On the Structures of δ Near-Rings

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ABSTRACT

If, in a ring (N, +, ·) we ignore the commutativity of '+' and one of the distributive laws, (N, +, ·) becomes a Near-Ring. If we do not stipulate the left distributive law, (N, +, ·) is a right Near-Ring. This research aims to introduce the concept of δ Near-Ring. There exists $x \in M$ such that $xyN = xNy^2$ for all *y* in N is called δ Near-Ring. The element wise characterization for δ Near-Ring will be investigated and shall establish theorems and properties in this Near-Ring.

Mathematics Subject Classification: 16Y30
Keywords: δ Near-Ring, near-field.

DOI Number: 10.48047/NQ.2022.20.9.NQ44940 **Neuroquantology 2025; 23(1): 8094-8100**

1 Introduction

A right near-ring is a non-empty set N together with two binary operations "+"and "·" such that (1) (N, +) is a group. (2) (N, ·) is a semi-group and (3) $(n_1 + n_2)n_3 = n_1 n_3 + n_2 n_3$ for all $n_1, n_2, n_3 \in N$.

Throughout this paper N stands for a right near-ring. $(N, +, \cdot)$ with at least two elements and '0' denotes the identity element of the group (N, +) Obviously, 0n = 0 for all n in N. N is said to be zero-symmetric if n0 = 0 for all n in N. As in [2], a subgroup of $(M, +)$ of $(N, +)$ is called an N-subgroup of N if $NM \subset M$ and an invariant N subgroup of N if, in addition, $MN \subset M$. In [5], N is defined to be pseudo commutative if $xyz = zyx$ for all x, y, z in N. The concept of a mate function in N has been introduced in [4] with a view to handling the regularity structure with considerable ease. A map 'f' from N into N is called (i) a mate function for N if $x = xf(x)x$, (ii) a P_3 mate function, if, in addition, $xf(x) = f(x)x$ for all x in N. A near-ring N is said to have property P_4 if for all ideals I of N and for all $x, y \in N, xy \in I$ implies $yx \in I$. By identity 1 of N, we mean only the multiplicative identity of N.

Basic concepts and terms used but left undefined in this paper can be found in [2].

2 Notations

(i) E denotes the set of all idempotent of N. (e in N is called an idempotent if $e^2 = e$)

(ii) L denotes the set of all nilpotent of N. (a in N is nilpotent if $a^k = 0$ for some positive integer k.)
(iii) $N_0 = \{n \in N / n0 = 0\}$ - zero-symmetric part of N.
(iv) $N_d = \{n \in N / n(x+y) = nx + ny$ for all x, y in $N\}$ - set of all distributive element of N.
(v) $C(N) = \{n \in N / nx = xn$ for all x in $N\}$ - centre of N.

3 Preliminary Results

We freely make use of the following results and designate them as **R(1)**, **R(2)**....etc.

R(1) N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals of N is again non-zero (Theorem 1.60, p.25 of [2])
R(2) N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N (Problem 14, p.9 of [3]).
R(3) If f is a mate function for N, then for every x in N, $xf(x)$, $f(x)x \in E$ and $Nx = Nf(x)x$, $xN = xf(x)N$ (Lemma 5 of [4]).
R(4) If $L = \{0\}$ and $N = N_0$, then (i) $xy = 0 \Rightarrow yx = 0$ for all x, y in N.
(ii) N has Insertion of factors property- IFP for short- i.e for x, y in N, $xy = 0 \Rightarrow xny = 0$ for all n in N. If N satisfies (i) and (ii) then N is said to have $(*$, IF P) (Lemma 2.3 of [4])
R(5) N has strong IFP if and only if for all ideals I of N and for all $x, y, n \in N, xy \in I$ implies $xny \in I$.
R(6) Any pseudo commutative near-ring with a right identity is weak commutative ((i.e). $xyz = zyx$ for all x, y, z in N [2]) (Proposition 2.9 of [5])
R(7) A zero-symmetric near-ring N is a near-field if $N_d \neq \{0\}$ and for all $n \in N - \{0\}, N_n = N$ (Theorem 8.3, p.249 of [2]).
R(8) If N is zero symmetric and reduced, then $abe = aeb$ for all $a, b \in N$ for all $e \in N$.

