# FUZZY OPEN SETS AND MAPS ON FUZZY BI-TOPOLOGICAL SPACES

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**ABSTRACT.**
In this paper we studied the concept of fuzzy open sets and maps on fuzzy bi-topological space and obtained some significant results in this context with help of various supporting examples.
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**1. INTRODUCTION**
Zadeh [10] has introduced the notion of fuzzy set which is a significant notion in the theory of fuzzy mathematics. Chang [3] has introduced the concept of fuzzy topological space as a generalization of topological space and Kandil [5] introduced fuzzy bi-topological spaces in 1989.
In this paper, we have introduced the concept of fuzzy open sets and maps on fuzzy bi-topological space and verify the results with the help of some counter examples. The basic definitions and concepts of fuzzy open sets in fuzzy bi-topological spaceand notations are discussed in Section 2. In section 3, we studied and introduced the concept of continuous maps on fuzzy bi-topological space and established several results. Finally, Section 4 concludes the paper.

**2. Fuzzy open sets on fuzzy bi-topological Spaces**
**Definition 2.1:** Let$(X, \mathcal{T}_1, \mathcal{T}_2)$ consisting of a universal set X with the fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is called **fuzzy bi-topological space**
**Example 2.1:** Let $X = \{x_1, x_2, x_3\}$ and A and B be fuzzy sets on X defined as, $A = \{(x_1, 0.9), (x_2, 0.8), (x_3, 1)\}$and $B = \{(x_1, 0.7), (x_2, 0.8), (x_3, 5)\}$. Let $\mathcal{T}_1 = \{0, A, 1\}$and $\mathcal{T}_2 = \{0, B, 1\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$is a fuzzy bi-topological space
**Definition 2.2:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - \text{semi} - \text{openset}$ if $A \subseteq \mathcal{T}_j - \text{cl}(\mathcal{T}_i - \text{int}(A))$
**Example 2.2:** Let $X = \{x_1, x_2\}$ and A, B, C and D be fuzzy sets on X defined as, $A = \{(x_1, 0.6), (x_2, 0.7)\}$, $B = \{(x_1, 0.5), (x_2, 0.6)\}$, $C = \{(x_1, 0.5), (x_2, 0.4)\}$ and $D = \{(x_1, 0.6), (x_2, 0.5)\}$. Let $\mathcal{T}_1 = \{0, A, B, 1\}$ and $\mathcal{T}_2 =$

$\{0, C, D, 1\}$ be two fuzzy topologies on X. Now consider, $E = \{(x_1, 0.7), (x_2, 0.8)\}$ be a fuzzy set. Clearly $E \subseteq \mathcal{T}_2 - cl(\mathcal{T}_1 - int(E))$ and hence E isfuzzy $(1,2) - semi - open$ set. Again $E \subseteq \mathcal{T}_1 - cl(\mathcal{T}_2 - int(E))$, implies E fuzzy $(2,1) - semi - open$ set. Hence E is fuzzy $(i, j) - semi - open$ set

**Remark 2.1:** In a fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$, every fuzzy $\mathcal{T}_i - open$ set$(i = 1,2)$ is fuzzy $(i, j) - semi - open$ set but not converse. In Example 2.2, E fuzzy $(i, j) - semi - open$ set but not fuzzy $\mathcal{T}_i - open$ set$(i = 1,2)$

**Definition 2.3:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - semi - closed$ set if $A^c = 1 - A$ is fuzzy $(i, j) - semi - open$ set

**Definition 2.4:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - pre - open$set if $A \subseteq \mathcal{T}_i - int(\mathcal{T}_j - cl(A))$

