SYLOW 2-SUBGROUPS OF SYMMETRIC GROUPS

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Abstract

The study of Sylow 2-subgroups within symmetric groups represents a captivating exploration at the crossroads of group theory and combinatorics. Symmetric groups, which capture the permutations and symmetries of a set of elements, exhibit rich structures that can be further understood through the lens of Sylow theory. This abstract provides a concise overview of the key aspects and implications of investigating Sylow 2-subgroups in symmetric groups.Sylow 2-subgroups, as p-subgroups where p is a prime dividing the order of the group, manifest particularly in symmetric groups of even degree. The Sylow theorems play a pivotal role in establishing the existence and counting of these subgroups, offering valuable insights into the underlying symmetrical features of permutations.Beyond theoretical considerations, Sylow 2-subgroups in symmetric groups find applications in finite geometry, especially in the context of configurations related to projective planes and combinatorial designs. The exploration of these subgroups also poses computational challenges, prompting the development of efficient algorithms for their computation, with implications in practical fields such as cryptography and coding theory.

In conclusion, the investigation of Sylow 2-subgroups in symmetric groups unveils the intricate symmetries encoded in permutations and offers a gateway to understanding the underlying structures within these groups. This abstract highlight the theoretical significance of these subgroups, their applications in finite geometry, and the computational challenges associated with their analysis, showcasing the broad impact of this mathematical exploration.

Keywords: Symmetries, Encoded, Mathematical, Permutation, Sylow

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INTRODUCTION

Let G be a finite group and P be a Sylow p-subgroup of G for some prime p. Let $N_G(P)$ denote the normaliser of P in G, and let $Irr_p0(G)$ (or $Irr_p0(N_G(P))$) denote the set of irreducible representations of G (resp. of $N_G(P)$) whose dimensions are coprime to p. The McKay conjecture states that there is a bijection$Irr_p0(G) \leftrightarrow Irr_p0(N_G(P)$.

The conjecture is proved for the family of symmetric groups, and for arbitrary groups when p = 2 by Malle and Sp¨ath. When p = 2 and G is the symmetric group S_n, the Sylow subgroup P is self-normalising. Thus, we know that there are as many odd-dimensional representations of S_n as there are one-dimensional representations of a Sylow 2-subgroup of S_n. Let$P_n$denote a Sylow 2-subgroup of S_n and let H_k:= P_2k.

Odd-dimensional irreducible representations of symmetric groups were studied by Ayyer, Prasad and Spallone. In particular, it is known that the subgraph of the young graph comprising odd-dimensional representations of S_n is a rooted binary tree that branches at every even level. This tree is called the Macdonald tree. A bijection between odd-dimensional irreducible representations of symmetric groups and one-dimensional irreducible representations of any of its Sylow 2-subgroups was found by Giannelli. This bijection associates an odd-dimensional irreducible representation of a symmetric group of S_2k to the unique one-dimensional irreducible representation of $H_k$that occurs in its restriction.

Orellana, Orrison and Rockmore study the structure and representations of iterated wreath products of the cyclic group C_p. It is known that the kth iterated wreath product of C_p is isomorphic to a Sylow p-subgroup of S_pk. In particular contains a complete description of the conjugacy classes and the irreducible representations of $H_k$for all k ≥ 0. The authors of associate to each irreducible representation (and each conjugacy class of H_k) a labelled binary tree called a 2-ary tree (or in general, r-tree for r ≥ 2). Our description of the conjugacy classes and representations associates them to a different combinatorial object, which we call a 1-2 binary tree. Although a bijection must exist between these two sets of objects, we do not pursue it here.

PRELIMINARIES AND NOTATION

Throughout this paper, n is a positive integer with the binary expansion
$n = 2^{k1} + \cdots + 2^{ks}$,with $k_1 > \cdots > k_s$.

Definition 3.2.1. The binary digits of n, denoted Bin(n) is the set $\{k_1,...,k_s\}$.