4 Definition and Examples

In this section we define δ Near-Ring and give certain examples of this new concept.
Definition 4.1 Let N be a right Near-ring. A Near-ring N is called δ Near-Ring if there exists $x \in N$ such that $xyN = xNy^2$ for all y in N.

Example 4.2 (a) The near-ring (N, +, ·) defined on the Klein's four group $N = \{0, a, b, c\}$ where multiplication is defined as per scheme 6, p.408, Pilz[2].

·	0	a	b	c
0	0	0	0	0
A	0	b	c	a
B	0	c	a	b
C	0	a	b	c

Is a δ Near-Ring of N.
(b) The near-ring (N, +, ·) where (N, +) is the group of integers modulo 5 and '·' defined as per scheme 6, p.408, Pilz[2].

·	0	1	2	3	4
0	0	0	0	0	0
1	0	0	4	1	0
2	0	0	3	2	0
3	0	0	2	3	0
4	0	0	1	4	0

eISSN: 1303-5150

is not a δ Near-Ring. Since $12N \neq 1N2^2$

(c) Let (N, +) be the group of integers modulo 6. We '·' defined as per scheme 36, p.408, Pilz[2].

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	2	0	4	2
2	0	2	4	0	2	4
3	0	0	0	0	0	0
4	0	4	2	0	4	2
5	0	2	4	0	2	4

This near-ring (N, +, ·) is a zero-symmetric δ Near-Ring with no identity.

5 Properties of δ **Near-Ring**

In this section, we study some of the important properties of δ Near-Ring

Proposition 5.1 If N is δ Near-Ring with identity 1, then N is zero-symmetric.

Proof

Let N be a δ Near-Ring, Then there exists $x \in N$ such that $xyN = xNy^2$ for all y in N. Putting$x = 1$, we get $yN = Ny^2$ for all y in N. When $y = 0$, $0N = N0 = \{0\}$, it follows that N is zero symmetric.

Proposition 5.2

Let N be a δ Near-Ring with identity, If N is left bipotent, then $yNy = Ny^2$ for all y in N.

Proof. Let N be a δ Near-Ring, Then there exists $x \in N$ such that $xyN = xNy^2$ for all y in N. implies $xyNy = xNy^2y$. Put $x = 1, yNy = 1Ny^2y = Nyy$ [Since N is left bipotent] $= Ny^2$. Therefore $yNy = Ny^2$ for all y in N.

Proposition 5.3 Let N be a Pseudo Commutative near-ring with identity 1, If N is left bipotent, then N is a δ Near-Ring.

Proof. Let N be a Pseudo Commutative near-ring with identity.

Then for all x, y, z in N, $xyz = zyx$….. (1). Since N is left bipotent, $Na = Na^2$ ….(2) for all a in N. If $a \in xyN$, then there exists $n \in N$ such that $a = xyn1 = x1ny$ [by Equation(1)] = $x1ny^2$[by Equation(2)]. Therefore $a \in xNy^2$. Thus $xyN \subseteq xNy^2$…… (3)

On the other hand, if $b \in xNy^2$, then there exists n' ∈ N, such that b = $xn'y^2$ = $xn'y$ [by Equation(2)] $= xyn'$. Therefore, $b \in xyN$. Consequently, $xNy^2 \subseteq xyN$ ……(4). Combining Equations (3) and (4), we get $xyN = xNy^2$and the desired result now follows.

Proposition 5.4 Any homomorphic image of a δ Near-Ring is also a δ Near-Ring.

As an immediate consequence, we have the following corollary.

Corollary 5.5 Let N be a δ Near-Ring. If I is any ideal of N, then N/I is also a δ Near-Ring.

