Product Of M-Bipolar Fuzzygraphs and Their Degree of Vertices

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Abstract

Fuzzy graph theory concepts are applied in different fields such as image capturing, image segmentation, networking, data mining, planning (landscape connectivity, air lines connectivity etc.), scheduling, clustering, artificial intelligence, decision making, multi-agent systems and auto meta theory. In this paper, we initiate the concept of m-bipolar fuzzy graphs (m-BPFGs). Properties of Cartesian product, composition, direct product, semi strong product, and strong product of two m-BPFGs have been studied. In addition, vertices degrees of the resultant graphs, which are attained by two given m-BPFGs G_1 and G_2 using the operations Cartesian product, composition, direct product, semi strong product, and strong product are calculated along with some basic theorems and examples.

Key Words: m-BPFG, Cartesian product, composition, direct product, semi strong product and strong product

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1. Introduction

FSs have been used in different domains in order to resolve issues associated with doubt and uncertainty in day to day applications of living conditions as shown by Zadeh [16] in 1965. The constraints in earlier model can be overcomes with the introduction of BFS idea in 1994 by Zhang [17]. Some concepts were later revamped by Chen et al. [4] into m-PF set theory.

It is well known that, a "graph" is a collection of points (known as vertices) and the lines between those points (known as edges) which can be used to characterize a physical situation comprising discrete objects with a relationship. In a view of its simplicity, the graph theory has various applications like analysing data, image segmentation, networking, clustering, planning, communication etc. But, in some cases, these graphs are unable to accurately represent several practical phenomena due to the ambiguity of different attributes and vagueness of the systems. This has led to define the FGs to over-come efficiently most of the real-world problems. In "FGs (Fuzzy graphs)", assigned values of vertices and edges removes uncertainty in the physical problems.

Rosenfeld [13] paved the path for the idea of fuzzy vertex, edges, path, subgraph and also complement of a FG. The works of Akram [1, 2] played a essential role in studying various major properties of BFGs, interval-valued FGs. Samanta and Pal [15] extended this FG theory technique to fuzzy planar graphs in order to other complex problems related to image segmentation using kernel contraction method. Ghorai and Pal [5–9] introduced the technique of generalized m-PFGs, planar graphs.

Talebi et al. [14] initiated edge regularity in m-polar interval-intuitionistic FGs. Further, Bera and Pal [3] studied about the statistical expressions like irregularity, regularity and density on these graphs. Mahapatra et al. [10, 11] investigated m-PF threshold graphs as well as their application on RPCS- resource power controlling system and interval-valued m-PF planar graphs. Ramakrishna et al.[12] gave the mark on the concept of m-BPFG, edge regularity on m-BPFGs with suitable examples. m-BPFGs and related definitions can refer in [12].

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This paper attempts to develop theory to analyze the parameters combining concepts from BFGs and m-PFGs as unique like. The resultant graph turned m-BPFG and some properties have been studied.

2. Preliminaries

Each and every vertex and edge of an m-PFG contains m components and these components are permanent. However these components possibly will be bipolar. With this concept, m-BPFG has been developed. Prior to defining m-BPFG, we use the subsequent equivalence relation while defining this concept.

For a set V, define an equivalence relation $\leftrightarrow$ on $V\times V-\{(q,q):q\in V\}$ as follows: $(q_1,r_1)\leftrightarrow(q_2,r_2)\Leftrightarrow$ either $(q_1,r_1)=(q_2,r_2)$ or $q_1=r_2,\ r_1=q_2$.

Definition 2.1. An m-BPFG of a graph $G^*=(V,E)$ is a pair of $G=(V,Q,R)$ where $Q=\left\langle\left[P_h\,o\Psi_Q^+,P_h\,o\Psi_Q^-\right]_{h=1}^m\right\rangle$, $P_h\,o\Psi_Q^+:\mathrm{V}\to[0,1]$ and $P_h\,o\Psi_Q^-:\mathrm{V}\to[-1,0]$ is an m-BPFS on V and $R=\left\langle\left[P_h\,o\Psi_R^+,P_h\,o\Psi_R^-\right]_{h=1}^m\right\rangle$, $P_h\,o\Psi_R^+:\overset{\text{sur}}{\mathrm{V}}{}^2\to[0,1]$ and $P_h\,o\Psi_R^-:\overset{\text{su}}{\mathrm{V}}{}^2\to[-1,0]$ is an m-BPFS in $\overset{\text{su}}{\mathrm{V}}{}^2$ such that $P_h\,o\Psi_R^+(qr)\le\min\{P_h\,o\Psi_Q^+(q),P_h\,o\Psi_Q^+(r)\}$, $P_h\,o\Psi_R^-(qr)\ge\max\{P_h\,o\Psi_Q^-(q),P_h\,o\Psi_Q^-(r)\}$ for all $qr\in\overset{\text{su}}{\mathrm{V}}{}^2, h=1\ to\ m$ and $P_h\,o\Psi_R^+(qr)=P_h\,o\Psi_R^-(qr)=0$ for all $qr\in\overset{\text{su}}{\mathrm{V}}{}^2-E$.

Definition 2.2. The degree of a vertex $q\in V$ in an m-BPFG $G=(V,Q,R)$ is defined as

$$d_G(q)=\left\langle\left[P_h\,od_G^+(q),P_h\,od_G^-(q)\right]_{h=1}^m\right\rangle=\left\langle\left[\sum_{\substack{q\neq r\\ qr\in E}}P_h\,o\Psi_R^+(qr),\sum_{\substack{q\neq r\\ qr\in E}}P_h\,o\Psi_R^-(qr)\right]_{h=1}^m\right\rangle.$$

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3. Degree of a vertex in Cartesian Product

Definition3.1. The cartesian product $G_1\times G_2$ of two m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of the graphs G_1^* and G_2^* respectively is defined as a triplet $(V_1\times V_2,\ Q_1\times Q_2,\ R_1\times R_2)$ such that for $h=1\ to\ m$

$(i)\,P_h o\Psi^+_{(Q_1\times Q_2)}(q_1,q_2)=\min\{P_h\,o\Psi^+_{Q_1}(q_1),P_h\,o\Psi^+_{Q_2}(q_2)\}$

$P_h o\Psi^-_{(Q_1\times Q_2)}(q_1,q_2)=\max\{P_h\,o\Psi^-_{Q_1}(q_1),P_h\,o\Psi^-_{Q_2}(q_2)\}$ for all $(q_1,q_2)\in V_1\times V_2$,

$(ii)\,P_h\,o\Psi^+_{(R_1\times R_2)}((q,q_2)(q,r_2))=\min\{P_h\,o\Psi^+_{Q_1}(q),P_h\,o\Psi^+_{R_2}(q_2r_2)\}$

$P_h\,o\Psi^-_{(R_1\times R_2)}((q,q_2)(q,r_2))=\max\{P_h\,o\Psi^-_{Q_1}(q),P_h\,o\Psi^-_{R_2}(q_2r_2)\}$ for all $q\in V_1,\ q_2r_2\in E_2$,

$(iii)\,P_h\,o\Psi^+_{(R_1\times R_2)}((q_1,s)(r_1,s))=\min\{P_h\,o\Psi^+_{R_1}(q_1r_1),P_h\,o\Psi^+_{Q_2}(s)\}$