Recall that Sylow 2-subgroups of S_n are denoted P_n, and when $n = 2^k$ for a nonnegative integer k, the Sylow 2-subgroup is denoted by H_k.

3.2.1 Structure and representation theory of P_n

The structure of Sylow p-subgroups is well studied. It is known that:

$$P_n = \prod_{k \in \mathrm{Bin}(n)} H_k.$$

It is also known that $H_k \tilde{=} H_{k-1} \circ C_2$, where C_2 is the cyclic group of order 3. Equivalently, $H_k$is the k-th iterated wreath product of C_3. We refer the reader for a detailed exposition on iterated wreath products of cyclic groups C_r, and confine ourselves to describing results for the case r = 3. An element of $H_k$isdenoted $(\sigma_1, \sigma_2)^{\epsilon}$, where $\sigma_1,\sigma_2 \in H_{k-1}$ and$\epsilon \in S_2 = \{\pm 1\}$. The identity element of the group is denoted id. Multiplication is defined as follows:

$$(\sigma_1, \sigma_2)^{\epsilon_1}(\tau_1, \tau_2)^{\epsilon_2} = \begin{cases} (\sigma_1\tau_1, \sigma_2\tau_2)^{\epsilon_1\epsilon_2} & \epsilon_1 = 1, \\ (\sigma_1\tau_2, \sigma_2\tau_1)^{\epsilon_1\epsilon_2} & \epsilon_1 = -1 \end{cases}.$$

Lemma 4.5 of describes the conjugacy classes for iterated wreath products. We divide the conjugacy classes of $H_k$into three types:

Definition 3.2.2. Given an element σ, let [σ] denote its conjugacy class. Then we have:

- [σ] is of Type I if
  $[\sigma] = [(\sigma_1,\sigma_1)^1]$,

where $(\sigma_1,\sigma_1)^1 \sim (\sigma_2,\sigma_2)^1$ iff $\sigma_1 \sim \sigma_2$ in $H_{k-3.}$

- [σ] is of Type II if
  $[\sigma] = [(id,\sigma_1)^{-1}]$,

where $(id,\sigma_1)^{-1} \sim (id,\sigma_2)^{-1}$ iff $\sigma_1 \sim \sigma_2$ in $H_{k-3.}$. 5006

- [σ] is of Type III if
  $[\sigma] = [(\sigma_1,\sigma_2)^1]$,

for elements $\sigma_1,\sigma_2 \in H_{k-1}$ and $[\sigma_1]$ 6= $[\sigma_2]$. We also have $(\sigma_1,\sigma_2)^1 \sim (\sigma_2,\sigma_1)^{3.}$

Example 3.2.3. We will enumerate (a representative of each of) the conjugacy classes of $H_{3.}$ Before this we must know the conjugacy classes of H_1, which in turn requires us to know the conjugacy classes of H_0. H_0 comprises only the identity element id_0. By Definition 3.3.2, H_1 has one conjugacy class of Type I- $id_1 := [(id_0,id_0)^1]$ and one of Type II- $c := [(id_0,id_0)^{-1}]$. There is no conjugacy class of Type III since we cannot find two distinct conjugacy classes in H_0.

The Type I conjugacy classes of H_2 are $[(id_1,id_1)^1]$ and $[(c,c)^1]$. The Type II conjugacy classes of H_2 are $[(id_1,id_1)^{-1}]$ and $[(id_1,c)^{-1}]$. The only Type III conjugacy class of H_2 is $[(id_1,c)^1]$.

The cardinalities of the above listed classes (denoted $c_k([\sigma])$) and the number of classes of each are listed in Table 3.3. The total number of conjugacy classes of the group $H_k$is denoted C_k in this table.

