Application of new wavelet for solving Burger's Fisher equation with collocation method

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Abstract

In this thesis, a novel wavelet method is successfully used to find the numerical solution to the Burger’s Fisher problem. This strategy exhibits reasonably rapid convergence when compared to other strategies currently in use. Examples are given as illustrations to show how the new wavelet method works and how strong it is. By comparing the results, finding the precise solution, and using the numerical output from the cas wavelet and the haar wavelet, we were able to develop the general formulas for n different wavelet integrals.

Keywords: Operational Matrix, Cosine and Sine (CAS) wavelet, collocation points, wavelet of Haar.

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Introduction 1.

A function may be expressed in terms of fundamental, spatially confined functions called wavelets according to a novel numerical idea called wavelet analysis. Wavelets are a type of data structure that was first introduced in the early 1980s. They piqued the curiosity of the mathematical community as well as members in a wide range of disciplines where wavelets had potential applications. As a result of this interest, multiple books on the subject have been published, as well as a vast number of research publications.[11]
\[ \Psi_{k,n}(t) = |a_0|^\frac{k}{2} \psi \left( a_0^k t - n b_0 \right) \quad \ldots \ldots \ (2) \]

Where the wavelet basis for \( L^2(R) \) is \( \Psi_{k,n}(t) \). When, in particular,
\[ a_0 = 2, b_0 = 1, \text{ then } \Psi_{k,n}(t) \text{ Establishes an orthonormal foundation } [2] \]

In the field of numerical methods, the wavelet method has gained popularity in recent years. For this, many types of wavelets and approximation functions have been applied. Using the Haar wavelet and CAS wavelet methods,

The following is the structure of the paper. The integrals of Haar wavelets are provided in section 2. Section 3 introduces CAS wavelets and their integrals. In section 4, we derive the general formulas of the integration of the new wavelet. In section 5, the steps of the suggested approach for solving nonlinear PDEs are carried out. After that, in section 6, the numerical solution of Burger’s equation are solved. Finally, the numerical results are summarized, there are also ideas for future research.

2. Haar wavelets as well as integrals

This section contains the Haar wavelet integrals analytically, Whereas, higher order differential equations can be solved numerically by these integrals.

The Haar wavelets family is specified as [4, 5, 12].

\[ \psi^H_{l,1} = \begin{cases} 1 & \text{for } t \in [\xi_1(i), \xi_2(i)] \\ -1 & \text{for } t \in [\xi_2(i), \xi_3(i)] \\ 0 & \text{elsewhere} \end{cases} \quad \ldots \ldots (3) \]

\[ \psi^H_{l,2}(x) = \begin{cases} \frac{t - \xi_1(i)}{\xi_3(i) - t} & , \quad t \in [\xi(i), \xi_2(i)] \\ \frac{1}{4m^2} - \frac{(\xi_3(i) - t)^2}{2} & , \quad t \in [\xi_2(i), \xi_3(i)] \end{cases} \]

\[ \psi^H_{l,3}(x) = \begin{cases} 0 & , \quad t \in [\xi(i), \xi_2(i)] \\ \frac{t - \xi_2(i)}{6m^2} - \frac{3(\xi_3(i) - t)^3}{6} & , \quad t \in [\xi_2(i), \xi_3(i)] \end{cases} \]

Where

\[ \xi_1(i) = \frac{k}{2j}, \xi_2(i) = \frac{k+0.5}{2j}, \xi_3(i) = \frac{k+1}{2j} \ldots (7) \]

Therefore, the general nature of the wavelet integration Haar[13]
\[ \psi^H_{l,5}(t) = \]
If we integrate equation (11) from (0) to (1) we get:  

$$P_{2^k(2M+1),1}(t) = \int_0^x \psi_{\mu,m}^{CAS}(x') dx'$$ \hspace{1cm} (12)  

For the interval \( [0, \frac{\mu}{2^k}] \), we have:  

$$\psi_{\mu,m}^{CAS}(t) = 0 \hspace{1cm} 0 \leq t < \frac{\mu}{2^k}$$  

For the interval \( \frac{\mu}{2^k} \leq t < \frac{\mu + 1}{2^k} \), we have:  

$$\psi_{\mu,m}^{CAS}(t) = \begin{cases} 0 & 0 \leq t < \frac{\mu}{2^k} \\ \left( 2^k \frac{1}{2^{k+1} \pi m} \left( \sin \left( 2\pi m \left( 2^k t - \mu \right) \right) \right) - \cos \left( 2\pi m \left( 2^k t - \mu \right) \right) \right) & \frac{\mu}{2^k} \leq t < \frac{\mu + 1}{2^k} \\ 2^k \frac{1}{2^{k+1} \pi m} (\sin(2\pi m) - \cos(2\pi m)) - 2^k \frac{1}{2^{k+1} \pi m} & \frac{\mu + 1}{2^k} \leq t < 1 \end{cases}$$ \hspace{1cm} (13)  