**Example 2.3:** Let $X = \{x_1, x_2\}$ and A, B, C and D be fuzzy sets on X defined as, $A = \{(x_1, 0.6), (x_2, 0.5)\}$, $B = \{(x_1, 0.3), (x_2, 0.5)\}$, $C = \{(x_1, 0.4), (x_2, 0.4)\}$ and $D = \{(x_1, 0.6), (x_2, 0.5)\}$. Let $\mathcal{T}_1 = \{0, A, C, 1\}$ and $\mathcal{T}_2 = \{0, B, 1\}$ be two fuzzy topologies on X. Clearly $D \subseteq \mathcal{T}_1 - cl(\mathcal{T}_2 - int(D))$ and hence D is fuzzy $(1,2) - pre - open$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.5:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - pre - closed$ set if $A^c = 1 - A$ is fuzzy $(i, j) - pre - open$ set . In Example 2.3, fuzzy set $E = \{(x_1, 0.4), (x_2, 0.5)\}$ is fuzzy $(1,2) - pre - closed$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.6:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - \beta - open$set if $A \subseteq \mathcal{T}_j - cl(\mathcal{T}_i - int\left(\mathcal{T}_j - cl(A)\right))$. In Example 2.3, fuzzy set D is fuzzy $(1,2) - \beta - open$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.7:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - \beta - closed$ set if $A^c = 1 - A$ isfuzzy $(i, j) - \beta - open$ set. In Example 2.3, fuzzy set D is fuzzy $(1,2) - \beta - closed$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Remark 2.2:** In a fuzzy bi-topological space$(X, \mathcal{T}_1, \mathcal{T}_2)$, every fuzzy $(i, j) - pre - open$setis fuzzy $(i, j) - \beta - open$ set but not converse 4282

**Example 2.4:** Let $X = \{x_1, x_2\}$ and A, B and C be fuzzy sets on X defined as, $A = \{(x_1, 0.6), (x_2, 0.3)\}$, $B = \{(x_1, 0.3), (x_2, 0.2)\}$ and $C = \{(x_1, 0.5), (x_2, 0.7)\}$ . Let $\mathcal{T}_1 = \{0, A, 1\}$ and $\mathcal{T}_2 = \{0, B, 1\}$ be two fuzzy topologies on X. Now we observe $C \subseteq \mathcal{T}_2 - cl(\mathcal{T}_1 - int(\mathcal{T}_2 - cl(C))$ which implies that C is fuzzy $(i, j) - \beta - open$ set but it is not fuzzy $(1,2) - pre - open$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.8:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - \alpha - open$ set if $A \subseteq \mathcal{T}_j - int(\mathcal{T}_i - cl\left(\mathcal{T}_j - int(A)\right))$

**Example 2.5:** Let $X = \{x_1, x_2\}$ and A, B, C and D be fuzzy sets on X defined as, $A = \{(x_1, 0.2), (x_2, 0.3)\}$, $B = \{(x_1, 0.5), (x_2, 0.5)\}$, $C = \{(x_1, 0.2), (x_2, 0.4)\}$ and $D = \{(x_1, 0.3), (x_2, 0.4)\}$. Let $\mathcal{T}_1 = \{0, A, B, 1\}$ and $\mathcal{T}_2 = \{0, C, 1\}$ be two fuzzy topologies on X. Clearly $D \subseteq \mathcal{T}_1 - int(\mathcal{T}_2 - cl(\mathcal{T}_1 - int(D)))$ and hence D is fuzzy $(1,2) - \alpha - open$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.9:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called fuzzy $(i, j) - \alpha - closed$ set if $A^c = 1 - A$ isfuzzy $(i, j) - \alpha - open$ set. In Example 2.5, fuzzy set $E = \{(x_1, 0.7), (x_2, 0.6)\}$ is fuzzy $(1,2) - \alpha - closed$ set in $(X, \mathcal{T}_1, \mathcal{T}_2)$

**Definition 2.10:** A fuzzy set A of fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called

- Fuzzy $(i, j) - regular - open$ set if $\mathcal{T}_j - int\left(\mathcal{T}_i - cl(A)\right) = A$

- Fuzzy $(i,j) - \text{regular} - \text{closed set if } \mathcal{T}_j - \text{cl}(\mathcal{T}_i - \text{int}(A)) = A$

