Two-dimensional Dirac-oscillator and its mapping to newly discovered special functions

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Abstract
We consider (2+1) dimensional Dirac oscillator in the framework of the minimal length generalized uncertainty principle (GUP), whose solutions are related with the classical Jacobi polynomials, and connect these solutions in terms of newly discovered Jacobi type polynomials.

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1 Introduction

In relativistic quantum theory, the Dirac equationplays an important has various applications in theoretical as well as experimental physics. In quantum theory, we use the Dirac equation for a given system and obtain their solutions numerically or analytically. In the case of the problems which are analytical solvable are generally mapped to the well-known forms of the differential equations whose solutions are already known or can be easily obtained. It was realized first time by Ito et al that as the momentum$p \to p - im\omega r$, the system reduces to the an equivalent oscillator type problem with the frequency of oscillator ω [1,2]. As we know, the quantum theories ordinarily describe particles as being point like, in the sense thatthere is uncertainty in position Δx be made arbitrarily small and satisfies $\Delta x \Delta p \approx \hbar$ with usual commutation relation$[\hat{x}, \hat{p}] = i\hbar$. The unification between the general theory of relativity and the quantum mechanics predicts the existence of a minimal measurable length on the order of the Planck length. It has been observed that near the Planck scale, the usual Heisenberg uncertainty principle should be reformulated and redefined as $[\hat{x}, \hat{p}] = i\hbar(1 + \gamma p^2)$, where γ is a small positive parameter known as deformation parameter. The modification of the uncertainty relation leads to the standard generalised Heisenberg uncertainty relation$\Delta\hat{x}\Delta\hat{p} \approx i\hbar(1 + \gamma(\Delta p)^2)$. Thus, in order to make the existence of minimum length consistent with quantum mechanics the usual canonical commutation relation has to be modifiedleading to theGeneralized Uncertainty Principle (GUP). The concept of GUP and minimum measurable length also originates from several studies [3-6]. The fundamental outcome of GUP is the appearance of an intriguing UV/IR mixing which is also noted in Ads/CFT correspondence as well as in Non-commutative field theories. In Ref. [7], they consider a 2-dimensional Dirac oscillator in the in non-commutative space with minimal length and reduced the problem in the form of Poschl-Teller potential, whose solutions are in the form of classical Jacobi polynomials.

In 2009, Ullate et al [8, 9] discovered two new infinite sequence of polynomial functions of a Strum-Liouville problem known as the X_1-Jacobi and the X_1-Laguerre exceptional orthogonal polynomials (EOPs), which are further generalised to the corresponding X_m-EOPs. Unlike the usual orthogonal polynomials, these EOPs start with degree $m \geq 1$ and still form a complete orthonormal set with respect to a positive definite inner product defined over a compact interval. After discovery of these new polynomials most of the known exactly solvable potentials have been extended rationally with their exact solutions using different approaches like supersymmetric (SUSY) in quantum mechanics [10], Point canonical transformation (PCT) [11,12],Darboux-Crum transformation [13,14]and Group theoretic approach [15-16] etc. In the present work, we continue the works discussed in Ref. [7], and extend the results in terms of X_1 Jacobi EOPs using the method of PCT approach.

Our paper is organized as follows. In section 2, we briefly discuss the point canonical transformation (PCT) approach to construct solvable potentials with their solutions. In section 3, we introduce the 2-dimensional Dirac oscillator and discuss the problem in the framework of minimal length. The problem is reduced in the form of the usual trigonometric Poschl-Teller potential and discusses its solutions in the form of classical Jacobi polynomials. In section 4, we consider the expression of the usual trigonometric Poschl-Teller potential and obtain its corresponding rational potential using PCT approach. It is further shown that the solution associated with this rationally extended potential is in the term of X_1-exceptional Jacobi polynomials. Finally, we summarize our results with some future scope of works in section 5.

