Locally convex topological vector lattices and their representation

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Abstract:
A locally convex topology is defined for a vector lattice having a weak order unit and a certain partition of the weak order unit, analogous to the order unit topology. The purpose of this paper is to generalize the theory of normed vector lattices in an analogous fashion, by introducing the concept of locally convex lattices. This work is motivated by the work of [15-17].

Keywords: Normed spaces, Normed vector lattices, Locally convex spaces, etc.
DOI Number: 10.48047/nq.2019.17.12.NQ19116 **NeuroQuantology2019;17(12):64-67**

Introduction:

The theory of vector lattices appeared in early thirties of last century and is connected with the names of L.V. Kantrovich, F. Riesz, and H. Ereudenthal (see [1]). The study of vector space equipped with a given norm structure was evidently motivated by the general circumstances that brought to life functional analysis in those years. Here, the general inclination to abstraction and uniform approach to studying functions, operation on functions, and equations related to them should be noted. A remarkable circumstances was that the comparison of the elements could be added to the properties of functional objects under consideration. At the same time, the general concept of Banach space ignored a specific aspect of the functional spaces-the existence of a natural order structure in them, which makes these spaces over-lattice. Along with the theory of ordered spaces, the theory of Banach algebra was being developed almost at the same time. Although at the beginning these two theories advanced in parallel, soon their paths parted. Banach algebra were found to be effective in function theory, in the spectral theory of operators, and in other related fields. The theory of vector lattices was developing more slowly and its achievements related to the characterization of various types of ordered spaces and to the description of operators acting in them was rather unpretentious and specialized (see [1-2]). In ordered to vector spaces there are several natural ways to define convergence using only the ordering. We will refer to them as order-convergence. Order convergence of nets is widely used. The study of normed vector lattices, it is used for order continuous norms (see [1-3]). It is also used in the theory of operators between vector lattices to define order continuous operators which are operators that are continuous with respect to order convergence (see [3]). The commonly use definition of order convergence for nets [4] originates from the definition for sequences. Authors [5] proposed, in the setting of vector lattices, a new and improved definition for ordered-convergence of nets.

Definition: Let F be a linear lattice over the field of real numbers and $\|\cdot,\cdot\|$ be real vector valued function on $F \times F$ which satisfy the following properties:

(i) $\|r,s\|=0 \Leftrightarrow$ they are linearly dependent

(ii) $\|r,s\|=\|s,r\|$

(iii) $\|r,\Delta s\|=|\Delta|\,\|r,s\|$

(iv) $\|r,\ s+t\|\leq\|r,\ s\|+\|r,\ t\|$

(v) if $|r|<|s|$

then $\|r,\ t\|\leq\|s,\ t\|$

thus $\|\cdot,\cdot\|$ is known as monotonic 2-norm and $(F,\|\cdot,\cdot\|)$ is known as 2-normed lattice.

Definition: Every ordered vector space which also satisfies all the conditions of topological vector space is only known as topological vector space (see [11]).

A topological vector lattice V over R is a Hausdorff topological vector space over R, a vector lattice, and locally solid. It means that there a neighbourhood base of O consisting of solid sets.

Notes: (i) A topological vector lattice V is a topological lattice.

(ii) Let V be a vector lattice and topological vector space. The following are equivalent

a) $\vee : V^2 \rightarrow V$ is continuous

b) $\wedge : V^2 + V$ is continuous

c) $+ : V \rightarrow V$ is given by $x^+ := x v o$ is continuous

d) $- : V \rightarrow V$ is given by $x^- := -x v o$ is continuous

e) $|\cdot| : V \rightarrow V$ given by $|x| := -x v x$ is continuous.

(iii) A topological vector lattice is an ordered topological vector space.

Proof: We need to show that the positive cone is a closed set. The positive cone is defined as $\{\zeta : 0\leq\zeta\}=\{\zeta:\zeta^-=0\}$, which is closed ($\because -$is continuous, and the positive cone is inverse image of a singleton, a closed set in R.

