, Characteristics of Ternary Semi rings, Int.J. of Engg. Res. and Mgt., 2(1) (2015), 3-6.

G. Srinivasa Rao, A. Nagamalleswara Rao, P.L.N. Varma, D.Madhusudhana Rao, Ch. Ramprasad, Prime Bi-interior ideals in TGSR, Malaya Journal of Mathematika, Vol.9, No.1, pp:542-546, 2021.

G. Srinivasa Rao, A. Nagamalleswara Rao, P.L.N. Varma, D. Madhusudhana Rao, Ch. Ramprasad, Bi-interior ideals in TGSR, Advances in Mathematics Scientific Journal, 10 (2021), No.3, pp: 1183-1195