More on Fuzzy Maximal, Minimal α-Open and α-Closed sets

M. Sankari†
Department of Mathematics
Lekshmipuram College of Arts and Science
Neyyoor, Kanyakumari
Tamil Nadu-629 802, India.
†Corresponding author: sankarisaravanan1969@gmail.com

Abstract

This article is to introduce the notion of fuzzy cleanly α -covered in fuzzy topological spaces. By means of fuzzy maximal α -open sets some properties of fuzzy cleanly α -covered fuzzy topological spaces are obtained and also by means of fuzzy maximal α -closed sets few similar results of a fuzzy topological spaces are investigated. Through fuzzy minimal α -open and fuzzy maximal α -closed sets, two strong fuzzy separation axioms are discussed.

Key words and phrases: Fuzzy minimal α -open set; Fuzzy maximal α -open set; Fuzzy minimal α -closed set; Fuzzy maximal α -closed set;

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1 Introduction and Preliminaries

Since the introduction of fuzzy sets by Zadeh[14], the concept of fuzzy topology introduced by Chang[4] in 1968. Mukherjee[1] introduced and studied some separation axioms through maximal, minimal open and closed sets. The idea of fuzzy minimal and maximal open introduced by Ittanagi and Wali in [5]. The notion of fuzzy minimal and fuzzy maximal open, fuzzy clopen, fuzzy mean open and fuzzy cut-point space studied in [[10, 11, 12, 13]]. In this paper, we study some properties and strong fuzzy separation axioms of fuzzy minimal and fuzzy maximal α -open sets.

Definition 1.1. *A fuzzy set* μ *in a fts is called a fuzzy* α *-open [3] if* $\mu \le int[cl(int\, \mu)]$ *and a fuzzy closed if* $cl[int(cl\, \mu)]$ *.*

Definition 1.2. *A fuzzy topological space X is said to be fuzzy* α *-connected [2] if it has no proper fuzzy* α *-clopen set.*

A fuzzy topological space X is said to be fuzzy α *-disconnected if it is not fuzzy* α *-connected.*

Definition 1.3. *([9]) A proper nonzero fuzzy* α *-open set* λ *of X is said to be a fuzzy maximal* α *-open set if* λ *and* 1_x *are the only fuzzy* α *–open sets containing* λ *.*

Definition 1.4. *([9]) A proper nonzero fuzzy* α *-open set* λ *of X is said to be a fuzzy minimal* α *-open set if* λ *and* 0_x *are the only fuzzy* α *-open sets contained in* λ *.*

Theorem 1.1. ([9]) If λ is a fuzzy minimal α -open set and μ is a fuzzy α -open set, then either $\lambda \wedge \mu = 0X$ or $\lambda < \mu$. If μ is a fuzzy minimal α -open set distinct from λ , then

$\lambda \wedge \mu = 0_x$.

Theorem 1.2. ([9]) If λ is a fuzzy maximal α -open set and μ is a fuzzy α -open set then, either $\lambda \vee \mu = 1_x$ or $\mu < \lambda$. If μ is a fuzzy maximal α -open set distinct from λ , then
$\lambda \vee \mu = 1_x$.

2 Fuzzy Maximal and Fuzzy Minimal α-Open and α-Closed Sets

Definition 2.1. *A fuzzy cover* **K** *of X is said to be a fuzzy minimal* α *-cover if for any*
$\lambda \in$**K** *,* **K** $-\{\lambda\}$ *is not a fuzzy* α *-cover of X . Then* **K** *is said to be a fuzzy minimal*α *-open (resp.* α *-closed) cover if each member of* **K** *is fuzzy* α *-open. (resp.* α *-closed)*

If **K** is a fuzzy minimal α -open cover of *X* , then there do not exist distinct λ, $\mu \in$**K** such that $\mu < \lambda$. Also, suppose a fuzzy α -open cover **K** of *X* consists of two distinct fuzzy α -open sets λ , μ such that $\mu < \lambda$, then **K** is not a fuzzy minimal α -open cover of *X* . Each fuzzy minimal α -open cover of a fuzzy α -compact space is finite and each fuzzy α -open cover of a fuzzy α -compact space has a finite fuzzy minimal α -open cover.