$P_h\,o\Psi^-_{(R_1\times R_2)}((q_1,s)(r_1,s))=\max\{P_h\,o\Psi^-_{R_1}(q_1r_1),P_h\,o\Psi^-_{Q_2}(s)\}$ for all $s\in V_2, q_1r_1\in E_1$,

$(iv)\,P_h\,o\Psi^+_{(R_1\times R_2)}((q_1,q_2)(r_1,r_2))=0$

$P_h\,o\Psi^-_{(R_1\times R_2)}((q_1,q_2)(r_1,r_2))=0$ for all $(q_1,q_2)(r_1,r_2)\in\overset{\text{suuuuuu}}{V_1\times V_2^2}-E.$

Definition 3.2. For any vertex $(q_1,q_2)\in V_1\times V_2$, the degree in Cartesian product is denoted by

$d_{(G_1\times G_2)}(q_1,q_2)=\left\langle\left[P_h\circ d^{+}_{(G_1\times G_2)}(q_1,q_2),P_h\circ d^{-}_{(G_1\times G_2)}(q_1,q_2)\right]_{h=1}^{m}\right\rangle$ and is defined by for $h=1\ to\ m$

$$P_h\circ d^{+}_{(G_1\times G_2)}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h\circ\Psi^{+}_{(R_1\times R_2)}\big((q_1,q_2)(r_1,r_2)\big)$$
$$= \sum_{q_1=r_1,\,q_2r_2\in E_2} P_h\circ\Psi^{+}_{Q_1}(q_1)\wedge P_h\circ\Psi^{+}_{R_2}(q_2r_2)+\sum_{q_2=r_2,\,q_1r_1\in E_1} P_h\circ\Psi^{+}_{Q_2}(q_2)\wedge P_h\circ\Psi^{+}_{R_1}(q_1r_1),$$
$$P_h\circ d^{-}_{(G_1\times G_2)}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h\circ\Psi^{-}_{(R_1\times R_2)}\big((q_1,q_2)(r_1,r_2)\big)$$
$$= \sum_{q_1=r_1,\,q_2r_2\in E_2} P_h\circ\Psi^{-}_{Q_1}(q_1)\vee P_h\circ\Psi^{-}_{R_2}(q_2r_2)+\sum_{q_2=r_2,\,q_1r_1\in E_1} P_h\circ\Psi^{-}_{Q_2}(q_2)\vee P_h\circ\Psi^{-}_{R_1}(q_1r_1)$$

Theorem 3.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs. If $R_2\subseteq Q_1$ and $R_1\subseteq Q_2$, then $d_{(G_1\times G_2)}(q_1,q_2)=d_{G_1}(q_1)+d_{G_2}(q_2)$ for all $(q_1,q_2)\in V_1\times V_2$.

Proof. For each $h=1,2,3,\dots,m$ we have,

$$P_h\circ d^{+}_{(G_1\times G_2)}(q_1,q_2) = \sum_{q_1=r_1,\,q_2r_2\in E_2} P_h\circ\Psi^{+}_{Q_1}(q_1)\wedge P_h\circ\Psi^{+}_{R_2}(q_2r_2)$$
$$+\sum_{q_2=r_2,\,q_1r_1\in E_1} P_h\circ\Psi^{+}_{Q_2}(q_2)\wedge P_h\circ\Psi^{+}_{R_1}(q_1r_1)$$
$$=\sum_{q_2r_2\in E_2} P_h\circ\Psi^{+}_{R_2}(q_2r_2)+\sum_{q_1r_1\in E_1} P_h\circ\Psi^{+}_{R_1}(q_1r_1)$$
$$=P_h\circ d^{+}_{G_1}(q_1)+P_h\circ d^{+}_{G_2}(q_2).$$

Similarly, $P_h\circ d^{-}_{(G_1\times G_2)}(q_1,q_2)=P_h\circ d^{-}_{G_1}(q_1)+P_h\circ d^{-}_{G_2}(q_2)$.

So $d_{(G_1\times G_2)}(q_1,q_2)=d_{G_1}(q_1)+d_{G_2}(q_2)$.

Theorem 3.2. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be two m-BPFGs such that $Q_1\subseteq R_{2,}$ then $R_1\subseteq Q_2$ and conversely.

Proof. From the definition of m-BPFGs, we get $P_h\circ\Psi^{+}_{R_k}(qr)\le\min\left\{P_h\circ\Psi^{+}_{Q_k}(q),P_h\circ\Psi^{+}_{Q_k}(r)\right\}$ for all $qr\in\tilde{V}^2$, $h=1,2,3\dots,m$ and $k=1,2$.

Therefore, $P_h\circ\Psi^{+}_{R_k}\le\max\left\{P_h\circ\Psi^{+}_{Q_k}\right\}$ and min $P_h\circ\Psi^{+}_{R_k}(qr)\le\min\left\{P_h\circ\Psi^{+}_{Q_k}(q)\right\}$ for $h=1,2,3,\dots,m$ and $k=1,2$.

Also, since $Q_1\subseteq R_{2,}$ $\max\left\{P_h\circ\Psi^{+}_{Q_1}\right\}\le\min\left\{P_h\circ\Psi^{+}_{R_2}\right\}$ for $h=1,2,3,\dots,m$

Hence, $P_h\circ\Psi^{+}_{R_1}\le\max\left\{P_h\circ\Psi^{+}_{Q_1}\right\}\le\min\left\{P_h\circ\Psi^{+}_{R_2}\right\}\le P_h\circ\Psi^{+}_{Q_2}$.

Similarly, $P_h\circ\Psi^{-}_{R_1}\ge P_h\circ\Psi^{-}_{Q_2}$ for $h=1,2,\dots,m$ i.e., $R_1\subseteq Q_2$.
In a similar way, the converse part can be proved.

Example 3.1. Let us consider the 2-BPFGs G_1,G_2 and their cartesian product $G_1\times G_2$ (see Figure1). For this graph $R_2\subseteq Q_1$ and $R_1\subseteq Q_2$. So by Theorem 3.1, $d_{(G_1\times G_2)}(q_1,q_2)=\langle[0.7,-0.5],[0.5,-0.3]\rangle$,

$d_{G_1}(q_1)+d_{G_2}(q_2)=\langle[0.7,-0.5],[0.5,-0.3]\rangle.$ In the same way, we have to find the degrees of all other vertices in $G_1 \times G_2$. This can be confirmed from Figure 1.

