Cohomologically Finite Generation Properties of Groups

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Abstract

Let G be a group and ZG be its integral group ring. Thus an additive group ZG is the free abelian group with the element of G, now for each group G and representation M of G there are abelian groups H^n(G,M) Where n = 0.1,2,3 called the n^{th}cohomology of G with coefficient in M. In this paper we explore the notion of Cohomologically finite generation properties of groups.

Keywords: Algebraic groups, noetherian rings, modules, projective modules.

DOI Number: 10.14704/nq.2022.20.9.NQ44740 **Neuro Quantology 2022; 20(9):6311-6315**

1. Introduction

The cohomology groups may be defined topologically and also algebraically. Now we will say something about topological approach of cohomology by Hurewicz theorem if X be path–connected space with π_nX= 0 for all n ≥ 2 such X is called aspherical, If cohomologyan aspherical space X is locally path connected the universal cover $\widetilde{X}$ *is* contractible and X= $\widetilde{X}/G$. Also $H^n(X)$ depend only on π_1(X) if G = π_1(X), we must thus define $H^n(G,Z) = H^n(X)$ and because X is determined upto cohomology equivalence the definition does not depend on X. The Hurewics theorem gives what the group cohomology's if there exist an aspherical space with the fundamental group, but it does not clearly defined that always such space exist. Various of the low–dimensional cohomology group had been studied earlier than the topologically defined groups or the general definition of group cohomology. In 1932 Baer studied $H^2(G,A)$as a group of equivalence classes of extensions. It was in 1945 that Eilenberg and Maclane introduced an algebraic approach which included these groups as special cases.

(1.1) Corollary: if G is finite group and M is a ZG-module then for all n ≥ 1, $H^n(G,M)$ is finite abelian group of exponent dividing |G|.

Proof: An abelian group A is uniquely divisible by an integer n if for each aϵA there exist a unique bϵA with a = nb. This happens if the homomorphism n: A → A is an isomorphism, we say that A is uniquely divisible if it is uniquely divisible by each positive integer n. For example Q and R are uniquely divisible, Q/Z divisible but not uniquely. At last we have if A is finite and g.c.d $(|A|, n)$ = 1 then A is uniquely divisible by n.

(1.2) Corollary: "If G is a finite group and M is a finitely generated ZG-module which is uniquely" divisible by |G| then $H^n(G,M) = 0$ for all n ≥ 1

Proof: Since multiplication |G| : M → M is an isomorphism, so |G| : $H^n(G,M) \to H^n(G,H)$, by functoriality of cohomology.This map is zero for each n ≥ 1, by proposition, we have $H^n(G,M) = 0$ for each n ≥ 1

(1.3) Corollary:

(1) $H^n(G,Z) \cong H^{n-1}(G, Q/Z) \cong H^{n-1}(G, C^\times)$ for each n ≥ 2, with similar isomorphism's in cohomology.

(2) if M is finitely generated RG-module in which |G| is invertible then $H^n(G,M) = 0$ for each n ≥ 1.

Now we show an application of this result known as the integral duality theorem which tell us a finite group that $(G,Z) \cong H_n(G,Z)$ when n ≥ 1, putting this together we have

$(G,Z) \cong H^3(G,Z) \cong H^2(G,C^{\times}) \cong H^2(G, Q/Z)$

These groups are all isomorphic to the Schur multiplier.

(1.4) Corollary: Let $1 \to M \to E \to G \to 1$ be a short exact sequence of finite groups where g.c.d $(|M|,|G|) = 1$. Then the extension is split, $E \cong M \times G$ and all subgroups of E of order $|G|$ are conjugate.

Proof: we proof when M is abelian. Here $H^2(G,M) = 0$ by corollary, so the result follows from our interpretation of second and first cohomology.

Let C be an abelian group and we call any module of the form $ZG \otimes_Z C$ an induced module and any module of the form $HOM_Z(ZG,C)$ a coinduced module. The latter is made into a ZG-module using the right action on ZG.

2. Lemma: if M is coinduced then $H^n(G, M) = 0$ for all $n \geq 1$. There is no restriction on G for this result.

Proof: if M= Home (ZG ,C) is coinduced module for any abelian group C we evaluate cohomology with coefficients in M by applying the functor $Hom_{ZG}(-,Hom_z(ZG,C))$ to a projective resolution. Now for some module P we have natural isomorphism functor $Hom_z(-,C)$ to a projective resolution Z we obtain cyclic complex because

$Hom_{ZG}(P, Hom_z(ZG,C)) \cong Hom_z(ZG_{\otimes ZG} P,C) \cong Hom_z(P ,C)$

If we apply the projective resolution then $H^n(G, M)= 0$ for all $n \geq 1$.