(iv) In order theory and functional analysis, a normed lattice is a topological vector lattice that is also a normed space whose unit ball is a solid. Normed lattices are important in the theory of topological vector lattices. They are closely related to Banach vector lattices, which are normed vector lattices that are also Banach spaces (see [12-13]).

Theorem 1: Every 2-normed lattices is locally convex topological vector lattices.

Proof: We have to show that $X_{\Sigma(0)} \leftarrow\in B(0)^1)$ is solid.

Let $|a|\leq|b|, b\in X_{\Sigma(0)}$, then $\|a,\ c\|\leq\|b,\ c\|\forall\ c\in Y$.

Let $\Sigma=\{(c_1,\Sigma_1),(c_2,\ \Sigma_2),\ldots,(c_n,\ \Sigma_n)\}$

$\therefore\ X_{\Sigma(0)}=(P\{c:d(0,c,c_j)<j=1$

$\Sigma_j(j=1,2,\ldots\};$

$b\in X_{\Sigma(0)}\Rightarrow d(0,c_j,b)<\Sigma_j$

$\Rightarrow d(0,c_i,a)<\Sigma_j$

$\Rightarrow a\in X_{\Sigma(0)}$

$\Rightarrow X_{\Sigma(0)}\in B(0)$ is solid by using [14] and given definition.

We get our required result.

Theorem 2: Every 2-normed lattice is a locally semi-convex topological vector lattice.

Proof: let $a,b\in X_{\Sigma(0)}\leftarrow\in B(0),$

$$\Sigma = \{(c_1, \Sigma_1), (c_2, \Sigma_2), \ldots, (c_n, \Sigma_n)\},$$

$$X_{\Sigma(0)} = \bigcap_{j=1}^{n} \{c : d(0, c_j, c) < \Sigma_j\}$$

$$a, b \in X_{\Sigma(0)} \Rightarrow d(0, a, c_j) < \Sigma_j$$

$$d(0, b, c_j) < \Sigma_j$$

now,

$$d(0, a+b, c_j) \leq d(0, a, c_j) + d(0, b, c_j)$$

$$< \Sigma_j + \Sigma_j = 2\Sigma_j$$

put $\mu = \frac{1}{2}$ then

$$d(0, \mu(a+b), c_j) < \Sigma_j$$

$$\Rightarrow \mu(a+b) \in X_{\Sigma(0)}$$

$$\Rightarrow a+b \in \frac{1}{\mu} X_{\Sigma(0)}$$

this completes the proof of the theorem.

Theorem: Let F be a Banach lattice and $a_n \xrightarrow{\upsilon} a$ satisfies iff an arbitrary sequence $\{a_{\ell_j}\}$ contains $\{a_{\ell_{j_r}}\}$ such that $a_{\ell_{j_r}} \to a$.

Proof: Let $a_n \xrightarrow{\upsilon} 0$. Let us consider a increasing sequence of such that $\|a_{\ell_r}\|^* \leq \frac{1}{r^3}$ where $\ell_1 < \ell_2 < \ldots < \ell_r < \ldots$.

The series $\sum_{r=1}^{\infty} r|a_{\ell_r}|$ is υ-converges,

Let $b = \upsilon - \sum_{r=1}^{\infty} r|a_{\ell_r}|$ then by hypothesis,

$$r|a_{\ell_r}| \leq b \forall r$$

$$\Rightarrow a_{\ell r} \xrightarrow{\gamma} 0.$$

Let us assume that $a_\ell^{\upsilon} \not\to 0$ then for a given $\Sigma > 0$ there exist a sub-sequence $\{a_{\ell_j}\}$ for all terms of which $\|a_{\ell_j}\|^* > \Sigma$.

As (γ)-convergence implies that υ-converges.

thus $\{a_\ell\}$ contains no subsequence which is (γ)-convergence to 0.

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