Proof. Let f : N → N/I be the canonical near-ring homomorphism given by f(x) = I + x. Clearly, f is onto. The rest of the proof is taken care of by Proposition 5.4.

Proposition 5.6 Every δ Near-Ring N is isomorphic to a subdirect product of subdirectly irreducible δ Near-Ring.

Proof. By R(1), N is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i' is and each N_i is a homomorphic image of N under the projection map π_i. Now the desired result follows from Proposition 5.4.

Theorem 5.7 Let N be a δ Near-Ring with identity 1 and left bipotent, then we have the following:

(i) Every N-subgroup is invariant

(ii) If N = N_d, then every left N-subgroup is an N-subgroup

Proof.

Let N be a δ Near-Ring, there exists some $x \in M$ such that $xyN = xNy^2$.. (1) for all y in N.

(i) Let T be any N-subgroups of N, Then $T = \sum_{x \in T} Nx$ ...(2). Now, $NxN = N1xN = N.1Nx^2(\text{by equation }(1)) = NNx$[since N is Left bipotent] $\subseteq Nx$. (ie.) $NxN \subseteq Nx$......(3). Therefore, $TN = (\sum_{x \in T} Nx)N$ (by equation (2)) $\subseteq \sum_{x \in T} Nx\,N = \sum_{x \in T} Nx$ (by equation (3)) $= T$ (ie.) $TN \subseteq T$. Consequently, T is invariant.

(ii) Let T be any left N-subgroups of N. Since N is distributive. We can write $T = \sum_{x \in T} xN$...(4). Now,$NxN = Nx^2N[\text{Since N is left bipotent}] = 1Nx^2N = 1xNN$[by equation (1)] $\subseteq xN$. [since xN is left N subgroup of N] (ie.) $NxN \subseteq xN$...(5). Therefore, $NT = N(\sum_{x \in T} xN)$ (by equation (4)) $\subseteq \sum_{x \in T} NxN$ [since N is distributive]= $\sum_{x \in T} xN$(by equation (4))= T. (ie.) $NT \subseteq T$. Consequently, T is an N-subgroup.

Proposition 5.8

Let N be a zero-symmetric δ Near-Ring, If N is left bipotent, then N has strong IFP.

Proof. Let N be a δ Near-Ring. Then there exists some non-zero $x \in M$ such that $xyN = xNy^2$...... (1) for all y in N. since N is weak commutative, $xyz = xzy$ for all x,y,z in N. Let I be an ideal of N. Since N is zero-symmetric $NI \subseteq I$..... (2) Let $ab \in I$. Now, for any $n \in N$, $anb \in aNb = 1aNb = 1Na^2b$ [by Equation (1)]= Nab[since N is left bipotent] $\subseteq NI \subseteq I$ [by Equation (2)]i.e. $anb \in I$. Now, R(5) guarantees that N has strong IFP.

We furnish below a necessary and sufficient condition for a δ Near-Ring to admit mate functions.

Lemma 5.9 Let N be a δ Near-Ring. Then N admits mate functions if and only if $y \in Ny^2$ for all y in N.

Proof. Since N is a δ Near-Ring Proposition 5.2 demands that, $yNy = Ny^2$ for all y in N..... (1). For the 'only if 'part, we assume that f is a mate function for N. Then for all y in N, $y = yf(y)y \in yNy$. Making use of Equation (1) we get $y \in Ny^2$.

For the 'if' part, let $y \in Ny^2$ for all y in N. Appealing to Equation (1), we get $y \in yNy$. I.e. $y = yny$ for some n in N. By setting n = f(y), we see that f is a mate function for N.

Throughout the rest of this section we assume that f serves as a mate function for N.

In the following theorem we prove twelve important properties of a δ Near-Ring admitting mate functions.

Theorem 5.10.