2 Point canonical transformation (PCT) approach

In this section, we discuss an important approach, the PCT approach [11,12] which is one of the useful method to obtain new solvable potential from the known usual potential. In other words, this method connects the classical special function (or polynomials) to the new special function (extended polynomials).If we have to find the expression of new potential $V_{new}(q)$ with its solutions (energy eigenvalue (E_{new}) and the eigenfunction ($\psi_{new}(q)$) , we consider the one-dimensional time-independent Schrodinger equation ($\hbar = 2m = 1$)

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$$-\psi_{new}(q)^{''}(q) + V_{new}(q)\psi_{new}(q) = E_{new}\psi_{new}(q). \qquad (2.1)$$

To solve this equation, we assume the solutions of the form

$$\psi_{new}(q) = u(q)\xi(g(q)), \qquad (2.2)$$

where$u(q)$ and $g(q)$are two undetermined functions and $\xi(g)$ is a polynomial type function, which satisfies a second-order differential equation

$$\xi^{''}(g) + J(g)\xi^{'}(g) + K(g)\xi(g) = 0. \qquad (2.3)$$

On using Eq. (2.2) in Eq. (2.1) and then compare with Eq. (2.3), we get

$$u(q) \propto \left(g^{'}(q)\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int J(g)dg\right). \qquad (2.4)$$

and

$$E_{new} - V_{new}(q) = \frac{1}{2}\left[\frac{g^{'''}(q)}{g^{'}(q)} - \frac{3}{2}\left(\frac{g^{''}(q)}{g^{'}(q)}\right)^2\right] + \left(g^{'}(q)\right)^2\left(K(g) - \frac{1}{2}J^{'}(g) - \frac{1}{4}J^2(g)\right), \qquad (2.5)$$

here a prime denotes a derivative with respect to x. To satisfy equation (2.5), one needs to find some function $g(x)$ ensuring the presence of a constant term on its right-hand side to compensate E_{new}on its left-hand one, while rest terms give rise to a potential $V_{new}(x)$ with well-behaved wavefunctions$\psi_{new}(x)$.

3Two-dimensional Dirac oscillator with a minimal length

The Dirac equation for the Dirac oscillator with the Hamiltonian H_D and the eigenfunction Φ_D is written as

$$\widehat{H}_D\Phi_D = E\Phi_D \quad (3.1)$$

where the Hamiltonian

$$\widehat{H}_D = c\boldsymbol{\alpha}.(\boldsymbol{p} - im\omega\tilde{\beta}\boldsymbol{r}) + \tilde{\beta}m_0c^2 \ . \quad (3.2)$$

Here $\alpha$and β are Dirac matrices, m_0 is the rest mass and$c$ is speed of light.

In the 2-dimensional case, the Dirac equation (2.1) is rewritten as [7]

$$[c\,\alpha_x(p_x - im_0\omega\tilde{\beta}x) + c\alpha_y(p_y - im_0\omega\tilde{\beta}y) + \tilde{\beta}m_0c^2]\Phi_D = E\Phi_D (3.3)$$

Here the matrices

$$\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_y = \sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{and } \tilde{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this way the Dirac eigenfunctionΦ_D will be

$$\Phi_D = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}.$$

In the minimal length formalism, the Heisenberg algebra is given by

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}(1 + \gamma p^2), (3.4)$$

where$\gamma > 0$is the minimal length parameter. The representation of $\hat{x}_i$ and $\hat{p}_j$which satisfies Eq.(3.4), may be taken as

$$\hat{x}_i = i\hbar(1 + \gamma p^2)\frac{\partial}{\partial p_i}, \ \hat{p}_i = p_i (3.5)$$

with$i = 1,2$i.e., $x_1 = x, x_2 = y$ and $p_1 = p_x, p_2 = p_y$.

Using this in Eq. (3.3), we get

$$[c^2P_-P_+ - (\epsilon^2 - m_0^2c^4)]\phi^{(1)} = 0. \quad (3.6)$$

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HereP_- and P_+ are

$$P_- = (p_x - ip_y) - \mu(1 + \gamma p^2)\left(\frac{\partial}{\partial p_x} - i\frac{\partial}{\partial p_y}\right), \quad (3.7)$$

and

$$P_+ = (p_x - ip_y) + \mu(1 + \gamma p^2)\left(\frac{\partial}{\partial p_x} + i\frac{\partial}{\partial p_y}\right), \quad (3.8)$$

with$\mu = m\omega h$ respectively. In the polar co-ordinates i.e.,

$$p_x = p\cos\theta, p_y = p\sin\theta \text{and} p = \sqrt{(p_x^2 + p_y^2)} \quad (3.9)$$