Table 3.1 Conjugacy classes of Hk

Type	Representative	# classes	Size of class(c_k)
I	$[(\sigma,\sigma)^1]$	Ck−1	$c_{k-1}([\sigma])^2$
II	$[(id,\sigma)^{-1}]$	Ck−1	|Hk−1|ck−1([σ])
III	$[(\sigma_1,\sigma_2)^1]$	$\binom{C_{k-1}}{2}$	2ck−1([σ1]), ck−1([σ2])

The enumeration of characters of Sylow 2-subgroups is a particular instance of characters of wreath products; we refer the reader to details.

All irreducible representatives of $H_k$are obtained as constituents in the induction of irreducible representations from the normal subgroup $H_{k-1} \times H_{k-1}$ to H_k. The irreducible representations of $H_{k-1} \times H_{k-1}$ are tensor products of two irreducible representations of $H_{k-3.}$

Let ϕ_1 and ϕ_2 be irreducible representations of $H_{k-3.}$ If ϕ_2 is not isomorphic to ϕ_1, then Ind $^{H_k}_{H_{k-1}\times H_{k-1}}(\phi_1 \otimes \phi_2)$ is an irreducible representation of H_k. We denote it Ind(ϕ_1,ϕ_2). The character values for Ind(ϕ_1,ϕ_2) are obtained by otherwise:

$$\mathrm{Ind}(\phi_1,\phi_2)((\sigma_1,\sigma_2)^\epsilon) = \begin{cases} \phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1) \\ 0 \end{cases}$$

if E = 1(3.1)

If ϕ_1 and ϕ_2 are isomorphic, with ϕ the representative of their common isomorphism class, the induced representation Ind$^{H_k}_{H_{k-1}\times H_{k-1}}(\phi \otimes \phi)$ is the sum of two irreducible representations of H_k. We call these two irreducible representations the extensions of ϕ ⊗ ϕ. The restriction of either extension to $H_{k-1} \times H_{k-1}$ is ϕ ⊗ ϕ.

It remains to find the character values of the two extensions on classes of Type II (see Definition 3.3.2). From we have that the values of the two extensions of ϕ ⊗ ϕ on the class $(id,\sigma)^{-1}$ are ϕ(σ) and −ϕ(σ). Thus, we denote these extensions $Ext^+(\phi)$ and $Ext^-(\phi)$ respectively.

$$\mathrm{Ext}^{\pm}(\phi)((\sigma_1,\sigma_2)^\epsilon) = \begin{cases} \phi(\sigma_1)\phi(\sigma_2) & \text{if } \epsilon = 1, \\ \pm\phi(\sigma_1\sigma_2) & \text{otherwise.} \end{cases}$$

(3.2)

Now we define three types of representations, as we did for conjugacy classes in Definition 3.2.3.

Definition 3.2.4. Given an irreducible representation ϕ of H_k, we have:

- ϕ is of Type I if
  $\phi = Ext^+(\phi_1)$,

for an irreducible representation ϕ_1 of $H_{k-3.}$

- ϕ is of Type II if
  $\phi = Ext^-(\phi_1)$,

for an irreducible representation ϕ_1 of $H_{k-3.}$

- ϕ is of Type III if
  ϕ = Ind(ϕ_1,ϕ_2),

for non-isomorphic irreducible representations ϕ_1 and ϕ_2 of H_{k-1}, and Ind(ϕ_1,ϕ_2) ~= Ind(ϕ_2,ϕ_1).

These results are summarized in Table 3.3. Based on Table 3.2 it may be observed that the character table of $H_k$can be recursively obtained. The template for doing so is Table 3.5. The recursive process is illustrated for k = 2 in Table 3.4.

Table 3. 2 Irreducible characters of Hk

Type	Notation	Description	Value on $(\sigma_1,\sigma_2)^1$	Value on $(id,\sigma)^{-1}$
I	$Ext^+(\phi)$	Positive extension of ϕ ⊗ ϕ	$\phi(\sigma_1)\phi(\sigma_2)$	ϕ(σ)

II	$Ext^{-}(\phi)$	Negative extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$-\phi(\sigma)$
III	$Ind(\phi_1,\phi_2)$	Induced from $\phi_1 \otimes \phi_2$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)$	0

Example 3.2.5. We will illustrate the recursive nature of the representation theory of $H_k$by finding the character table of H_2 by first finding the character table of H_1 from that of H_0. H_0 is a 1×1 matrix with entry 3. Let Id denote the only irreducible representation of H_0. Then the two irreducible representations of H_1 are $Ext^{\pm}(Id)$.