Definition 2.2. *A fuzzy cover* **K** *of X is said to be fuzzy* α *-disconnected if for each* $\lambda \in$**K** *, there exists a* $\mu \in$ **K** *such that* $\lambda \wedge \mu = 0_x$ *.*

Theorem 2.1. A fuzzy minimal α -open cover consists of a fuzzy minimal α -open set is fuzzy α -disconnected.

Proof. Consider **K** be a fuzzy minimal α -open cover of *X* . Let $\lambda \in$**K** be a fuzzy minimal α -open set. As λ is a proper nonzero fuzzy α -open set and **K** is a fuzzy α - cover of *X* , there exists at least one more fuzzy α -open set $\mu \in$**K** . By Theorem 1.1,
we have $\lambda \wedge \mu = 0_x$ or $\lambda \lneqq \mu$. **K** being a fuzzy minimal α -open cover, $\lambda \lneqq \mu$ is not possible.

Corollary 2.2. A fuzzy minimal α -open cover having only fuzzy minimal α -open sets is fuzzy α -disconnected.

We need the following definition when we have atleast two nonzero fuzzy sets in a fuzzy α -open cover of a fuzzy topological space.

Definition 2.3. *A fuzzy topological space X is said to be fuzzy cleanly* α *-covered if each fuzzy* α *-open cover of X has a fuzzy minimal* α *-open cover consisting of exactly two fuzzy* α *-open sets.*

Hence, a fuzzy cleanly α -covered fuzzy topological space is a fuzzy α -compact space.
But the converse need not be true.

Example 2.3. Let $X = \{a, b, c, d\}$. Then fuzzy sets

$$\gamma_1 = \frac{0}{a}+\frac{0}{b}+\frac{0}{c}+\frac{1}{d};\ \gamma_2 = \frac{1}{a}+\frac{0}{b}+\frac{0}{c}+\frac{0}{d};\ \gamma_3 = \frac{0}{a}+\frac{0}{b}+\frac{1}{c}+\frac{0}{d};\ \gamma_4 = \frac{0}{a}+\frac{1}{b}+\frac{0}{c}+\frac{1}{d};$$

$$\gamma_5 = \frac{0}{a}+\frac{1}{b}+\frac{1}{c}+\frac{0}{d};\ \gamma_6 = \frac{1}{a}+\frac{0}{b}+\frac{1}{c}+\frac{0}{d};\ \gamma_7 = \frac{1}{a}+\frac{0}{b}+\frac{1}{c}+\frac{1}{d};\ \gamma_8 = \frac{0}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

$$\gamma_9 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{0}{d}$$

are defined as follows: Consider the fuzzy topology $\tau = \{0_x, \gamma_1, \gamma_4, \gamma_7, 1_x\}$. Hence (X, τ) is fuzzy cleanly α -covered and the fuzzy space having no fuzzy maximal and fuzzy minimal α -open sets.

Example 2.4. From example 2.3, consider the fuzzy topology $\tau_1 = \{0_x, \gamma_1, \gamma_5, \gamma_8, \gamma_9, 1_x\}$ and the fuzzy space (X, τ_1) is fuzzy compact but not fuzzy cleanly α -covered because it is having many fuzzy minimal α -open sets and two fuzzy maximal α -open sets γ_8 and γ_9 which covers 1_x.

Theorem 2.5. If each fuzzy α -open cover of *X* contains a fuzzy maximal α -open set, then *X* is fuzzy cleanly α -covered.