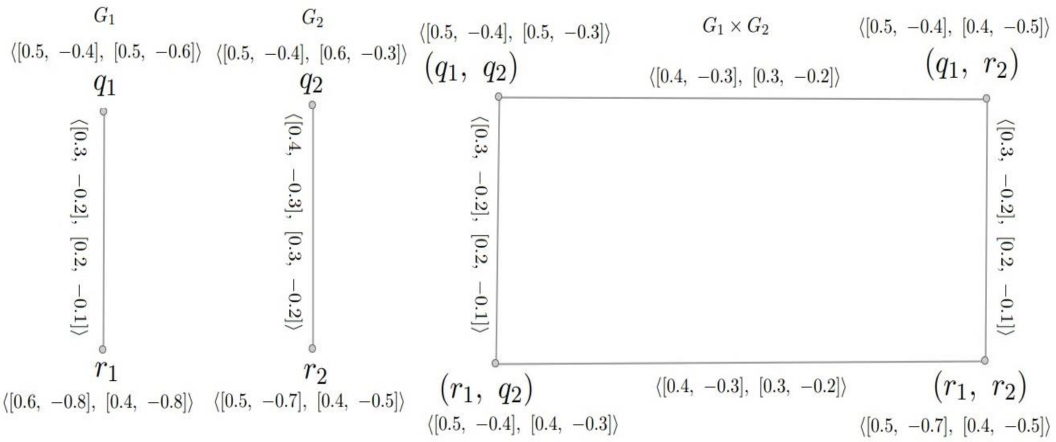

Figure 1 Cartesian product of $(G_1 \times G_2)$ of two m-BPFGs G_1 and G_2

Theorem 3.3. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be two m-BPFGs

If $Q_1 \subseteq R_2$ and Q_1 is constant with $Q_1(q)=\left\langle\left[P_h \mathrm{o}\Psi_{Q_1}^+(q), P_h \mathrm{o}\Psi_{Q_1}^-(q)\right]_{h=1}^m\right\rangle=\left\langle\left[c_h^+,c_h^-\right]_{h=1}^m\right\rangle=c$ for all $q \in V_1$, then $d_{(G_1 \times G_2)}(q_1,q_2)=d_{G_1}(q_1)+cd_{G_2^*}(q_2).$

If $Q_2 \subseteq R_1$ and Q_2 is constant with $Q_2(q)=\left\langle\left[P_h \mathrm{o}\Psi_{Q_2}^+(q), P_h \mathrm{o}\Psi_{Q2}^-(q)\right]_{h=1}^m\right\rangle=\left\langle\left[k_h^+,k_h^-\right]_{h=1}^m\right\rangle=k$ for all $q \in V_2$, then $d_{(G_1 \times G_2)}(q_1,q_2)=d_{G_2}(q_2)+kd_{G_1^*}(q_1).$

Proof. (i) Because $Q_1 \subseteq R_2$, by theorem 3.2 $R_1 \subseteq Q_2$. Then for $h=1,2,\ldots,m$

$$P_h \mathrm{o}d_{(G_1 \times G_2)}^+(q_1,q_2) = \sum_{q_1=r_1,\, q_2r_2 \in E_2} P_h \mathrm{o}\Psi_{Q_1}^+(q_1) \wedge P_h \mathrm{o}\Psi_{R_2}^+(q_2r_2) + \sum_{q_2=r_2,\, q_1r_1 \in E_1} P_h \mathrm{o}\Psi_{Q_2}^+(q_2) \wedge P_h \mathrm{o}\Psi_{R_1}^+(q_1r_1)$$

$$= \sum_{q_2r_2 \in E_2} P_h \mathrm{o}\Psi_{Q_1}^+(q_1) + \sum_{q_1r_1 \in E_1} P_h \mathrm{o}\Psi_{R_1}^+(q_1r_1)$$

$$= \sum_{q_2r_2 \in E_2} c_h^+ + P_h \mathrm{o}\Psi_{G_1}^+(q_1) = c_h^+ d_{G_2^*}(q_2) + P_h \mathrm{o}d_{G_1}^+(q_1).$$

Similarly, $P_h \mathrm{o}d_{(G_1 \times G_2)}^-(q_1,q_2)=c_h^- d_{G_2^*}(q_2)+P_h \mathrm{o}d_{G_1}^-(q_1).$

Hence, $d_{(G_1 \times G_2)}(q_1,q_2)=d_{G_1}(q_1)+cd_{G_2^*}(q_2).$

(ii) Similar to the above case.

4. Degree of A Vertex in Composition

Definition 4.1. The composition $G_1[G_2]$ of two m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of the graphs G_1^* and G_2^* respectively is defined as a triplet $(V_1 \times V_2, Q_1 \mathrm{o} Q_2, R_1 \mathrm{o} R_2)$ such that for $h=1,2,\ldots,m$

$(i)\, P_h \circ \Psi^{+}_{(Q_1 \circ Q_2)}(q_1,q_2)=\min\{P_h \circ \Psi^{+}_{Q_1}(q_1), P_h \circ \Psi^{+}_{Q_2}(q_2)\}$

$P_h \circ \Psi^{-}_{(Q_1 \circ Q_2)}(q_1,q_2)=\max\{P_h \circ \Psi^{-}_{Q_1}(q_1), P_h \circ \Psi^{-}_{Q_2}(q_2)\}$ for all $(q_1,q_2)\in V_1 \times V_2$,

$(ii)\, P_h \circ \Psi^{+}_{(R_1 \circ R_2)}((q,q_2)(q,r_2))=\min\{P_h \circ \Psi^{+}_{Q_1}(q), P_h \circ \Psi^{+}_{R_2}(q_2 r_2)\}$

$P_h \circ \Psi^{-}_{(R_1 \circ R_2)}((q,q_2)(q,r_2))=\max\{P_h \circ \Psi^{-}_{Q_1}(q), P_h \circ \Psi^{-}_{R_2}(q_2 r_2)\}$ for all $q\in V_1,\ q_2 r_2 \in E_2$,

$(iii)\, P_h \circ \Psi^{+}_{(R_1 \circ R_2)}((q_1,s)(r_1,s))=\min\{P_h \circ \Psi^{+}_{R_1}(q_1 r_1), P_h \circ \Psi^{+}_{Q_2}(s)\}$

$P_h \circ \Psi^{-}_{(R_1 \circ R_2)}((q_1,s)(r_1,s))=\max\{P_h \circ \Psi^{-}_{R_1}(q_1 r_1), P_h \circ \Psi^{-}_{Q_2}(s)\}$ for all $s\in V_2,\ q_1 r_1 \in E_1$,

$(iv)\, P_h \circ \Psi^{+}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))=\min\{P_h \circ \Psi^{+}_{Q_2}(q_2), P_h \circ \Psi^{+}_{Q_2}(r_2), P_h \circ \Psi^{+}_{R_1}(q_1 r_1)\}$

$P_h \circ \Psi^{-}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))=\max\{P_h \circ \Psi^{-}_{Q_2}(q_2), P_h \circ \Psi^{-}_{Q_2}(r_2), P_h \circ \Psi^{-}_{R_1}(q_1 r_1)\}$, for all $((q_1,q_2)(r_1,r_2))\in E^0 - E$, where $E=\{(q,q_2)(q,r_2): q\in V_1, q_2 r_2 \in E_2\}\cup\{(q_1,s)(r_1,s): s\in V_2, q_1 r_1 \in E_1\}$ and $E^0 = E \cup \{(q_1,q_2)(r_1,r_2): q_1 r_1 \in E_1, q_2 \neq r_2\}$,

$(v)\, P_h \circ \Psi^{+}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))=0,\ P_h \circ \Psi^{-}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))=0$ for all $(q_1,q_2)(r_1,r_2)\in V_1 \times V_2^2 - E^0$.