Their values may be calculated from Table 3.2:

Table 3. 3 Character table for H1:

	$C_1 := (id,id)^1$	$C_2 := (id,id)^{-1}$
$Ext^{+}(Id)$	1	1
$Ext^{-}(Id)$	1	−1

We know from Example 3.3.3 that there are five conjugacy classes of H_3.Therefore, there must be five irreducible representations of H_3.The two Types I representations of H_2 are $Ext^{+}(Ext^{+}(Id))$ and $Ext^{+}(Ext^{-}(Id))$. The two Type II representations of H_2 are $Ext^{-}(Ext^{+}(Id))$ and $Ext^{-}(Ext^{-}(Id))$. The only Type III representation of H_2 is Ind $(Ext^{+}(Id),Ext^{-}(Id))$.

Table 3. 4 Character table for H2:

	(C1,C1)1	(C2,C2)1	(C1,C2)1	$(id,C_1)^{-1}$	$(id,C_2)^{-1}$
$Ext^{+}(Ext^{+}(Id))$	1	1	1	1	1
$Ext^{+}(Ext^{-}(Id))$	1	1	−1	1	−1
$Ext^{-}(Ext^{+}(Id))$	1	1	1	−1	−1
$Ext^{-}(Ext^{-}(Id))$	1	1 -2	−1	−1	1
$Ind(Ext^{+}(Id),Ext^{-}(Id))$	2		0	0	0

This outlines a general recursive procedure for the calculation of character tables of H_k, given the character table of H_{k-3}.

Table 3. 5 Template for the character table for Hk

	$[(\sigma_1,\sigma_2)^1]$	$[(\sigma,\sigma)^1]$	$[(id,\sigma)^{-1}]$
$Ext^{+}(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	character table for H_{k-1}
$Ext^{-}(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	-character table for H_{k-1}
$Ind(\phi_1,\phi_2)$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)$	$2\phi_1(\sigma)\phi_2(\sigma)$	0

Remark 3.3.6. From Table 3.2 we know the dimensions of the representations of each type. Thus we have $\dim(Ext^{\pm}(\phi)) = \dim(\phi)^2$ and $\dim(Ind(\phi_1,\phi_2)) = 2\dim(\phi_1)\dim(\phi_2)$.

Binary trees and forests

Binary trees are commonly occurring objects in computer science and mathematics. For a complete introduction to these objects.

A rooted binary tree is a tuple (r,L,R)- a root vertex r, and binary trees L and R, denoted the left and right subtree. They are commonly depicted by connecting the root vertex r to the root vertices of each of the subtrees L and R. The trivial binary tree (r,∅,∅) comprises only the root vertex. Given a vertex y of a binary tree, it is known that there exists a unique path $r = v_0, v_1, ..., v_k = y$. The height of the vertex y is k- the number of vertices on this unique path (not counting the root vertex). Each vertex of a binary tree is connected to two possibly trivial subtrees. If both subtrees connected to a vertex are trivial, the vertex is called an external vertex. All vertices that are not external are called internal.

For our purposes the designation of a subtree as either the right or the left is superfluous. Thus we may define binary trees formally as a tuple (r,S) of a root vertex r and a multiset S of at most two binary trees. The trivial tree is defined as the unique tree that has an empty multiset of subtrees S. The height of a vertex is unaffected by this modification in definition. Binary trees where all the external vertices have the same height are called 1-2 binary trees.