the above Eqs. (3.7) and (3.8) reduced to

$$P_- = e^{i\theta}[p - \mu(1 + \gamma p^2)\left(\frac{\partial}{\partial p} - \frac{i}{p}\frac{\partial}{\partial\theta}\right) \quad (3.10)$$

and

$$P_+ = e^{i\theta}[p + \mu(1 + \gamma p^2)\left(\frac{\partial}{\partial p} + \frac{i}{p}\frac{\partial}{\partial\theta}\right) \quad (3.11)$$

respectively. Now using P_- and P_+ in Eq. (3.6), we get

$$p^2 + 2(1 + \gamma p^2)\{\mu\,(i\frac{\partial}{\partial\theta} - 1) - \gamma\,\mu^2(p\frac{\partial}{\partial p} + \frac{i\partial}{\partial\theta})\} - \mu^2(1 + \gamma p^2)^2(\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{1}{p^2}\frac{\partial^2}{\partial\theta^2}) - \zeta^2\}\Phi_1 = 0$$

(3.12)

with

$$\zeta^2 = \frac{\epsilon^2 - m_0^2c^4}{c^2} \ .$$

To solve this equation, we put

$\Phi_1 = h(p)e^{im\theta}$ (3.13)

and get

$[-a(p)\frac{d^2}{dp^2} + b(p)\frac{d}{dp} + c(p) - \zeta^2]h(p) = 0$ (3.14)

Where the functions $a(p), b(p)$ and $c(p)$ are

$$a(p) = \mu^2(1+\gamma p^2)^2$$

$$b(p) = -2\gamma\,\mu^2(1+\gamma p^2)p - \frac{\mu^2(1+\gamma\,p^2)^2}{p}$$

and

$c(p) = p^2 - 2\mu\,(m+1)(1+\gamma p^2) + 2\gamma\,\mu^2\,m(1+\gamma p^2) + \mu^2(1+\gamma\,p^2)^2\frac{m^2}{p^2}.$ (3.15)

To solve the Eq. (14), the following transformations have been used [7] i.e.,

$$h(p) = \rho(p)\psi\,(q), \qquad q = \int \frac{1}{\sqrt{a(p)}}dp,$$

with

$\rho(p) = e^{\int \chi(p)dp}$ (3.16)

On using these transformations, we obtain a form similar to the Schrödinger differential equation

$$\left[-\frac{d^2}{dq^2} + V(q)\right]\phi^{(1)}(q) = \zeta^2\phi^{(1)}(q),$$

where

$$\chi(p) = \frac{2\,b + a'(p)}{4a} = -\frac{1}{2p}$$

and

$$V(q) = p^2 - 2\mu\,(m+1)(1+\gamma p^2) + 2\gamma\mu^2\,m(1+\gamma p^2) + \gamma\,\mu^2(1+\gamma p^2) + \frac{\mu^2}{(1+\gamma p^2)^2}\left(m^2 - \frac{1}{4}\right)$$

(3.17)

.

Again on using the transformation $p = \frac{1}{\sqrt{\gamma}}tan(q\mu\sqrt{\gamma})$, the potential $V(q)$ is further reduced in the form of a well-known trigonometric Poschl-Teller potential like [17]

$V(q) = -\frac{1}{\gamma} + \gamma\mu^2\left(\frac{s(s-1)}{sin^2(q\mu\,\sqrt{\gamma})} + \frac{\lambda\,(\lambda-1)}{cos^2(q\mu\,\sqrt{\gamma})}\right)$ (3.18)

where

$$s = m + \frac{1}{2},$$

and

$$\lambda = m - \frac{1}{\gamma\mu} + \frac{3}{2}$$

Thus the energy eigenfunctions and the energy eigenvalues corresponding to the Schrodinger like equation (17) in term of classical Jacobi polynomial $P_n^{(\alpha,\beta)}(g(q))$ are given by

$$\psi(q) = (1-g(q))^{\frac{\lambda}{2}}(1+g(q))^{\frac{s}{2}}P_n^{(\alpha,\beta)}(g(q))$$

and

$E_n = \zeta^2 + \frac{1}{\gamma} = (s\delta + \lambda\delta + 2n\delta)^2$(3.19)

respectively. Here$\alpha = \lambda - \frac{1}{2}$, $\beta = s - \frac{1}{2}$ and $g(q) = \cos(2q\delta)$ with $\delta = \mu\sqrt{\gamma}$.