Definition 4.2. For any vertex $(q_1,q_2)\in V_1 \times V_2$, the degree in composition is denoted by $d_{(G_1[G_2])}(q_1,q_2)=\left\langle \left[P_h \circ d^{+}_{(G_1[G_2])}(q_1,q_2), P_h \circ d^{+}_{(G_1[G_2])}(q_1,q_2) \right]_{h=1}^{m} \right\rangle$ and is defined by for $h=1\ to\ m$

$$P_h \circ d^{+}_{(G_1[G_2])}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h \circ \Psi^{+}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))$$

$$= \sum_{q_1=r_1, q_2 r_2 \in E_2} P_h \circ \Psi^{+}_{Q_1}(q_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2 r_2) + \sum_{q_2=r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{Q_2}(q_2) \wedge P_h \circ \Psi^{+}_{R_1}(q_1 r_1)$$

$$+ \sum_{q_2\neq r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{Q_2}(q_2) \wedge P_h \circ \Psi^{+}_{Q_2}(r_2) \wedge P_h \circ \Psi^{+}_{R_1}(q_1 r_1),$$

$$P_h \circ d^{-}_{(G_1[G_2])}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h \circ \Psi^{-}_{(R_1 \circ R_2)}((q_1,q_2)(r_1,r_2))$$

$$= \sum_{q_1=r_1, q_2 r_2 \in E_2} P_h \circ \Psi^{-}_{Q_1}(q_1) \vee P_h \circ \Psi^{-}_{R_2}(q_2 r_2) + \sum_{q_2=r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{-}_{Q_2}(q_2) \vee P_h \circ \Psi^{-}_{R_1}(q_1 r_1)$$

$$+ \sum_{q_2\neq r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{-}_{Q_2}(q_2) \vee P_h \circ \Psi^{-}_{Q_2}(r_2) \vee P_h \circ \Psi^{-}_{R_1}(q_1 r_1).$$

Theorem 4.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be two m-BPFGs. If $R_2 \subseteq Q_1$ and $R_1 \subseteq Q_2$ then $d_{(G_1[G_2])}(q_1,q_2) = |V_2| d_{G_1}(q_1) + d_{G_2}(q_2)$ for all $(q_1,q_2)\in V_1 \times V_2$.

Proof. For each $h=1\ to\ m$, we have

$$P_h \circ d^{+}_{(G_1[G_2])}(q_1,q_2) = \sum_{q_1=r_1, q_2 r_2 \in E_2} P_h \circ \Psi^{+}_{Q_1}(q_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2 r_2) + \sum_{q_2=r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{Q_2}(q_2) \wedge P_h \circ \Psi^{+}_{R_1}(q_1 r_1)$$

$$+ \sum_{q_2\neq r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{Q_2}(q_2) \wedge P_h \circ \Psi^{+}_{Q_2}(r_2) \wedge P_h \circ \Psi^{+}_{R_1}(q_1 r_1)$$

$$= \sum_{q_2 r_2 \in E_2} P_h \circ \Psi^{+}_{R_2}(q_2 r_2) + \sum_{q_2=r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{R_1}(q_1 r_1) + \sum_{q_2\neq r_2, q_1 r_1 \in E_1} P_h \circ \Psi^{+}_{R_1}(q_1 r_1)$$

(Since $P_h \, \mathrm{o}\Psi^{+}_{Q_1} \geq P_h \, \mathrm{o}\Psi^{+}_{R_2}$ and $P_h \, \mathrm{o}\Psi^{+}_{Q_2} \geq P_h \, \mathrm{o}\Psi^{+}_{R_1}$)

$=|V_2| P_h \, \mathrm{o}d^{+}_{G_1}(q_1) + P_h \, \mathrm{o}d^{+}_{G_2}(q_2).$

Similarly, $P_h \, \mathrm{o}d^{-}_{(G_1[G_2])}(q_1, q_2) = |V_2| P_h \, \mathrm{o}d^{-}_{G_1}(q_1) + P_h \, \mathrm{o}d^{-}_{G_2}(q_2)$

So $d_{(G_1[G_2])}(q_1, q_2) = |V_2| d_{G_1}(q_1) + d_{G_2}(q_2)$

Theorem 4.2. Let $G_1 = (V_1, Q_1, R_1)$ and $G_2 = (V_2, Q_2, R_2)$ be 2 m-BPFGs.

If $Q_1 \subseteq R_2$, and Q_1 is constant with $Q_1(q) = \left\langle \left[P_h \, \mathrm{o}\Psi^{+}_{Q_1}(q), P_h \, \mathrm{o}\Psi^{-}_{Q_1}(q) \right]_{h=1}^{m} \right\rangle = \left\langle \left[c_h^{+}, c_h^{-} \right]_{h=1}^{m} \right\rangle = c$ for all $q \in V_1$, then $d_{(G_1[G_2])}(q_1, q_2) = |V_2| d_{G_1}(q_1) + c d_{G_2^{*}}(q_2).$

If $R_2 \subseteq Q_1$, and Q_2 is constant with $Q_2(q) = \left\langle \left[P_h \, \mathrm{o}\Psi^{+}_{Q_2}(q), P_h \, \mathrm{o}\Psi^{-}_{Q2}(q) \right]_{h=1}^{m} \right\rangle = \left\langle \left[k_h^{+}, k_h^{-} \right]_{h=1}^{m} \right\rangle = k$ for all $q \in V_2$, then $d_{(G_1[G_2])}(q_1, q_2) = d_{G_2}(q_2) + k|V_2| d_{G_1^{*}}(q_1).$

Proof. (i) Because $Q_1 \subseteq R_2$, by Theorem 3.2 $R_1 \subseteq Q_2$. Now for $h = 1, 2, \ldots, m$

$$P_h \, \mathrm{o}d^{+}_{(G_1[G_2])}(q_1, q_2) = \sum_{q_1 = r_1, q_2 r_2 \in E_2} P_h \, \mathrm{o}\Psi^{+}_{Q_1}(q_1) \wedge P_h \, \mathrm{o}\Psi^{+}_{R_2}(q_2 r_2)$$

$$+ \sum_{q_2 = r_2, q_1 r_1 \in E_1} P_h \, \mathrm{o}\Psi^{+}_{Q_2}(q_2) \wedge P_h \, \mathrm{o}\Psi^{+}_{R_1}(q_1 r_1) + \sum_{q_2 \neq r_2, q_1 r_1 \in E_1} P_h \, \mathrm{o}\Psi^{+}_{Q_2}(q_2) \wedge P_h \, \mathrm{o}\Psi^{+}_{Q_2}(r_2) \wedge P_h \, \mathrm{o}\Psi^{+}_{R_1}(q_1 r_1)$$

$$= \sum_{q_2 r_2 \in E_2} P_h \, \mathrm{o}\Psi^{+}_{Q_1}(q_1) + \sum_{q_2 = r_2, q_1 r_1 \in E_1} P_h \, \mathrm{o}\Psi^{+}_{R_1}(q_1 r_1) + \sum_{q_2 \neq r_2, q_1 r_1 \in E_1} P_h \, \mathrm{o}\Psi^{+}_{R_1}(q_1 r_1)$$

$$= \sum_{q_2 r_2 \in E_2} c_h^{+} + |V_2| \sum_{q_1 r_1 \in E_1} P_h \, \mathrm{o}\Psi^{+}_{R_1}(q_1 r_1) = c_h^{+} d_{G_2^{*}}(q_2) + |V_2| P_h \, \mathrm{o}d^{+}_{G_1}(q_1)$$

Similarly, $P_h \, \mathrm{o}d^{-}_{(G_1[G_2])}(q_1, q_2) = c_h^{-} d_{G_2^{*}}(q_2) + |V_2| P_h \, \mathrm{o}d^{-}_{G_1}(q_1).$

Hence, $d_{(G_1[G_2])}(q_1, q_2) = |V_2| d_{G_1}(q_1) + c d_{G_2^{*}}(q_2).$

(ii) Similarly to the above case.