**Remark 2.3:** In fuzzy bi-topological space $(X, \mathcal{T}_1, \mathcal{T}_2)$

- Every fuzzy $(i,j) - \text{regular} - \text{open set is fuzzy } \mathcal{T}_j - \text{open but not converse}$
- Every fuzzy $(i,j) - \text{regular} - \text{closed set is fuzzy } \mathcal{T}_j - \text{closed but not converse}$

**3. Fuzzy continuous maps on fuzzy bi-topological Spaces**

**Definition 3.1:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy continuous if $\mathcal{F}: (X, \mathcal{T}_1) \to (Y, \psi_1)$ and $\mathcal{F}: (X, \mathcal{T}_2) \to (Y, \psi_2)$ are fuzzy continuous maps

**Example 3.1:** Let $X = \{x_1, x_2\}$and $Y = \{y_1, y_2\}$. Suppose and A and B are fuzzy sets on X and C and D are fuzzy sets on Y which are defined as, $A = \{(x_1, 0.2), (x_2, 0.1)\}$ , $B = \{(x_1, 0.6), (x_2, 0.7)\}$ , $C = \{(y_1, 0.2), (y_2, 0.1)\}$ and $D = \{(y_1, 0.6), (y_2, 0.7)\}$. Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0, B, 1\}$, $\psi_1 = \{0, C, 1\}$ and $\psi_2 = \{0, D, 1\}$be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$ and $\mathcal{F}(x_2) = y_2$ which is pairwise fuzzy continuous

**Definition 3.2:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy semi-continuous map if inverse image of every $\psi_i - \text{fuzzy open set in Y is fuzzy } (i,j) - \text{semi} - \text{open set in X}$

**Example 3.2:** Let $X = \{x_1, x_2\}$and $Y = \{y_1, y_2\}$. Suppose, A and B are fuzzy sets on X and Y respectively which are defined as $A = \{(x_1, 0.2), (x_2, 0.3)\}$ and $B = \{(y_1, 0.3), (y_2, 0.4)\}$Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0,1\}$, $\psi_1 = \{0, B, 1\}$ and $\psi_2 = \{0,1\}$ be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$ and $\mathcal{F}(x_2) = y_2$ . Clearly $\mathcal{F}^{-1}(B) = \{(x_1, 0.3), (x_2, 0.4)\}$ and $A \subseteq \mathcal{F}^{-1}(A) \subseteq \mathcal{T}_2 - \text{cl}(A) = 1$, which implies $\mathcal{F}^{-1}(B)$ is fuzzy $(1,2) - \text{semi} - \text{open set in X}$. Again $\mathcal{F}^{-1}(0)$ and $\mathcal{F}^{-1}(1)$ are fuzzy $(1,2) - \text{semi} - \text{open set in X}$. Thus the mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is pairwise fuzzy semi-continuous map.

**Definition 3.3:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy pre-continuous map if inverse image of every $\psi_i - \text{fuzzy open set in Y is fuzzy } (i,j) - \text{pre} - \text{open set in X}$

**Example 3.3:** Let $X = \{x_1, x_2, x_3\}$and $Y = \{y_1, y_2, y_3\}$. Suppose and A, B, C and D are fuzzy sets on X and Y which are defined as $A = \{(x_1, 0.5), (x_2, 0.5), (x_3, 0.5)\}$ , $B = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}$ , $C = \{(x_1, 0.1), (x_2, 0.1), (x_3, 0.1)\}$and $D = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.2)\}$ Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0, B, 1\}$, $\psi_1 = \{0, C, 1\}$ and $\psi_2 = \{0, D, 1\}$ be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (X, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$ , $\mathcal{F}(x_2) = y_2$ and $\mathcal{F}(x_3) = y_3$ . Thus the mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is pairwise fuzzy pre-continuous map.