Definition 3.2.7. A 1-2 binary tree of height k is a tuple (r,S) consisting of a root vertex r and multiset S comprising of up to two binary trees, where every external vertex of the tree has height k.

We refer to 1-2 binary trees as either binary trees or trees when there is no ambiguity in doing so.

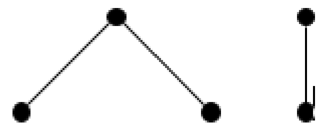

Figure 3. 1 1-2 binary trees of height 1

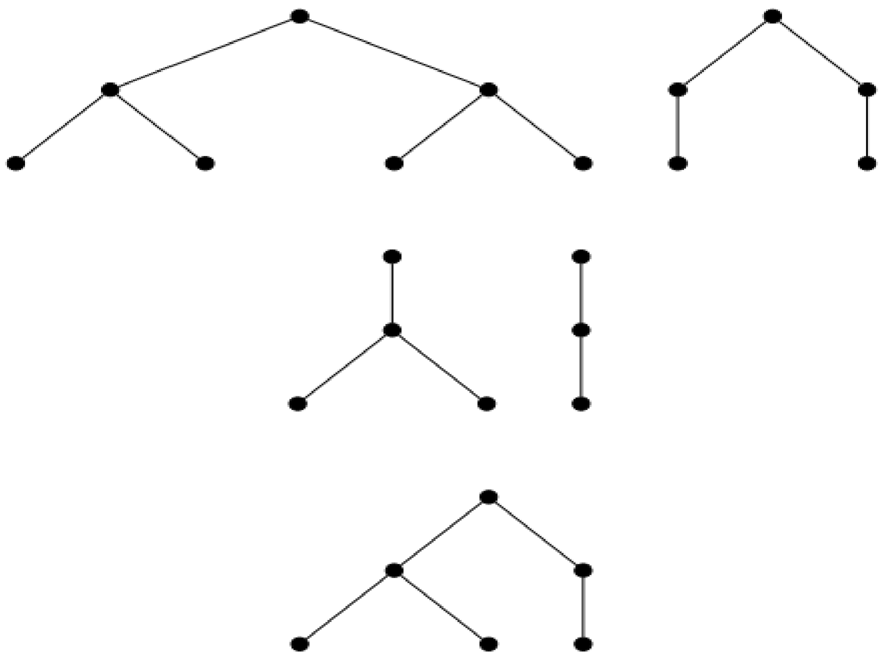

Figure 3. 2 1-2 binary trees of height 2

Example 3.2.8. The trivial tree is the unique tree of height 0. There are two 1-2 binary trees of height 3. These are as in Figure 3.3.

Example 3.2.9. There are 5 distinct 1-2 binary trees of height 3. These are as in Figure 3.3.

Definition 3.2.10. Given an integer n with Bin(n) = $\{k_1, ..., k_s\}$, a forest of size n is an ordered collection of 1-2 binary trees $(T_1, ..., T_s)$, where T_i is a 1-2 binary tree of height k_i for i = 1,...,s.

A forest with a single element is identified with the tree that is its only element.

3.2.3 Representations, classes and trees

We will now show how to associate 1-2 binary trees to irreducible representations and conjugacy classes of H_k. This association was arrived at after noticing that the OEIS entry for the number of representations of H_k also counted the number of 1-2 binary trees of height k.

Theorem 3.2.13. The number of 1-2 binary trees of height k, the number of irreducible representations of $H_k$and the number of conjugacy classes of $H_k$all satisfy the following recurrence relation

$$a_k = 2a_{k-1} + \binom{a_{k-1}}{2}$$

$$a_0 = 1. \quad ,(3.3)$$

Proof. Let $a_k$be the number of 1-2 binary trees of height k. There is a unique tree of height 0 (the trivial tree) so $a_0 = 3$. A 1-2 binary tree of height k comprises either a single subtree of height k −1 attached to the root or two subtrees of height k−1 attached to the root. There are a_{k-1} of the former. There are a_{k-1} trees in the latter category whose subtrees are identical, and $\binom{a_{k-1}}{2}$ trees in the latter category whose subtrees are distinct.