Example 4.1. Let us consider the 2-BPFGs G_1, G_2 and their composition $G_1[G_2]$ (see Figure 2). For this graph $R_2 \subseteq Q_1$ and $R_1 \subseteq Q_2$. So, by Theorem 4.1, $d_{(G_1[G_2])}(q_1, q_2) = \langle [0.7, -1.2], [1.6, -1.1] \rangle$, $d_{G_1}(q_1)|V_2| + d_{G_2}(q_2) = \langle [0.7, -1.2], [1.6, -1.1] \rangle$. Similarly, we can find the degrees of all other vertices in $G_1[G_2]$. This can be verified from Figure 2.

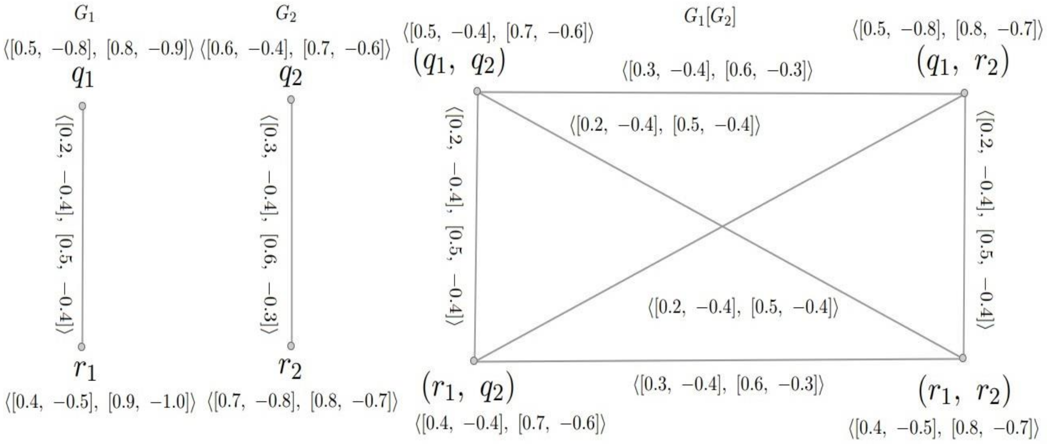

Figure 2: Composition of $(G_1[G_2])$ of2 m-BPFGs G_1 and G_2.

5. Vertex Degree in Direct Product

Definition 5.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs of the graphs G_1^* and G_2^* respectively such that $V_1\cap V_2=\phi$. The direct product of G_1 and G_2 is defined to be the m-BPFG $G_1\cap G_2=(Q_1\cap Q_2,R_1\cap R_2)$ of the graph $G^*=(V_1\times V_2,E)$ where $E=\{(q_1,r_1)(q_2,r_2)\,|\,q_1q_2\in E_1,r_1r_2\in E_2\}\subseteq \overset{\text{SUUUUUU}}{V_1\times V_2}{}^{2}$ and for each $h=1\ to\ m$

$(i)\,P_h\circ\Psi^+_{(Q_1\cap Q_2)}(q,r)=\min\{P_h\circ\Psi^+_{Q_1}(q),P_h\circ\Psi^+_{Q_2}(r)\}$

$P_h\circ\Psi^-_{(Q_1\cap Q_2)}(q,r)=\max\{P_h\circ\Psi^-_{Q_1}(q),P_h\circ\Psi^-_{Q_2}(r)\}$ for all $(q,r)\in V_1\times V_2$,

$(ii)\,P_h\circ\Psi^+_{(R_1\cap R_2)}((q_1,r_1)(q_2,r_2))=\min\{P_h\circ\Psi^+_{R_1}(q_1q_2),P_h\circ\Psi^+_{R_2}(r_1r_2)\}$

$P_h\circ\Psi^-_{(R_1\cap R_2)}((q_1,r_1)(q_2,r_2))=\max\{P_h\circ\Psi^-_{R_1}(q_1q_2),P_h\circ\Psi^-_{R_2}(r_1r_2)\}$ for all $q_1q_2\in E_1$, $r_1r_2\in E_2$,

$(iii)\,P_h\circ\Psi^+_{(R_1\cap R_2)}((w,x)(y,z))=0,\ P_h\circ\Psi^-_{(R_1\cap R_2)}((w,x)(y,z))=0$ for all $(w,x)(y,z)\in\overset{\text{SUUUUUU}}{V_1\times V_2}{}^{2}-E.$

Definition 5.2. For any vertex $(q_1,q_2)\in V_1\times V_2$, the degree is in direct product denoted by $d_{(G_1\cap G_2)}(q_1,q_2)=\left\langle\left[P_h\circ d^+_{(G_1\cap G_2)}(q_1,q_2),P_h\circ d^-_{(G_1\cap G_2)}(q_1,q_2)\right]_{h=1}^{m}\right\rangle$ and is defined by for $h=1\ to\ m$

$$P_h\circ d^+_{(G_1\cap G_2)}(q_1,q_2)=\sum_{(q_1,q_2)(r_1,r_2)\in E}P_h\circ\Psi^+_{(R_1\cap R_2)}((q_1,q_2)(r_1,r_2))$$

$$=\sum_{q_1r_1\in E_1,q_2r_2\in E_2}P_h\circ\Psi^+_{R_1}(q_1r_1)\wedge P_h\circ\Psi^+_{R_2}(q_2r_2),$$

$$P_h\circ d^-_{(G_1\cap G_2)}(q_1,q_2)=\sum_{(q_1,q_2)(r_1,r_2)\in E}P_h\circ\Psi^-_{(R_1\cap R_2)}((q_1,q_2)(r_1,r_2))$$

$$=\sum_{q_1r_1\in E_1,q_2r_2\in E_2}P_h\circ\Psi^-_{R_1}(q_1r_1)\vee P_h\circ\Psi^-_{R_2}(q_2r_2)$$

Theorem 5.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs of the graphs. If $R_1\subseteq R_2$, then

$d_{(G_1 \sqcap G_2)}(q_1,q_2)=d_{G_1}(q_1)$. Also, if $R_2 \subseteq R_1$, then $d_{(G_1 \sqcap G_2)}(q_1,q_2)=d_{G_2}(q_2)$ for all $(q_1,q_2) \in V_1 \times V_2$.

Proof. Let $R_1 \subseteq R_2$ i.e., $P_h \circ \Psi^+_{R_2} \geq P_h \circ \Psi^+_{R_1}$ for each $h=1 \ to \ m$. Then we have

$$P_h \circ d^+_{(G_1 \sqcap G_2)}(q_1,q_2) = \sum_{q_1r_1 \in E_1, q_2r_2 \in E_2} P_h \circ \Psi^+_{R_1}(q_1r_1) \wedge P_h \circ \Psi^+_{R_2}(q_2r_2)$$

$$= \sum_{q_1r_1 \in E_1} P_h \circ \Psi^+_{R_1}(q_1r_1) = P_h \circ d^+_{G_1}(q_1)$$

Similarly, $P_h \circ d^-_{(G_1 \sqcap G_2)}(q_1,q_2) = P_h \circ d^-_{G_1}(q_1)$

Hence, $d_{(G_1 \sqcap G_2)}(q_1,q_2)=d_{G_1}(q_1)$

Similarly, if $R_2 \subseteq R_1$, then $d_{(G_1 \sqcap G_2)}(q_1,q_2)=d_{G_2}(q_2)$

Example 5.1. Let us consider the 2-BPFGs G_1, G_2 and their direct product $G_1 \sqcap G_2$ (see Figure 3). For this graph $R_1 \subseteq R_2$ so by Theorem 5.1, $d_{(G_1 \sqcap G_2)}(q_1,q_2) = \langle [0.4,-0.3],[0.3,-0.2] \rangle$, $d_{G_1}(q_1) = \langle [0.4,-0.3],[0.3,-0.2] \rangle$. In the same way, we can find the degrees of all other vertices in $G_1 \sqcap G_2$ This can be confirmed from the Figure 3.