Suggested wave value  

$$CAS_m(x) = \cos(m \pi t) + \sin(m \pi t)$$ \hspace{1cm} (11)
As CAS waves are derived from trigonometric functions that are regularly and finitely integrated, we now derive the new wave formula that results from the link between Haar and CAS waves. Note that the commutative property is realized in the (convolution)
5. Derivation of a general formula for the integration of the new wavelet

The integrals for the new wavelet given in equation (17) are obtained analytically. Trigonometric functions with periodic integration are used to characterize the new wavelet. We can discover a general formula for each of these wavelets' integrals for $n$.

\[
P_{2k(2M+1),1}(t) = \int_0^x \Psi_{\mu,m}^{\text{New}}(x^*) dx^* \quad \text{then}
\]

\[
P_{2k(2M+1),2}(t) = \int_0^x P_{2k(2M+1),1}(t) dx^*
\]


\[
\begin{align*}
0, 0 \leq t < \frac{\mu}{2^k} & \\
\left\{ \begin{array}{l}
\frac{-sk - 3}{2^3 \pi^3 m^3} [\sin (2m\pi (2^k t - \mu)) - \cos (2m\pi (2^k t - \mu))] + \\
+ (4t^2 (\pi m)^2 \cos (\pi m) + 2t^2 (\pi m)^2 (2^k)^2 + \\
\frac{\mu}{2^k} \right.
\end{array} \right.
\end{align*}
\]

We find \( P_{l,s} (t) \) by repeating the integration \( n \) times:

\[
\begin{align*}
0, 0 \leq t < \frac{\mu}{2^k} & \\
\left\{ \begin{array}{l}
\left( \frac{-sk - 3}{2^3 \pi^3 m^3} \right) [2m\pi (2^k t - \mu)] + \\
\sum_{j=0}^{s} \left( \frac{(-1)^{r+j+1}(2)^{r+1}}{(r-j)!j!} \cos (2m\pi) (m\pi)^r n^r + \\
\begin{array}{l}
\begin{aligned}
\frac{1}{(2^k)^r (m\pi)^{r+1}} & (2m\pi (2^k t - \mu)) + \\
\sum_{j=1}^{r} & \frac{(-1)^{r+j+1}(2)^{r+1}}{(r-j)!j!} \cos (m\pi) (t2^k)^j \mu r-j (m\pi)^r + \\
+ \frac{2^{r+1}}{j!} (m\pi)^r \sin (m\pi) + \\
\frac{(-1)^{r+1}(2)^{r+1}}{(r-j)!j!} & (m\pi)^r - \frac{2^{r+1}}{j!} (m\pi)^r \cos (m\pi) + \\
\end{aligned}
\end{array}
\end{array} \right.
\end{align*}
\]

\[
\left( \frac{-k - 3}{2^3 \pi^3 m^3} \right) \left( \frac{1}{(2^k)^r (m\pi)^{r+1}} \right) \cdot 
\begin{array}{l}
\sum_{j=1}^{r} \left( (-1)^{r-j} (2m\pi)^r \cos (2m\pi) + (-1)^{r-j} (2m\pi)^r \sin (2m\pi) - \frac{2^{r+1}}{j!} (m\pi)^r \cos (m\pi) + \\
+ \frac{2^{r+1}}{j!} (m\pi)^r \sin (m\pi) + \\
\sum_{u=0}^{r-1} \frac{2^r}{(j-u)!} (m\pi)^{r-u} \right) (-1)^{a_p}
\end{array}
\]

\[
\mu + 1 \leq t < 1
\]
6. The suggestion method of solution for partial differential equations

The novel wavelet approach is used in this section to solve nonlinear partial differential equations. Nonlinear PDE's generic form

\[ F(\hat{x}, t, u, Du, D^2u, \ldots, D^{\alpha+\beta}u) = f(\hat{x}, t) \quad \ldots \ldots (26) \]

\[ D^{\alpha+\beta}u = \frac{\partial^{\alpha+\beta}u(\hat{x}, t)}{\partial t^\alpha \partial x^\beta} \]

Where \( x \in [a, b] \) and \( t \in (0, T] \) because the variables \( x \) and \( t \) belong to the \( \Omega \) interval.