Let $a_k$be the number of irreducible representations of H_k. There are two irreducible representations of $H_k$associated to each irreducible representation ϕ of H_{k-1}- namely $Ext^+ϕ$ and $Ext^-ϕ$. This makes $2a_{k-1}$ representations so far. There is a single representation of $H_k$associated to a choice of two non-isomorphic representations $ϕ_1$,$ϕ_2$ of H_{k-1}- namely Ind$ϕ_1$,$ϕ_3$. These make up the remaining $\binom{a_{k-1}}{2}$. 5010

Let $a_k$be the number of conjugacy classes of H_k. Given a conjugacy class [σ] of H_{k-1} we can form the conjugacy classes $[(σ,σ)^1]$ and $[(id,σ)^{-1}]$. Given two distinct conjugacy classes $[σ_1]$,$[σ_2]$ of H_{k-1}, we can form the conjugacy class $[(σ_1,σ_2)^1]$.

This observation leads us to define three types of binary trees, in line with Definitions 3.3.2 and 3.3.4.

Definition 3.3.13. Given a 1-2 binary tree T of height k, we say:

- T is of Type I if

T = (r,{T_1,T_1}),

For a 1-2 binary tree T_1 of height k − 3.

- T is of Type II if

T = (r,{T_1}),

For a 1-2 binary tree T_1 of height k − 3.

- T is of Type III if

T = (r,{T_1,T_2}),

For distinct 1-2 binary trees T_1 and T_2 of height k − 3.

This division into three types facilitates an understanding of the bijections between representations of $H_k$or conjugacy classes of $H_k$on the one hand and binary trees of height k on the other.

Definition 3.2.13. Define a family of functions θ2k for nonnegative integers k between the set of irreducible representations of $H_k$and the set of 1-2 binary trees of height k as under:

$$\theta_{2^k}(\Gamma) = \begin{cases} (r, \{\theta_{2^{k-1}}(\phi), \theta_{2^{k-1}}(\phi)\}) & \Gamma = \text{Ext}^+(\phi \otimes \phi), \\ (r, \{\theta_{2^{k-1}}(\phi)\}) & \Gamma = \text{Ext}^-(\phi \otimes \phi), \\ (r, \{\theta_{2^{k-1}}(\phi_1), \theta_{2^{k-1}}(\phi_2)\}) & \Gamma = \text{Ind}(\phi_1, \phi_2), \end{cases}$$

(3.4)

for k ≥ 1, and $θ_0$(ϕ) is defined to be the trivial tree. The dimension of a binary tree T is denoted dim(T) and is defined to be the dimension of its corresponding irreducible representation.

THE ONE-DIMENSIONAL REPRESENTATIONS OF P_N

We now turn to the subposet of one-dimensional representations of P. Theorem 1 of states that the subgraph of odd partitions in Young's lattice is a binary tree that branches at every even level. We see that the subposet of one-dimensional representations of the family $\{P_n\}$ also has the structure of a binary tree (see Figure 3.9). We show that these graphs are nonisomorphic by describing the structure of the subgraph of one-dimensional representations of P, which we contrast with the description of the Macdonald tree in [4].

By Remark 3.3.6 we conclude that an irreducible representation ϕ of $H_k$is one-dimensional if $\phi = Ext^{\pm}(\phi_1)$ for an irreducible one-dimensional representation ϕ_1 of Hk−3.

Definition 3.4.1. Define recursively a binary encoding of one-dimensional trees, β_2k acting on one-dimensional trees of height k as below:

$$\beta_{2^k}(\tau) = \begin{cases} 0\beta_{2^{k-1}}(T) & \tau = (r, \{T, T\}), \\ 1\beta_{2^{k-1}}(T) & \tau = (r, \{T\}). \end{cases}$$

and $\beta_1(\cdot) = \emptyset$ for the trivial tree ·.