4Connection toa new type of special function (the X_1-Jacobi EOPs)

The second-order differential equation satisfied by this X_1-Jacobi polynomial exceptional$\hat{P}_n^{(\alpha,\beta)}(g)$ is given by ($n \geq 1$) [12]

$$\hat{P}_n^{(\alpha,\beta)''}(g) - \left(\frac{(\beta+\alpha+2)g-(\beta-\alpha)}{(g^2-1)} + \frac{2(\beta-\alpha)}{(\beta+\alpha)-(\beta-\alpha)g}\right)\hat{P}_n^{(\alpha,\beta)''}(g) + \left(\frac{(\beta-\alpha)g-(n-1)(n+\beta+\alpha)}{g^2-1} + \frac{(\beta-\alpha)^2}{(\beta+\alpha)-(\beta-\alpha)g}\right)\hat{P}_n^{(\alpha,\beta)}(g) = 0 \qquad (4.1)$$

On comparing Eqs. (2.3) and (4.1) , we get

$$J(g) = -\frac{(\beta+\alpha+2)g-(\beta-\alpha)}{1-g^2} - \frac{2(\beta-\alpha)}{(\beta-\alpha)g-(\beta+\alpha)} \qquad (4.2)$$

$$K(g) = -\frac{\left((\beta-\alpha)g-(n-1)(n+\beta+\alpha)\right)}{(1-g^2)} - \frac{(\beta-\alpha)^2}{(\beta-\alpha)g-(\beta+\alpha)}. \qquad (4.3)$$

and

$$F(g) = \hat{P}_n^{(\alpha,\beta)}(g(q)) \qquad (4.4)$$

Now on using $\alpha = \lambda - \frac{1}{2}$, $\beta = s - \frac{1}{2}$,$g(q) = \cos(2q\delta)$ and $J(g)$ and $K(g)$ (from Eqs. (4.2) and (4.3)) and by replacing $n \to n+1$ in Eq. (2.5), we get the expression of the new potential

$$V_{new}(q) = V(q) + V_{rat}(q) \qquad (4.5)$$

where

$$V_{rat}(q) = \frac{8\left(A\delta + B\delta - \frac{\delta^2}{4}\right)}{\left(\frac{A}{\delta} - \frac{B}{\delta}\right)\cos^2(2q\delta) - \left(\frac{A}{\delta} + \frac{B}{\delta} - \frac{1}{4}\right)} + \frac{32\delta^2\left(\frac{A}{\delta} - \frac{\delta^2}{2}\right)\left(\frac{B}{\delta} - \frac{\delta^2}{2}\right)}{\left[\left(\frac{A}{\delta} - \frac{B}{\delta}\right)cos^2(2q\delta) - \left(\frac{A}{\delta} + \frac{B}{\delta} - \frac{1}{4}\right)\right]^2}$$

and $A = s\delta, B = \lambda\delta$. The associated normalizableeigenfunction becomes

$$\psi_n(q) \to \phi_{n,new}^{(1)}(q) \propto \frac{(1-cos(2q\delta))^{\left(\frac{\alpha}{2}+\frac{1}{4}\right)}(1+cos(2q\delta))^{\left(\frac{\beta}{2}+\frac{1}{4}\right)}}{[(\beta-\alpha)cos(2q\delta)-(\beta+\alpha)]}\hat{P}_{n+1}^{(\alpha,\beta)}(g(q)); n \geq 0 \qquad (4.6)$$

with the same energy eigenvalues as given by Eq. (3.19). Following the same procedure, we can easily generalize these results to the X_m ($m = 0,1,2\ldots$) case.

5 Conclusions

In this work, we consider the problem of the 2-dimensional Dirac oscillator in the framework of minimal length and reduced the problem equivalent to the one dimensional Schrodinger equation with the trigonometric Poschl-Teller like potential. The bound state solutions of this equation are obtained analytically in the form of classical Jacobi polynomials. We further extend this problem considering the form of the potential and construct the corresponding rational counterparts using point canonical approach. The energy eigenvalues and the eigenfunctions of this extended potential are also obtained and shown that the eigenfunctions are related in the form of X_1-exceptional Jacobi polynomials, which can be further extended to the X_m-case for any positive values of m.

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