Proof. Assume that **K** be a fuzzy α -open cover of X and ω ∈ **K** be a fuzzy maximal α -open set. Choose if β ∈ **K** is another fuzzy maximal α -open set distinct from ω , then we have $\omega \vee \beta = 1_x$ by Theorem 1.2 . Hence {ω, β} forms fuzzy α -subcover of **K** for X . Let ω be the only fuzzy maximal α -open set in **K** . Now ω being a proper nonzero fuzzy α -open set, there exist at least one more fuzzy α -open set β ∈**K** distinct from ω to cover X . If $\beta = 1_x$, then **K** is a trivial fuzzy α -open cover of X . Hence we assume that $\beta \neq 1_x$. By Theorem 1.2, we have $\omega \vee \beta = 1_x$ or $\beta \subsetneqq \omega$. If for all Ω ∈ **K** , we have $\Omega \subsetneqq \omega$, then **K** cannot be a fuzzy α -open cover of X . Hence it follows that there exists a fuzzy α -open set β ∈ **K** distinct from ω such that $\omega \vee \beta = 1_x$.

Recall that a collection **K** of fuzzy subsets of X is called fuzzy locally finite if each $x\alpha \in X$ has a fuzzy neighborhood meeting only fuzzy finitely many members of **K** . Also if each $\{\Omega k \mid k \in \Lambda\}$ is a fuzzy locally finite collection of fuzzy sets in X , then
$Cl(\bigvee k{\in}\Lambda\ \Omega_k) = \bigvee k{\in}\Lambda Cl(\Omega_k)$.

Theorem 2.6. If $\{\Omega k \mid \Omega{\in}\Lambda\}$ is a collection of distinct fuzzy minimal α -open sets in a fuzzy locally finite space X , then $Cl(\bigvee\Omega{\in}\Lambda\ \Omega_k) = \bigvee\Omega{\in}\Lambda Cl(\Omega_k)$.

Proof. As Ωk is a fuzzy minimal α -open set for each $k \in\Lambda$, we have $\Omega k1 \wedge \Omega k2 = 0_x$ for $k_1, k_2 \in \Lambda$ with $k_1 k_2$. For each $x\alpha \in X$, there exists a finite fuzzy α -open set Ω such that $x\alpha \in\Omega$. Since Ω is a finite fuzzy α -open set and $\Omega k1 \wedge \Omega k2 = 0_X$ for $k_1, k_2 \in \Lambda$ with $k_1 \neq k2$, Ωintersects only finitely many members of $\{\Omega k \mid k \in\Lambda\}$. Therefore, $\{\Omega k \mid \Omega{\in}\Lambda\}$ is fuzzy locally finite. Hence the result follows.

Definition 2.4. *([9]) A proper nonzero fuzzy α -closed γ of X is said to be a fuzzy minimal α -closed set if any fuzzy α -closed set which is contained in γ is* 0_X *or γ .*

Theorem 2.7. ([9]) If γ is a fuzzy minimal α -closed set and ϑ is any fuzzy α -closed set, then either $\gamma \wedge \vartheta = 0_X$ or $\gamma < \vartheta$.

Definition 2.5. *([9]) A proper nonzero fuzzy α -closed γ of X is said to be a fuzzy maximal α -closed set if any fuzzy α -closed set which contains γ is* 1_X *or γ .*

Theorem 2.8. ([9]) If γ is a fuzzy maximal α -closed set and ϑ is a fuzzy α -closed set, then either $\gamma \vee \vartheta = 1_X$ or $\vartheta < \gamma$.*xd*

As Theorem 2.9, Corollary 2.10, Theorem 2.11 and Theorem 2.12 are similar to Theorem 2.1, Corollary 2.2, Theorem 2.5 and Theorem 2.6 , the proofs are omitted.

Theorem 2.9. A fuzzy minimal α -closed cover consists of a fuzzy minimal α -closed set is fuzzy α -disconnected

Corollary 2.10. A fuzzy minimal α -closed cover consists of only fuzzy minimal α -closed sets is fuzzy α -disconnected.

Theorem 2.11. If each fuzzy α -closed cover F of X contains a fuzzy maximal α -closed set γ , then there exists an ϑ ∈ F such that $\gamma \vee \vartheta = 1_X$.

Theorem 2.12. If $\{\Omega k \mid k \in\Lambda\}$ is a collection of distinct fuzzy minimal α -closed sets in a fuzzy locally finite space X , then $Cl(\bigvee k{\in}\Lambda\Omega k) = \bigvee k{\in}\Lambda Cl(\Omega k)$.