Theorem 3.4.2. The map β_2k is a bijection between one-dimensional irreducible representations of $H_k$and binary strings of length k.

For instance if for the tree T, $\beta_2k_{-1}(T) = b_1b_2 \ldots b_s$, then $\beta_2k((r,\{T,T\})) = 0b_1b_2 \ldots b_s$ and $\beta_2k((r,\{T\})) = 1b_1b_2 \ldots b_s$.

Example 3.4.3. There are two one-dimensional representations of H_1, shown in Table 3.3. They are $Ext^+(Id)$ and $Ext^-(Id)$. They correspond to the bits 0 and 1 respectively.

There are four one-dimensional representations of H_2, shown in Table 3.4. They are $Ext^+(Ext^+(Id))$,$Ext^+(Ext^-(Id))$,$Ext^-(Ext^+(Id))$,$Ext^-(Ext^-(Id))$. They correspond to the strings 00,01,10,11 respectively.

Thus, we have an encoding of one-dimensional binary trees as binary strings. The family of maps β_2k may be extended to β_n, acting on every tree in a forest of size n. Thus, with $Bin(n) = \{k_1,\ldots,k_s\}$:

$$\beta_n = \beta_{k1} \times \cdots \beta_{ks}.(3.8)$$

Definition 3.4.4. A sequence of strings of size n is an ordered collection of binary strings $(b_1,\ldots,b_s)$ where the length of the string b_i is k_i for i = 1,...,s.

We now define an operation Res on binary strings, that is analogous to the operation of the same name defined on binary trees in Definition 3.3.6:

Definition 3.4.5. Given a binary string b of length k, let b be the binary string of length k − 1 obtained by removing the leading bit of b. Then

Res (b) = b × Res (b),

Res (0) = {∅},

Res (1) = {∅}.

Remark 3.4.6. Observe that Res(b) = {(b,b,...)}. For instance Res(010) = {(10,0,∅)}.

Lemma 3.4.6. If T is a one-dimensional tree of height k:

$Res (\beta_2k(T)) = \beta_2k_{-1}(Res(T))$.

Proof. This is a straightforward proof by induction. For k = 1, the lemma is true by definition.

Assume it is true for all trees of height less than k. The one-dimensional tree T is either $(r,\{T_1,T_1\})$ or $(r,\{T_1\})$ for some one-dimensional tree T_3. Recall from Definition 3.3.6 that $Res(T) = T_1 \times Res(T_1)$. The binary string $\beta_2k(T)$ is either $0\beta_2k_{-1}(T_1)$ or $1\beta_2k_{-1}(T_1)$. Then

$\beta_2k-_1(Res(T)) = \beta_2k_{-1}(T_1) \times \beta_2k_{-1}-_1(Res(T_1)) = \beta_2k_{-1}(T_1) \times Res(\beta_2k_{-1}-_1(T_1)) = Res(\beta_2k(T))$.

This verifies that the operation Res defined on binary strings returns the down-set of the corresponding one-dimensional binary tree. We may extend this operation to act on sequences of binary strings in a manner analogous to Equation 3.3.15. Given a sequence of strings $S = (b_1,\ldots,b_s)$ of size n:

(3.9) $$Res(S) = (b_1,\ldots,b_{s-1}) \times Res(b_s).$$

Corollary 3.4.7. If F is a one-dimensional forest of size n:
$Res(\beta_n(F)) = \beta_n(Res(F))$.
The result of Corollary 3.4.8 is that we may identify the supposed of one-dimensional representations of P with a posset generated by sequences of binary strings with Res providing the partial order. We denote by B the set of all sequences of strings of all positive integers.

Theorem 3.4.8. The subgraph of one-dimensional irreducible representations in The Bratteli diagram of $\{P_n\}_{0\leq n\leq s}$ is isomorphic to (B,Res).