3. Cohomologically finite generation property of groups

Let G is a group and R is a ring of unit 1, the graded R-algebra of a group G is cohomologically finite if there are noetherian R-algebras and finitely produced $H^n(G, M)$ modules in R-algebras. For any RG-module M which is generated finitely over R. Several representational features of group theory are compromised with limited generation. Finitely generated cohomology rings can be found in [2, 7]. Is the ring of invariants A^G a Noetherian one? This is an important question of invariance theory. All Noetherian associative rings A must have the finite generation property if a group meets this requirement. When a group is cohomological type, the finite generation feature can be discovered in its cohomological form. Finite generation properties are the same for all extensions of finite generation groups. Groups of the cohomological type that extend finite groups are cohomological of finite type when the second basic statement is true. [12]'s invariant subring finiteness criteria apply to numerous groups, and some are closed under group extensions. For the major results to be shown, the cohomological generalization to limited generations of invariant subrings is necessary. Cohomological adaptation for group schemes over a field k was investigated by Van der Kallen in [17], $H^n(G, A)$ represents the finitely generated k-algebra G over k when A is a finitely generated (commutative) k-algebra with a (rational) G-action as k-algebra automorphisms? CFG is the acronym used in [22] to describe this type of G group (cohomologically finitely generated). As long as finite numbers generateA^G, whether A is countably generated by k-algebra, G can be called a group scheme. According to [19], all finite group schemes have the same attribute, CFG. Algebraic group schemes over a field and invariants and cohomology rings are discussed further in [25]. As recently proven in [31], an algebraic group G can only possess the characteristic CFG if it also possesses the property F.G. Many reductive groups, including all reductive groups, are CFG subgroups.

Automorphisms of G on a finitely generated R-algebra are only possible when the fixed point subalgebra A^G is likewise an R-algebra. If we're talking about Noetherian rings that are commutative and finitely generated R-algebras, we call this F.G. An A^G algebra in particular, can be generated infinitely. G has the CFG property if the graded R-algebra $H^n(G, A)$ is a finitely generated R-algebra and the Noetherian RG-module homomorphism $H^n(G, M)$ is a Noetherian RG-module homomorphism for any Noetherian commutative ring R and commutative R-algebra A on which G acts as R-algebra automorphisms. G is Cohomological finiteness is a group property with the CFG property (by taking A and R with trivial G-action). Because $H^0(G, A)$ is a quotient of H, A^G

is a finitely generated R-algebra (G, A). In other words, the property CFG entails the property F.G.

It is said that group G has cohomological finiteness if the R-module H^n(G, M) is finitely constructed for any Noetherian multiplicative ring R and RG-module M. which is finitely generated as an R-module.

We find Groups that are finite in cohomological type are known as cohomologically finite groups. Contrary to popular belief, the opposite of this is not true by a result of Evens [15] and Venkov [32]. They showed that if G is finite then G is cohomologically of finite type but a finite group G is not cohomologically finite because cohomology groups H^n(G, R) are infinite dimensions when R is a field with fundamental splitting equal to the order of G.

A mathematical group G has the attribute CFG if and only if the rational action in FG can be applied to it, as per Touzé and van der Kallen [35]. According to the Mumford Conjecture, an algebraic group G can only have the attribute F.G. if it is reducible Touzé and van der Kallen might simply assume that G is a reductive group. In the case of characteristic zero, all rational representations can be reduced to zero. The cohomological dimension of G is zero, which means that F.G. and CFG are accurate. There are no more reasons to examine Frobenius kernels and twists in spectral sequences. Thus we can now focus on the positive characteristic situation.

For finite cohomological dimensions, whether G entails CFG can be asked. A Noetherian ring A and a finitely generated A^G-module M about which H^k(G, M) is a finite number generated A^G-module are required for G to have the attribute CFG. When k is 0, then G has the property FG.

H^k(G, A) is not finitely formed over A^G when the cohomological dimensionality is 1, such as the free group.

4. Virtual finite projective resolution of Groups

Type VFP (virtual finite projective resolution) is applied to a group G when it contains a subgroup of finite index that is an F.P. group. When it comes to VFP's, all F.P.'s are VFP's. There are finitely created abelian groups in all degrees of cohomology groups H^n(G, Z). Groups with nontrivial torsion components can have cohomology groups H^n(G, R) of arbitrarily high degrees. Therefore it's important to remember that these groups are not invariably cohomologically finite. An argument in [24] uses the stratification of cohomological variations in the perspective of elementary abelian p-subgroups. Ian Leary supplied the following rationale. If G is a finite subgroup and Z is a cyclic subgroup of prime order p, then H^n (Z, Fp) is a finitely generated module of H^n(G, Fp). A non-zero image can be found in infinitely many degrees, for example, using H^n(G, Fp), H^n(Z, Fp). If G has a finite VCD, it has a torsion-free subgroup of the finite index. The finite cyclic subgroup Z = (g) is embedded in G/H via the composition G → G/H as a torsion-free normal subgroup H with a finite index (of prime order p). Because the constraint map H^k(G/H, Fp) →H^k((g), F_p) has an unlimited number of degrees, H^k(g, Fp) is never 0 for any degree in H^k(G, Fp). Because H^k(G, Fp) factors into the restriction map, the group of Z-rational points is affected. For every reductive algebraic group, G, GL(n, C) built over the field of rational numbers Q, G(Z), G(Q), GL(n, Z) is of type VFP and usually contains nontrivial torsion members. Other natural groups include the translating class group and the outer predictiveness group. Nontrivial torsion elements can be found in Coxeter groups and Out(Fn) of free groups Fn, but they are still type VFP. It was shown in the preceding section that they are not cohomologically finite.

CONCLUSION

Because of the existence of groups of type F.P, we determine that the groupings in the present paper are cohomologically finite.

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