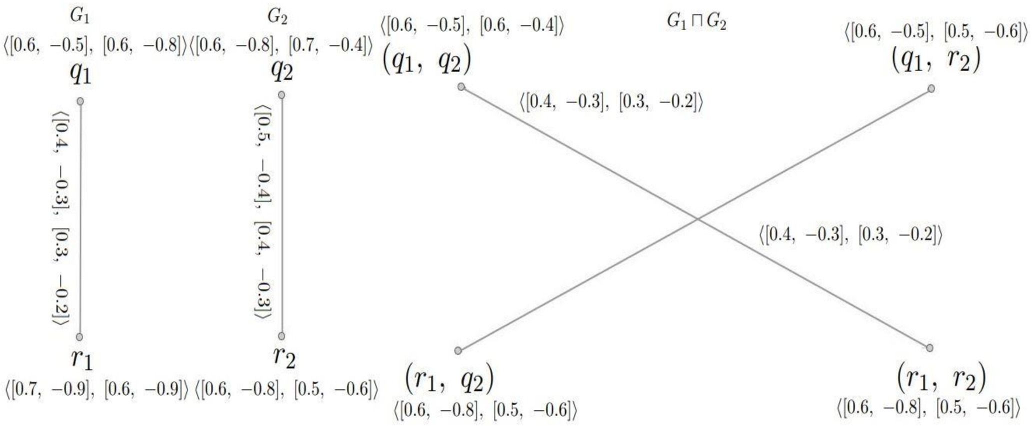

Figure.3 Direct Product of $(G_1 \sqcap G_2)$ of 2 m-BPFGs G_1 and G_2

6. Vertex Degree in Semi Strong Product

Definition 6.1. The semi strong product of 2 m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of G_1^* and G_2^* where it is suppose that $V_1 \cap V_2 = \phi$, is defined to be the m-BPFG $G_1 \bullet G_2 = (Q_1 \bullet Q_2, R_1 \bullet R_2)$ of $G^*=(V_1 \times V_2, E)$ where $E=\{(q,r_1)(q,r_2) \mid q \in V_1, r_1r_2 \in E_2\} \cup \{(q_1,r_1)(q_2,r_2) \mid q_1q_2 \in E_1, r_1r_2 \in E_2\} \subseteq V_1 \times V_2$ fulfilling the following : for each $h=1 \ to \ m$

$(i) P_h \circ \Psi^+_{(Q_1 \bullet Q_2)}(q,r) = \min\{P_h \circ \Psi^+_{Q_1}(q), P_h \circ \Psi^+_{Q_2}(r)\}$

$P_h \circ \Psi^-_{(Q_1 \bullet Q_2)}(q,r) = \max\{P_h \circ \Psi^-_{Q_1}(q), P_h \circ \Psi^-_{Q_2}(r)\}$ for all $(q,r) \in V_1 \times V_2$,

$(ii) P_h \circ \Psi^+_{(R_1 \bullet R_2)}((q,r_1)(q,r_2)) = \min\{P_h \circ \Psi^+_{Q_1}(q), P_h \circ \Psi^+_{R_2}(r_1r_2)\}$

$P_h \circ \Psi^-_{(R_1 \bullet R_2)}((q,r_1)(q,r_2)) = \max\{P_h \circ \Psi^-_{Q_1}(q), P_h \circ \Psi^-_{R_2}(r_1r_2)\}$ for all $q \in V_1$, $r_1r_2 \in E_2$,

$(iii)\, P_h \circ \Psi^{+}_{(R_1 g R_2)}\left((q_1,r_1)(q_2,r_2)\right) = \min\left\{P_h \circ \Psi^{+}_{R_1}(q_1q_2), P_h \circ \Psi^{+}_{R_2}(r_1r_2)\right\}$

$P_h \circ \Psi^{-}_{(R_1 g R_2)}\left((q_1,r_1)(q_2,r_2)\right) = \max\left\{P_h \circ \Psi^{-}_{R_1}(q_1q_2), P_h \circ \Psi^{-}_{R_2}(r_1r_2)\right\}$ for all $q_1q_2 \in E_1, r_1r_2 \in E_2$,

$(iv)\, P_h \circ \Psi^{+}_{(R_1 g R_2)}\left((w,l)(t,q)\right) = 0,\ P_h \circ \Psi^{-}_{(R_1 g R_2)}\left((w,l)(t,q)\right) = 0,$ for all $(w,l)(t,q) \in V_1 \times V_2^2 - E$.

Definition 6.2. For any vertex $(q_1,q_2) \in V_1 \times V_2$, the degree in semi strong product is denoted by $d_{(G_1 g G_2)}(q_1,q_2) = \left\langle \left[P_h \circ d^{+}_{(G_1 g G_2)}(q_1,q_2), P_h \circ d^{-}_{(G_1 g G_2)}(q_1,q_2)\right]_{h=1}^{m} \right\rangle$ and is defined by for $h = 1\ to\ m$

$$P_h \circ d^{+}_{(G_1 g G_2)}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h \circ \Psi^{+}_{(R_1 g R_2)}\left((q_1,q_2)(r_1,r_2)\right)$$

$$= \sum_{q_1=r_1,\, q_2r_2\in E_2} P_h \circ \Psi^{+}_{Q_1}(q_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2r_2) + \sum_{q_1r_1\in E_1,\, q_2r_2\in E_2} P_h \circ \Psi^{+}_{R_1}(q_1r_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2r_2)$$

$$P_h \circ d^{-}_{(G_1 g G_2)}(q_1,q_2) = \sum_{(q_1,q_2)(r_1,r_2)\in E} P_h \circ \Psi^{-}_{(R_1 g R_2)}\left((q_1,q_2)(r_1,r_2)\right)$$

$$= \sum_{q_1=r_1,\, q_2r_2\in E_2} P_h \circ \Psi^{-}_{Q_1}(q_1) \wedge P_h \circ \Psi^{-}_{R_2}(q_2r_2) + \sum_{q_1r_1\in E_1,\, q_2r_2\in E_2} P_h \circ \Psi^{-}_{R_1}(q_1r_1) \wedge P_h \circ \Psi^{-}_{R_2}(q_2r_2).$$

Theorem 6.1. Let $G_1 = (V_1, Q_1, R_1)$ and $G_2 = (V_2, Q_2, R_2)$ be 2 m-BPFGs. If $R_1 \subseteq R_2 \subseteq Q_1$, then $d_{(G_1 g G_2)}(q_1,q_2) = d_{G_1}(q_1) + d_{G_2}(q_2)$ for all $(q_1,q_2) \in V_1 \times V_2$.