Let N be a zero-symmetric δ Near-Ring with a P_3 mate function f. Then we have,

(i) L = {0} i.e. N has no non-zero nilpotent.
(ii) N has (*, IFP)
(iii) $E \subseteq C(N)$ if $E \subseteq N_d$ i.e. idempotent are central.
(iv) Every left ideal I of N is an N-subgroup of N.
(v) N has property P_4.
(vi) $NxNy = Nx \cap Ny = Nxy$ for all x, y in N. if N is left bipotent.
(vii) Every N-subgroup of N is an ideal.
(viii) $A = \sqrt{A}$ for every N-subgroup A of N.

Proof.

(i) Since f is a mate function for N, Lemma 5.9 demands that $y \in Ny^2$ for all y in N. Therefore, $y = ny^2$ for some n in N. Suppose $y^2 = 0$. Clearly, then $y = 0$. Now, R(2) guarantees that L = {0}.

(ii) By (i) L = {0}. Let $x, y \in N$ and $xy = 0$. Then $(yx)^2 = (yx)(yx) = y(xy)x = y0x = 0$ [since $N = N_0$]. Since N reduced, we get $yx = 0$. Further, for every $n \in N$, $(xny)^2 = (xny)(xny) = xn(yx)ny = xn0ny = 0$. It follows that $xny = 0$[since N is reduced]. Thus N is ($*$, IFP).

(iii) Let $e \in E$. Since N is δ Near-Ring, $eN = eNe^2 = eNe$. Therefore for some n in N, $ene = eu$ and $en =$

eve for some u, v in N. Now, $ene = eue$ and $ene = eve$. Thus $ene = en$ for all n in N. ....(1). Also we have $e(ene - en) = 0$. This implies $e(ene - en) = 0$ and $ne(ene - en) = n.0 = 0$ [since N is zero-symmetric]. Now, $ne(ne - ene) - ene(ne - ene) = 0$. Consequently, $(ne - ene)^2 = 0$ and (i) guarantees $ne - ene = 0$. Therefore, $ene = ne$ for all n in N..... (2) From Equations (1) and (2) we get $en = ne$ for all n in N. Thus $E \subseteq C(N)$.

(iv) Let $n \in N$ and $i \in I$. Since I is left ideal, for all $n' \in N, n(n' + i) = nn' \in I$. Substituting $n' = 0$ and using the hypothesis that N is zero-symmetric, we get $ni \in I$. i.e) $NI \in I$ and hence I is an N-subgroup of N.

(v) Let I be an ideal of N. Then $IN \subseteq I$. Since N is zero symmetric, $NI \subseteq I$. [by Theorem 5.10 (iv)]. Let $xy \in I$. Now, $(yx)^2 = (yx)(yx) = y(xy)x \in NIN \subseteq IN \subseteq I$. i.e. $(yx)^2 \in I$. Appealing to (iv) we get $yx \in I$ and hence (v) follows.

(vi) First we show that for any two N-subgroups A and B of N, $A \cap B = AB$. By Theorem 5.7 (i), A and B are invariant N-subgroups. Now, for $a \in A\ and\ b \in B, ab \in AN \subseteq A$. Therefore, $AB \subseteq A$ ...(3). Also, $ab \in NB \subseteq B$. Therefore, $AB \subseteq B$ ....(4). Combining Equations (3) and (4), we get $AB \subseteq A \cap B$.... (5). On the other hand, if $x \in A \cap B$ then, since f is a mate function for N, $x = xf(x)x \in (AN)B \subseteq AB$. Consequently, $A \cap B \subseteq AB$ .....(6) Combining Equations (5) and (6), we get $A \cap B = AB$ ....(7) for all N-subgroups A, B of N. Clearly, Nx and Ny are N-subgroups of N. Therefore, by replacing A, B respectively by Nx and Ny. we get $Nx \cap Ny = NxNy$ for all x, y in N.... (8) Next we show that $NxNy = Nxy$ for all x, y in N. Since f is a mate function for N, $Nx = Nxf(x)x \subseteq NxN$ i.e. $Nx \subseteq NxN \Rightarrow Nxy \subseteq NxNy$. For the reverse inclusion, $NxNy = NxNy^2$ [since N is left bipotent] $= Nxy\,N$ [since N is δ Near-Ring]$= Nxyf(y)N = NxNyf(y)$[since $E \subseteq C(N)$] $= NxNf(y)y$ [Since f is P_3 mate function] $\subseteq Nxf(x)Ny = NNxf(x)y$[since $E \subseteq C(N)$] $\subseteq Nf(x)xy$[Since f is P_3 mate function] $\subseteq Nxy$. Therefore, $NxNy \subseteq Nxy$. Thus $NxNy = Nxy$ for all x, y in N.... (9). Combining Equations (8) and (9), we get $NxNy = Nx \cap Ny = Nxy$ for all x, y in N.