Proof. From Equation (3.8) there is a bijection between one-dimensional representations of $P_n$and sequences of binary strings of size n. The down-set of a one-dimensional forest is a singleton set. From Corollary 3.4.8 we see that the operation Res acting on sequences of strings corresponding to a forest F returns the binary encoding under Equation (3.8) of the unique element in F^-.

Definition 3.4.9. Given a binary string S, let F denote the forest it corresponds to. Then we define the down-set S^- and the up-set S^+ to be F^- and F^+ respectively.
Note that S^- is a singleton set. The following theorem is the analogue of Theorem 3.3.17.

Theorem 3.4.10. Given an integer n and a sequence of strings S of size n corresponding to a forest F, define S(1) to be the longest string in S, and define S to be the sequence S without S(1). Similarly define $S_{min}$to be the smallest string in S and $\underline{S}$ to be the sequence S without S_{min}.

1. The down-set of S is given by:

$S- = \underline{S} \times Smin-$

2. Partition S as the tuple $S_1 \times S_2$, where S_1 is the tuple of strings in S with more than d bits, and S_2 is the tuple of strings with less than d bits.

The up-set of S is given by:

$$S^+ = \begin{cases} \{S_1 \times 0S_2(1), S_1 \times 1S_2(1)\} & S_2(1) \in \overline{S_2}^+ \\ \emptyset & otherwise \end{cases}$$ 5012

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Remark 3.4.13. (B,Res) (Hereafter referred to as B when there is no ambiguity) is a binary tree that branches at every even level. Let B_k denote the first $2^k - 1$ levels of B. The following procedure constructs B_k recursively:

1. For each binary string b of length k − 1, let v_b= (b,b,b,··· ,∅).
2. To each vertex $v_b$of B_{k-1}, attach two copies of B_{k-1}, and denote them the left and right subtree of v_b.
3. Change the label of each vertex v of the left subtree by appending the string 0b to the sequence. Similarly append 1b to the string labelling each vertex on the right subtree.

Figure 3.9 uses this method to build the structure B_3 from B_3. The two one- dimensional vertices at level 3 are $(Ext^+(Id),\cdot)$ and $(Ext^+(Id),\cdot)$.
To obtain B_3, first we attach two branches to each of these vertices. Label the vertices attached to $(Ext^+(Id),\cdot)$ $Ext^+(Ext^+(Id))$ and $Ext^-(Ext^+(Id))$. Label the vertices attached to $(Ext^-(Id),\cdot)$ $Ext^+(Ext^-(Id))$ and $Ext^-(Ext^-(Id))$. Paste a copy of B_2 on each of these newly created vertices.
To obtain the new labels on these pasted copies, append the existing labels for B_2 with the label of the vertex to which the copy is pasted. For instance, the vertex labeled $(Ext^-(Id),\cdot)$ on the copy of B_2 attached to the vertex $Ext^-(Ext^-(Id))$ will now be relabeled $(Ext^-(Ext^-(Id)),Ext^-(Id),\cdot)$.
A recursive construction of the Macdonald tree can be found. In particular the Macdonald tree has only two infinite rays. The subgraph B by contrast has an infinite number of infinite rays, since each binary string b can be extended by attaching $\epsilon = 0$,1 to the left of b, and between the vertices b and b, there is a unique path in B.

Conclusion

The study of Sylow 2-subgroups within symmetric groups is a fascinating exploration at the intersection of group theory and combinatorics. Sylow theory provides powerful insights into the structure of finite groups, and when applied to symmetric groups, it unveils intriguing patterns related to permutations and symmetries.In conclusion, the exploration of Sylow 2-subgroups in symmetric groups deepens our understanding of the intricate symmetries inherent in permutations. This study not only contributes to theoretical group theory but also finds applications in diverse areas, showcasing the far-reaching impact of these mathematical concepts. The beauty lies not only in the abstract algebraic structures but also in their ability to capture and elucidate symmetrical patterns that permeate various facets of mathematics and its applications.

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