Proof. Let $R_1 \subseteq R_2 \subseteq Q_1$ i.e., $P_h \circ \Psi^{+}_{Q_1} \geq P_h \circ \Psi^{+}_{R_2} \geq P_h \circ \Psi^{+}_{R_1}$ for each $h = 1, 2, \ldots, m$ and $(q_1,q_2) \in V_1 \times V_2$.

$$P_h \circ d^{+}_{(G_1 g G_2)}(q_1,q_2) = \sum_{q_1=r_1,\, q_2r_2\in E_2} P_h \circ \Psi^{+}_{Q_1}(q_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2r_2) + \sum_{q_1r_1\in E_1,\, q_2r_2\in E_2} P_h \circ \Psi^{+}_{R_1}(q_1r_1) \wedge P_h \circ \Psi^{+}_{R_2}(q_2r_2)$$

$$+ \sum_{q_2r_2\in E_2} P_h \circ \Psi^{+}_{R_2}(q_2r_2) + \sum_{q_1r_1\in E_1} P_h \circ \Psi^{+}_{R_1}(q_1r_1)$$

$$= P_h \circ d^{+}_{G_1}(q_1) + P_h \circ d^{+}_{G_2}(q_2)$$

Similarly, $P_h \circ d^{-}_{(G_1 g G_2)}(q_1,q_2) = P_h \circ d^{-}_{G_1}(q_1) + P_h \circ d^{-}_{G_2}(q_2)$

This shows that $d_{(G_1 g G_2)}(q_1,q_2) = d_{G_1}(q_1) + d_{G_2}(q_2)$ for all $(q_1,q_2) \in V_1 \times V_2$.

Example 6.1. Let us consider the 2-BPFGs G_1, G_2 and their semi strong product $G_1 g G_2$ (see Figure 4). For this graph $R_1 \subseteq R_2 \subseteq Q_1$ So by Theorem 6.1, $d_{(G_1 g G_2)}(q_1,q_2) = \langle [0.7, -0.4], [0.3, -0.2] \rangle$, $d_{G_1}(q_1) + d_{G_2}(q_2) = \langle [0.7, -0.4], [0.3, -0.2] \rangle$. In the same way, we have to find the degrees of all other vertices in $G_1 g G_2$. This can be confirmed from Figure 4.

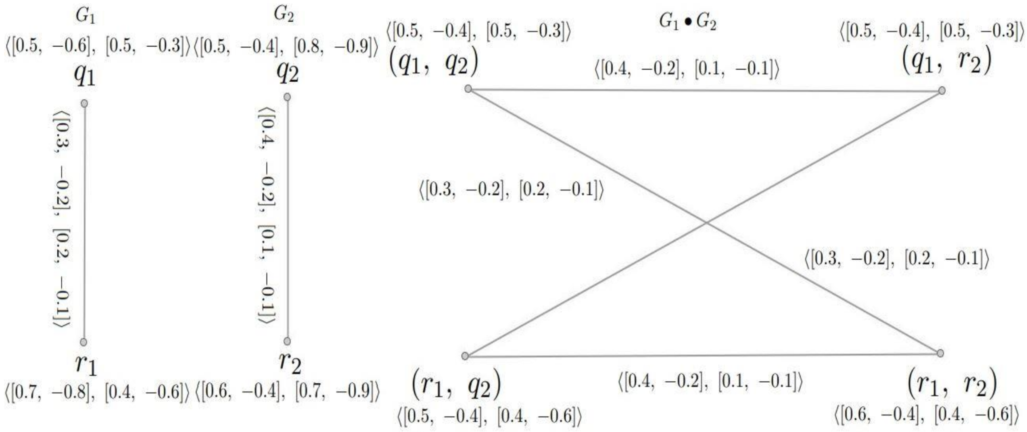

Figure 4: Semi Strong Product of $(G_1 g G_2)$ of2 m-BPFGs G_1 and G_2

7. Vertex Degree in Strong Product

Definition 7.1. The strong product of 2 m-BPFGs $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ of G_1^* and G_2^* suh that $V_1 \cap V_2=\phi$, is defined to be the m-BPFG $G_1 \otimes G_2=(Q_1 \otimes Q_2, R_1 \otimes R_2)$ of $G^*=(V_1 \times V_2, E)$ where $E=\{(q,r_1)(q,r_2) \mid q \in V_1, r_1 r_2 \in E_2\} \cup \{(q_1,w)(q_2,w) \mid w \in V_2, r_1 r_2 \in E_1\}$ $\cup \{(q_1,r_1)(q_2,r_2) \mid q_1 q_2 \in E_1, r_1 r_2 \in E_2\} \subseteq V_1 \times V_2^2$ satisfying the following : for each $h=1 \ to \ m$

$(i)\, P_h \circ \Psi^+_{(Q_1 \otimes Q_2)}(q,r)=\min\{P_h \circ \Psi^+_{Q_1}(q), P_h \circ \Psi^+_{Q_2}(r)\}$

$P_h \circ \Psi^-_{(Q_1 \otimes Q_2)}(q,r)=\max\{P_h \circ \Psi^-_{Q_1}(q), P_h \circ \Psi^-_{Q_2}(r)\}$ for all $(q,r) \in V_1 \times V_2$,

$(ii)\, P_h \circ \Psi^+_{(R_1 \otimes R_2)}((q,r_1)(q,r_2))=\min\{P_h \circ \Psi^+_{Q_1}(q), P_h \circ \Psi^+_{R_2}(r_1 r_2)\}$

$P_h \circ \Psi^-_{(R_1 \otimes R_2)}((q,r_1)(q,r_2))=\max\{P_h \circ \Psi^-_{Q_1}(q), P_h \circ \Psi^-_{R_2}(r_1 r_2)\}$ for all $q \in V_1,\ r_1 r_2 \in E_2$,

$(iii)\, P_h \circ \Psi^+_{(R_1 \otimes R_2)}((q_1,w)(q_2,w))=\min\{P_h \circ \Psi^+_{R_1}(q_1 q_2), P_h \circ \Psi^+_{Q_2}(w)\}$

$P_h \circ \Psi^-_{(R_1 \otimes R_2)}((q_1,w)(q_2,w))=\max\{P_h \circ \Psi^-_{R_1}(q_1 q_2), P_h \circ \Psi^-_{Q_2}(w)\}$ for all $w \in V_2, q_1 q_2 \in E_1$,

$(iv)\, P_h \circ \Psi^+_{(R_1 \otimes R_2)}((q_1,r_1)(q_2,r_2))=\min\{P_h \circ \Psi^+_{R_1}(q_1 q_2), P_h \circ \Psi^+_{R_2}(r_1 r_2)\}$

$P_h \circ \Psi^-_{(R_1 \otimes R_2)}((q_1,r_1)(q_2,r_2))=\max\{P_h \circ \Psi^-_{R_1}(q_1 q_2), P_h \circ \Psi^-_{R_2}(r_1 r_2)\}$ for all $q_1 q_2 \in E_1, r_1 r_2 \in E_2$,

$(v)\, P_h \circ \Psi^+_{(R_1 \otimes R_2)}((w,l)(t,q))=0,\ P_h \circ \Psi^-_{(R_1 \otimes R_2)}((w,l)(t,q))=0,$ for all $(w,l)(t,q) \in V_1 \times V_2^2 - E$.