(vii) Since N is a δ Near-Ring, we have by Proposition 5.2 $yNy = Ny^2$ ...(10) Let $y \in N$. Since f is a mate function for N, $y = yf(y)y$. Let $f(y)y = e$. Then by R(3), $e \in E$ and $Ne = Ny$ ....(11). Let $S = \{n - ne/n \in N\}$. We claim that $(0 : S) = Ne$. Since $(n - ne)e = 0$ for all $n \in N, (n - ne)Ne = \{0\}$ [by (ii)]. Consequently, $Ne \subseteq (0 : S)$ .....(12). For the reverse inclusion, let $z \in (0 : S)$. Since f is a mate function for N, $z = zf(z)z \in zNz = Nz^2$ [by Equation (10)]. Then for some $w \in N$, $z = wz^2$ ....(13). Clearly, $wz - wze \in S$. Since $z \in (0 : S), z(wz - wze) = 0$. By(ii) $(wz - wze)z = 0$. This implies that $wz^2 - wzez = 0$. Now by (i) and R(8), $wz^2 - wz^2 e = 0$. This implies $z - ze = 0$ [by Equation (13)]. Hence $z = ze \in Ne$. i.e. $z \in Ne$. It follows that $(0 : S) \subseteq Ne$ ....(13). Combining Equations (11), (12) and (13), we get $(0 : S) = Ny$. Using Proposition 2.5, we get Nx is an ideal of N. Now, if M is any N-subgroup of N, then $M = \sum_{y \in M} Ny$. Thus M becomes an ideal of N.

(viii) Let A be an N-subgroup of N and let $y \in \sqrt{A}$. Then there exists some positive integer k such that $y^k \in A$.

eISSN: 1303-5150

Since f is a mate function for N, $y = yf(y)y \in yNy = Ny^2$ [by Proposition 5.2]. Then $y = ny^2$ for some y in N. This implies $y = ny.y = n(ny^2)y = n^2y^3 = \ldots = n^{(k-1)}y^k \in NA \subseteq A$. Therefore, $\sqrt{A} \subseteq A$. Obviously, $A \subseteq \sqrt{A}$. Hence (viii) follows. We furnish below a significant result concerning the behavior of ideals in a δ_2 Near-Ring .

Theorem 5.11 Let N be a zero-symmetric δ Near-Ring with mate functions and P be a proper ideal of N. Then the following are equivalent:

i. P is a prime ideal
ii. P is a completely prime ideal
iii. P is a maximal ideal.

Proof. (i)⇒(ii): Let $xy \in P$. By Theorem 5.10(vi), $NxNy = Nxy \subseteq NP \subseteq P$. By Theorem 5.10(vii), Nx and Ny are ideals in N. Since P is prime, $NxNy \subseteq P$ implies $Nx \subseteq P$ or $Ny \subseteq P$. Since f is a mate function for N, for all x, y in N, $x = xf(x)x \in Nx \subseteq P$ and $y = yf(y)y \in Ny \subseteq P$. Therefore either $x \in P$ or $y \in P$. Hence (ii) follows.

(ii)⇒(i) is obvious.