**Step (1):** Using the relationship, we convert the period \([a, b]\) containing \( \hat{x} \) to the period \([0, 1]\) containing \( x \).

\[ x = \frac{1}{L}(\hat{x} - a) \quad L = b - a \quad \ldots \ldots (27) \]

**Step (2):** The period \((0, T]\) is divided into \( N \) equal pieces of length \( \Delta t = \frac{T}{N} \) and time \( t_s = (s - 1)\Delta t, s = 1, 2, \ldots, N \). The \([0, 1]\) interval containing \( x \) is also divided into \( M_1 \) assembly collocation points of similar length.

\[ x_i = \frac{l - 0.5}{M_1} \quad l = 12, \ldots, M_1 \quad \ldots \ldots (28) \]

**Step (3):** Assume that

\[ \frac{\partial^{\alpha+\beta}u(x, t)}{\partial t^\alpha \partial x^\beta} = \sum_{i=0}^{M_1-1} c_i W_i(x, t) \]

Where \( c_i \) are the coefficients of the wavelet.

**Step (4):** We get by integrating with respect to \( t \) from \( t_s \) to \( t \) and integrating with respect to \( x \) from 0 to \( x \).

\[ u(x, t) = \frac{(t - t_s)^\alpha}{a!} \sum_{i=0}^{M_1-1} c_i p_{\beta, i}(x) + \delta(x, t) \]

Where \( \delta(x, t) \) is determined by the beginning and boundary conditions, and \( P_{\beta, i}(x) \) is determined by the wavelet type employed in the solution (Haar, CAS, New)

**Step (5):** We use the problem to replace the answer \( u(x, t) \) and its derivatives with regard to \( t \) and \( x \).

**Step (6):** The wavelet coefficients \( c_i \) are calculated by solving a linear set of algebraic equations.

**Step (7):** We get the numerical solution \( u(x, t) \) according to the equation.

7. Numerical Experiments

We will combine the new wavelet method with the Haar and CAS wavelet method to create an approximation of the solution to the Burger's Fisher equation in order to demonstrate the effectiveness of the suggested approach. Utilizing MATLAB R2013a (8.1.0.604), all calculations have been carried out.
We use the error norm [13]:

\[ \delta_{e} = \frac{1}{N} \| u_{\text{exact}} - u_{\text{num}} \|_{2} \]

Table (1): Compares the approximate and precise solutions for \( m = 16 \) of example utilizing a Ha-Wavelets, CA-Wavelets, and novel wavelet at \( t = 0.02 \)

<table>
<thead>
<tr>
<th>(x/32)</th>
<th>Haar wavelet</th>
<th>CAS wavelet</th>
<th>New wavelet</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.49998595000071</td>
<td>0.499985950000734</td>
<td>0.499985950000764</td>
<td>0.499985950000794</td>
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<td>0.4999907824999909</td>
<td>0.499990782500001</td>
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<tr>
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<tr>
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<td>0.4999673450000219</td>
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<tr>
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<td>0.4998814075022203</td>
<td>0.4998814075022239</td>
</tr>
</tbody>
</table>

Table (2): Compares the approximate and precise solutions for \( m = 8 \) at \( t = 0.02 \)

<table>
<thead>
<tr>
<th>(x/16)</th>
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<th>CAS wavelet</th>
<th>New wavelet</th>
<th>Exact solution</th>
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</table>
The error scale for each of the Haar, CAS, and New wavelets, as well as numerous examples of their behavior, is shown in the table (3).

Table (3): The error norm $\delta_e$ for the approximation solutions at $t = 0.02$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Haar Wavelet</th>
<th>CAS Wavelet</th>
<th>New wavelet</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$7.064410040542e-14$</td>
<td>$7.223613024467e-14$</td>
<td>$7.8845385877e-15$</td>
</tr>
<tr>
<td>16</td>
<td>$1.070312596213e-13$</td>
<td>$1.075900756604e-13$</td>
<td>$5.497747014119e-15$</td>
</tr>
<tr>
<td>32</td>
<td>$1.564095819891e-13$</td>
<td>$1.566067878690e-13$</td>
<td>$3.898754018037e-15$</td>
</tr>
<tr>
<td>64</td>
<td>$2.247641505110e-13$</td>
<td>$2.248338379179e-13$</td>
<td>$2.754936881085e-15$</td>
</tr>
</tbody>
</table>

Figure (1): Compared the numerical answers to the Burger's Fisher equation's precise solution. at $t = 0.02$, $m=16$
In this paper, the general formula of integration for the new wavelet previously generated by the Haar and CAS convolution is derived to find the numerical solution of nonlinear PDEs, and the three-wavelet method is applied to solve the nonlinear Berger’s equation. Through the numerical solutions in the tables (1, 2, 3) and the figure, we notice that the solutions using the new wavelet give results that are closer to the exact solution compared to the Haar and CAS wavelets. It’s worth mentioning that the new wavelet solutions provide excellent results even for small values of \( m \) and \( k \) as note in figure (2). Also, when \( 2^k(2m+1) = 36, 2^k(2m+1) = 44, \ldots \), we can obtain the results closer to the exact values.

9. Conclusions

Acknowledged by the writers for helping to improve the caliber of this work.

References

Figure (2): Illustrates the convergence of the numerical solution by new wavelet for different values of \( M \).
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