Theorem 2.13. Let η, γ be fuzzy α -closed sets in X such that $\eta \wedge \gamma \neq 0_X$. Then $\eta \wedge \gamma$ is a fuzzy minimal α -closed set in (η, F η) if γ is a fuzzy minimal α -closed set in (X, F) .

Proof. Similar to Proof of Theorem 4.11 [9].

3 Two strong fuzzy separation axioms

It is observed that if there exists two fuzzy minimal α -open sets λ, μ in a fuzzy topological space X , then $\lambda \wedge \mu = 0_X$ [5]. By means of this result, we introduce the following two strong fuzzy separation axioms.

Definition 3.1. *A fuzzy topological space X is said to be fuzzy strongly α -regular i*f f *for every singleton in X and every fuzzy α -closed set ϑ in X such that* $x\alpha \leq co\ \vartheta$ *, there exists distinct fuzzy minimal α -open sets λ, μ such that* $x\alpha \leq \lambda$, $\vartheta \leq \mu$ *and* $\lambda \leq co\ \mu$ *.*

Clearly, a fuzzy strongly α -regular topological space is fuzzy α -regular.

Definition 3.2. *A fuzzy topological space X is said to be fuzzy strongly* α *-normal i*f f *for any two* α *-closed sets* γ, ϑ *such that* $\gamma \le co\, \vartheta$ *, there exists two disjoint fuzzy minimal* α *-open sets* λ, μ *such that* $\gamma < \lambda$, $\vartheta \le \mu$ *and* $\lambda \le co\, \mu$

For a subset ω of a fuzzy topological space *X* , we define

$$MinInt(\omega) = \begin{cases} 0_x \text{ if } \omega \text{ contains no fuzzy minimal } \alpha - \text{open set} \\ \bigvee\{ \alpha \mid \alpha \text{ is a fuzzy minimal } \alpha - \text{open set contained in } \omega\} \end{cases}$$

$$MaxCl(\omega) = \begin{cases} 1_X \text{ if } \omega \text{ contained in no fuzzy maximal } \alpha - \text{closed set} \\ \bigwedge\{\gamma \mid \gamma \text{ isa fuzzy maximal } \alpha - \text{closed set containing } \omega\} \end{cases}$$

The union of finitely many distinct fuzzy minimal α -open sets is not fuzzy minimal α -open set and the intersection of finitely many distinct fuzzy maximal α -closed sets is not fuzzy maximal α -closed set. From this point of view, *MinInt*(ω) (resp. *MaxCl*(ω)) may not be fuzzy minimal α -open (resp. fuzzy maximal α -closed).

Example 3.1. From example 2.3, consider the fuzzy topology $\tau_2 = \{0_x, \gamma_2, \gamma_3, \gamma_6, \gamma_8, 1_x\}$. Now *MinInt*($\gamma_7$)= γ_6 which is not fuzzy minimal α -open set and *MaxCl*(γ_4)= γ_4.

Theorem 3.2. For a fuzzy subset ω of *X* , $1_X - MinInt(\omega) = MaxCl(1_X - \omega)$.

Proof. Let us assume that $MinInt(\omega) = 0_X$. This means that ω contains no fuzzy minimal α -open set. If possible, suppose that $1_X - \omega$ contains a fuzzy maximal α –closed set γ . Then $1_X - \gamma$ is a fuzzy minimal α -open set contained in ω , a contradiction. So in
this case we have $1_X - MinInt(\omega) = MaxCl(1_X - \omega)$. Now suppose that ω contains a fuzzy minimal α -open set α . Then $1_X - \omega < 1_X - \alpha$. In addition, $1_X - \alpha$ is a fuzzy maximal
α -closed set. Hence we get $MaxCl(1_X - \omega) 1X$ if and only if $MinInt(\omega) 0_X$. It is easy to see that if $\{\alpha\}$ is a collection of all fuzzy minimal α -open sets contained in ω , then $\{1_X - \alpha\}$ is the collection of all fuzzy maximal α -closed sets containing $1X - \omega$ and vice-versa. Hence we have

$$\begin{aligned} 1_X - MinInt(\omega) &= 1_X - \bigvee \{\alpha \mid \alpha \text{ is fuzzy minimal } \alpha \text{ -open contained in } \omega\} \\ &= \bigwedge \{1_X - \alpha \mid 1_X - \alpha \text{ is fuzzy maximal } \alpha \text{ -closed containing } 1_X - \omega\} \\ &= MaxCl(1_X - \omega) . \end{aligned}$$