(i)⇒(iii):

Let J be an ideal of N such that $J \neq P$ and that $P \subseteq J \subseteq N$. Let $x \in J - P$. Since f is a mate function for N, for any x in N, $x = x(f(x)x) = f(x)xx$ [by Theorem 5.10(iii)]. Thus for all n in N, $nx = nf(x)x^2$ and this implies $(n - nf(x)x)x = 0$. Since N has $(*, IFP)$, we get $(n - nf(x)x)zx = 0$ and $z(n - nf(x)x)zx = z.0 = 0$. [since $N = N_0$] for all $z \in N$. Consequently, $N(n - nf(x)x)Nx = N0 = \{0\}$ [since $N = N_0$]] If $y = n - nf(x)x$, then $NyNx = \{0\} \subseteq P$. Since P is a prime ideal and Nx, Ny are ideals in N. [by Theorem 5.10(vii)], $Nx \subseteq P$ or $Ny \subseteq P$. If $Nx \subseteq P$, then $x = xf(x)x \in Nx \subseteq P$. Therefore, $x \in P$ which is a contradiction. Hence $Ny \subseteq P$. Then $Ny \subseteq J$ and this demands that for all y in N, $y = yf(y)y \in Ny \subseteq J$. Therefore $y \in J$. i.e. $n - nf(x)x \in J$. Now, since $x \in J, nf(x)x \in NJ \subseteq J$ and therefore $n \in J$. Hence $J = N$ and (iii) follows.

(iii)⇒(i) is obvious.

6 Structure Theorem of δ Near-Ring

In this section we establish a structure theorem for δ Near-Ring. Let us first furnish a characterization theorem for δ Near-Rings.

Theorem 6.1. Let N be a zero-symmetric near-ring with a mate function f. Then N is δ Near-Ring if and only if $yN = Ny^2$ for all y in N and $E \subseteq C(N)$.

Proof.

For the 'only if' part, first we observe that $E \subseteq C(N)$ follows from Theorem 5.12(iii). For any y in N, if $a \in Ny^2$, then, since N is a δ Near-Ring, Proposition 5.3 demands that $a \in yNy \subseteq yN$. Therefore, $Ny^2 \subseteq yN$ …(1). On the other hand, if $b \in yN$, then for some n in N, $b = yn = yf(y)yy = yyf(y)y$ [since E ⊆ C(N)] $\in yNy = Ny^2$ [by Proposition 5.2]. Consequently, $yN \subseteq Ny^2$ …(2). Combining Equations (1) and (2), we get $yN = Ny^2$ for all y in N.

For the "if" part, first we show that f is a P_3 mate function. For any $y \in N$ we have, $y = yf(y)y \in yN = Ny^2$ [by assumption]. Therefore, $y = n_1y^2$ for some n_1 in N. And $yf(y)y = n_1y.yf(y)y = f(y)yn_1y^2$ [since $E \subseteq C(N)$] $= f(y)y^2$. This implies $(yf(y) - f(y)y)y = 0$. Therefore, by Theorem 5.10(ii), $y(yf(y) - f(y)y) = 0$ and $yf(y)(yf(y) - f(y)y) = 0$. Also, $f(y)y(yf(y) - f(y)y) = f(y).0 = 0$ [since $N = N_0$]. And $yf(y)(yf(y) - f(y)y) - f(y)y(yf(y) - f(y)y) = 0$. Consequently, $(yf(y) - f(y)y)^2 = 0$ and hence by Lemma 2.2.7, $yf(y) = f(y)y$ ….(3). Hence f is a P_3 mate function.

Now, $xNy^2 = x(Ny) = x(Nf(y)y)$ [by R(3),] $= x(f(y)yN)$ [since $E \subseteq C(N)$] $= x(yf(y)N)$ [by Equation (3)] $= xyN$ [by Lemma R(3),]. i.e. $xNy^2 = xyN$ for all x, y in N. Thus N is a δ Near-Ring.

eISSN: 1303-5150

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