**Definition 3.4:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy irresolute map if inverse image of every fuzzy $(i,j) - \text{semi} - \text{open set in Y is fuzzy } (i,j) - \text{semi} - \text{open set in X}$

**Example 3.4:** Let$X = \{x_1, x_2, x_3\}$and $Y = \{y_1, y_2\}$. Suppose, A and B are fuzzy sets on X and Y respectively which are defined as $A == \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.5)\}$and $B = \{(y_1, 0.5), (y_2, 0.7)\}$. Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0,1\}$, $\psi_1 = \{0, B, 1\}$ and $\psi_2 = \{0,1\}$ be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$ and $\mathcal{F}(x_2) = y_2$ . Thus the mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is pairwise fuzzy irresolute map.

**Definition 3.5:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy α-continuous map if inverse image of every $\psi_i - \text{fuzzy open set in Y is fuzzy } (i,j) - \alpha - \text{open set in X}$

**Example 3.5:** Let $X = \{x_1, x_2, x_3\}$and $Y = \{y_1, y_2, y_3\}$. Suppose and A, B, C, D and E are fuzzy sets on X and Y which are defined as $A = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}$, $B = \{(x_1, 0.5), (x_2, 0.5), (x_3, 0.5)\}$, $C = \{(y_1, 0.4), (y_2, 0.0), (y_3, 0.4)\}$, $D = \{(y_1, 0.5), (y_2, 0.0), (y_3, 0.5)\}$ and $E = \{(y_1, 0.5), (y_2, 0.5), (y_3, 0.5)\}$ Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0, B, 1\}$, $\psi_1 = \{0, C, E, 1\}$ and $\psi_2 = \{0, D, 1\}$ be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$, $\mathcal{F}(x_2) = y_2$ and $\mathcal{F}(x_3) = y_3$. Clearly$\mathcal{F}^{-1}(C) = 0$, $\mathcal{F}^{-1}(D) = 0$ and $\mathcal{F}^{-1}(E) = B$ are fuzzy $\mathcal{T}_2 - \alpha -$ open seton $(X, \mathcal{T}_1, \mathcal{T}_2)$. Thus the mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is pairwise fuzzy α-continuous map.

**Definition 3.6:** A mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is said to be pairwise fuzzy β-continuous map if inverse image of every $\psi_i -$ fuzzy open set in Y is fuzzy $(i, j) - \beta -$ open set in X

**Example 3.6:** Let $X = \{x_1, x_2\}$and $Y = \{y_1, y_2\}$. Suppose and A, B, C and D are fuzzy sets on X and Y which are defined as $A = \{(x_1, 0.6), (x_2, 0.3)\}$, $B = \{(x_1, 0.3), (x_2, 0.2)\}$, $C = \{(y_1, 0.5), (y_2, 0.7)\}$ and $D = \{(x_1, 0.2), (x_2, 0.6)\}$ Let $\mathcal{T}_1 = \{0, A, 1\}$, $\mathcal{T}_2 = \{0, B, 1\}$, $\psi_1 = \{0, C, 1\}$ and $\psi_2 = \{0, D, 1\}$ be two fuzzy topologies defined on X and Y. Then we define a mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (X, \psi_1, \psi_2)$ such that $\mathcal{F}(x_1) = y_1$ and $\mathcal{F}(x_2) = y_2$. Clearly $\mathcal{F}^{-1}(C) = 0$, $\mathcal{F}^{-1}(0)$, $\mathcal{F}^{-1}(C)$ and $\mathcal{F}^{-1}(D)$fuzzy $\mathcal{T}_i - \beta -$ open set on $(X, \mathcal{T}_1, \mathcal{T}_2)$ $(i = 1,2)$. Thus the mapping $\mathcal{F}: (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \psi_1, \psi_2)$ is pairwise fuzzy β-continuous map.

## 4. Conclusion

In this Paper we have studied the concept ofopen sets and maps onfuzzy bi-topological spaces in which many important results have been obtained. Further we have established the relationships with the help of some counter examples.

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