Theorem 3.3. For a fuzzy subset ω of *X* , *MinInt*(ω) is fuzzy minimal α -open if and only if ω contains one and only one fuzzy minimal α -open set.

Proof. The sufficiency follows easily. Let *MinInt*(ω) be fuzzy minimal α -open set and ω contains two fuzzy minimal α -open sets α, β . Then, $MinInt(\omega) = \alpha \vee \beta < \omega$. Since $\alpha, \beta < \alpha\vee\beta$ and *MinInt*(ω) is fuzzy minimal α -open , we have $\alpha = \alpha\vee\beta$ and $\beta = \alpha\vee\beta$. $\alpha = \alpha \vee\beta$ implies that $\beta < \alpha$ and $\beta = \alpha \vee\beta$ implies that $\alpha < \beta$. Hence we have $\alpha = \beta$.

Theorem 3.4 is a dual of Theorem 3.3. The proof of the theorem is omitted as the proof is similar to that of Theorem 3.3.

Theorem 3.4. For a fuzzy subset ω of *X* , *MaxCl*(ω) is fuzzy maximal α -closed if and only if ω contained in one and only one fuzzy maximal α -closed set.

Theorem 3.5. A fuzzy topological space *X* is fuzzy strongly α -regular if and only if for each fuzzy α -open set β and each $x\alpha \in \beta$, there exists a fuzzy minimal α -open set λ and a fuzzy maximal α -closed set γ such that $x\alpha \in \lambda < MaxCl(\lambda) < \gamma < \beta$.

Proof. Let X be a fuzzy strongly α -regular space and β be a fuzzy α -open set. For $x\alpha \in \beta$, we obtain by fuzzy strong α -regularity of X two distinct fuzzy minimal α -open sets λ, μ such that $x\alpha \in \lambda$, $1_X - \beta < \mu$. λ, μ being distinct fuzzy minimal α -open sets, we have $\lambda \wedge \mu = 0_X$. Now $\lambda \wedge \mu = 0_X \Rightarrow \lambda < 1_X - \mu$. Since μ is fuzzy minimalα -open, $1X - \mu$ is fuzzy maximal α-closed and so $MaxCl(1_X - \mu) = 1_X - \mu$.

Thus $MaxCl(\lambda) < MaxCl(1X - \mu) = 1_X - \mu < \beta$. Putting $\gamma = 1_X - \mu$, we see that γ is a fuzzy maximal α -closed and $x\alpha \in \lambda < MaxCl(\lambda) < \gamma < \beta$. Conversely, let γ be a fuzzy α -closed set and $x\alpha \in X$ such that $x\alpha\gamma$. As $1_X - \gamma$ is a fuzzy α -open set with $x\alpha \in 1_X - \gamma$, there exist a fuzzy minimal α -open set λ and a fuzzy maximal α -closed set ϑ such that $x\alpha \in \lambda < MaxCl(\lambda) < \vartheta < 1_X - \gamma$. We put $\mu = 1_X - \vartheta$. Then μ is a fuzzy minimal α -open set with $\gamma < \mu$ and $\lambda \wedge \mu = 0_X$.

Theorem 3.6. A fuzzy topological space X is fuzzy strongly α -normal if and only if for a fuzzy α -closed set γ and for a fuzzy α -open set β with $\gamma < \beta$, there exist a fuzzy minimal α -open set λ and a fuzzy maximal α -closed set ϑ such that $\gamma < \lambda < MaxCl(\lambda) < \vartheta < \beta$.

Proof. Similar to the proof of Theorem 3.5.

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