Definition 7.2. For any vertex $(q_1,q_2) \in V_1 \times V_2$, the degree in strong product is denoted by $d_{(G_1 \otimes G_2)}(q_1,q_2)=\left\langle \left[P_h \circ d^+_{(G_1 \otimes G_2)}(q_1,q_2), P_h \circ d^-_{(G_1 \otimes G_2)}(q_1,q_2)\right]_{h=1}^m \right\rangle$ and is defined for $h=1,2,\ldots,m$

$$P_h \circ d^+_{(G_1 \otimes G_2)}(q_1,q_2)=\sum_{(q_1,q_2)(r_1,r_2) \in E} P_h \circ \Psi^+_{(R_1 \otimes R_2)}((q_1,q_2)(r_1,r_2))$$

$$=\sum_{q_1=r_1, q_2 r_2 \in E_2} P_h \circ \Psi^+_{Q_1}(q_1) \wedge P_h \circ \Psi^+_{R_2}(q_2 r_2)+\sum_{q_1 r_1 \in E_1, q_2 r_2 \in E_2} P_h \circ \Psi^+_{R_1}(q_1 r_1) \wedge P_h \circ \Psi^+_{R_2}(q_2 r_2)$$

$$+\sum_{q_1 r_1 \in E_1, q_2=r_2} P_h \circ \Psi^+_{Q_2}(q_2) \wedge P_h \circ \Psi^+_{R_1}(q_1 r_1)$$

$$P_h\,\mathrm{o}d^-_{(G_1\otimes G_2)}(q_1,q_2)=\sum_{(q_1,q_2)(r_1,r_2)\in E} P_h\,\mathrm{o}\Psi^-_{(R_1\otimes R_2)}((q_1,q_2)(r_1,r_2))$$
$$=\sum_{q_1=r_1,\,q_2r_2\in E_2} P_h\,\mathrm{o}\Psi^-_{Q_1}(q_1)\wedge P_h\,\mathrm{o}\Psi^-_{R_2}(q_2r_2)+\sum_{q_1r_1\in E_1,\,q_2r_2\in E_2} P_h\,\mathrm{o}\Psi^-_{R_1}(q_1r_1)\wedge P_h\,\mathrm{o}\Psi^-_{R_2}(q_2r_2)$$
$$+\sum_{q_1r_1\in E_1,\,q_2=r_2} P_h\,\mathrm{o}\Psi^-_{Q_2}(q_2)\wedge P_h\,\mathrm{o}\Psi^-_{R_1}(q_1r_1)\quad \text{for } h=1,2,\dots,m$$

Theorem 7.1. Let $G_1=(V_1,Q_1,R_1)$ and $G_2=(V_2,Q_2,R_2)$ be 2 m-BPFGs of the graphs. If $R_2\subseteq Q_1$ $R_1\subseteq Q_2$ and $R_1\subseteq R_2$, then $d_{(G_1\otimes G_2)}(q_1,q_2)=|V_2|d_{G_1}(q_1)+d_{G_2}(q_2)$ for all $(q_1,q_2)\in V_1\times V_2$.

Proof. For $h=1,2,\dots,m$ and $(q_1,q_2)\in V_1\times V_2$, we have

$$P_h\,\mathrm{o}d^+_{(G_1\otimes G_2)}(q_1,q_2)=\sum_{(q_1,q_2)(r_1,r_2)\in E} P_h\,\mathrm{o}\Psi^+_{(R_1\otimes R_2)}((q_1,q_2)(r_1,r_2))$$
$$=\sum_{q_1=r_1,\,q_2r_2\in E_2} P_h\,\mathrm{o}\Psi^+_{Q_1}(q_1)\wedge P_h\,\mathrm{o}\Psi^+_{R_2}(q_2r_2)$$
$$+\sum_{q_1r_1\in E_1,\,q_2=r_2} P_h\,\mathrm{o}\Psi^+_{Q_2}(q_2)\wedge P_h\,\mathrm{o}\Psi^+_{R_1}(q_1r_1)$$
$$+\sum_{q_1r_1\in E_1,\,q_2r_2\in E_2} P_h\,\mathrm{o}\Psi^+_{R_1}(q_1r_1)\wedge P_h\,\mathrm{o}\Psi^+_{R_2}(q_2r_2)$$
$$+\sum_{q_2r_2\in E_2} P_h\,\mathrm{o}\Psi^+_{R_2}(q_2r_2)+\sum_{q_2=r_2,\,q_1r_1\in E_1} P_h\,\mathrm{o}\Psi^+_{R_1}(q_1r_1)+\sum_{q_1r_1\in E_1} P_h\,\mathrm{o}\Psi^+_{R_1}(q_1r_1)$$
$$=|V_2|P_h\,\mathrm{o}d^+_{G_1}(q_1)+P_h\,\mathrm{o}d^+_{G_2}(q_2)$$

Similarly, $P_h\,\mathrm{o}d^-_{(G_1\otimes G_2)}(q_1,q_2)=|V_2|P_h\,\mathrm{o}d^-_{G_1}(q_1)+P_h\,\mathrm{o}d^-_{G_2}(q_2)$

This shows that $d_{(G_1\otimes G_2)}(q_1,q_2)=|V_2|d_{G_1}(q_1)+d_{G_2}(q_2)$

Example 7.1. Let us consider the 2-BPFGs G_1,G_2 and their strong product $G_1\otimes G_2$ (see Figure 5). For this graph $R_2\subseteq Q_1$ $R_1\subseteq Q_2$ and $R_1\subseteq R_2$ so by Theorem 7.1., $d_{(G_1\otimes G_2)}(q_1,q_2)=\langle[0.7,-0.4],[0.4,-1.0]\rangle$, $|V_2|d_{G_1}(q_1)+d_{G_2}(q_2)=\langle[0.7,-0.4],[0.4,-1.0]\rangle$. In the same way, we have to find the degrees of all other vertices in $G_1\otimes G_2$ This can be confirmed from figure 5.

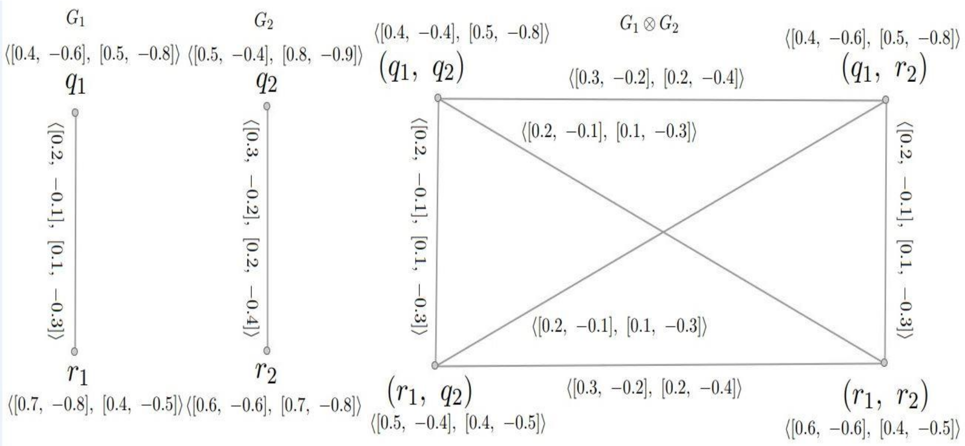

Figure 5: Strong Product of $(G_1 \otimes G_2)$ of2 m-BPFGs G_1 and G_2

Conclusions

In this article, Cartesian product $(G_1 \times G_2)$, composition $(G_1[G_2])$, direct product $(G_1 \mathrm{I}\ G_2)$, semi strong product $(G_1 \otimes G_2)$ and strong product $(G_1 g G_2)$ of two m-BPFGs are defined and also calculated the vertex degrees of the m-BPFGs G_1 and G_2 under few properties and elucidate them with examples.

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