SQUARE HARMONIC MEAN LABELING OF DISCONNECTED GRAPHS

L. S. Bebisha Lenin[1], M. Jaslin Melbha[2]
[1]Research Scholar, Department of Mathematics, Women's Christian College, Nagercoil-629001, Tamil Nadu, India
[2]Assistant Professor, Department of Mathematics, Women's Christian College, Nagercoil-629001, Tamil Nadu, India
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012.
E-mail: [1]bebishalenin8497@gmail.com, [2]mjaslinmelbha@gmail.com

Abstract

If there is an injective function $h : V(G) \to \{1,2,\dots,q+1\}$ such that an induced edge function $h^* : E(G) \to \{1,2,\dots,q\}$ defined by $h^*(e=uv) = \left\lceil \frac{2h(u)^2h(v)^2}{h(u)^2+h(v)^2} \right\rceil$ or $\left\lfloor \frac{2h(u)^2h(v)^2}{h(u)^2+h(v)^2} \right\rfloor$ is bijective, thena graph $G=(V,E)$ with p vertices and q edges is called a *square harmonic labeling*. A graph which admits a square harmonic mean labeling is called a *square harmonic mean graph*.

The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph$G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. In this paper, we introduced square harmonic mean labeling of disconnected graphs.

Keywords. Graphs, Disconnected graphs, square harmonic mean graph.
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1.Introduction

All graphs considered here are simple, finite, connected and undirected graph. Graph labeling plays an important role in graph theory. A graph labelling is an assignment of integers to the vertices, edges or both of a graph with certain conditions. We cite J. A. Gallian [1] for an extensive examination of graph labeling. We adhere to Harary's [2] conventions for all other terms and notations. S. Somasundaram, R. Ponraj and S.S. Sandhya [4] established the notion of harmonic mean labeling. The aforementioned studies served as our inspiration as we introduced square harmonic mean labeling and also investigate square harmonic mean labeling behaviour of some simple graphs.

Definition 1.1. *Path*refers to a walk where each of the vertices$u_0, u_1, \dots, u_n$ are distinct. A path on n vertices is denoted by P_n.

Definition 1.2. *Cycle* of G refers to a closed path. The symbolC_n stands for a cycle on n vertices.

Definition 1.3. The graph $G = G_1 \cup G_2$formed by taking one copy of G_1 and $V(G_1)$ copies of G_2, where the i^{th} vertex of G_1 is next to every vertex in the i^{th} copy of G_2 is known as the *corona*$G_1 \odot G_2$ of two graphs G_1 and G_2.

Definition 1.4. *Comb*refers to the graph formed by connecting a single pendant edge to each path vertex.

Definition 1.5. Any cycle with a pendant edge attached at each vertex is called a *Crown Graph*and it is denoted by $C_n \odot K_1$.

Definition 1.6. The Cartesian product of a path on two vertices and another path on n vertices is called the *Ladder Graph* L_n, which has the form $P_2 \times P_n$.

2. Main Results

Theorem 2.1. $C_m \cup L_n$ admits a square harmonic mean graph for $m \geq 3$ and $n \geq 2$.

Proof. Let C_m be the cycle $u_1, u_2, \ldots, u_m, u_1$ and L_n be a ladder connecting two paths $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ respectively.Let $G = C_m \cup L_n$ be the union of cycle C_m and ladder L_n. Let $V(G) = \{u_\alpha, v_\beta, w_\beta : 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, v_\beta v_{\beta+1}, w_\beta w_{\beta+1}, v_\beta w_\beta, v_n w_n : 1 \leq \alpha \leq m, 1 \leq \beta \leq n-1\}$. Then $|V(G)| = m + 2n$ and $|E(G)| = m + 3n - 2$.A function $h : V(G) \rightarrow \{1,2, \ldots, q+1\}$ is defined by $h(u_\alpha) = \alpha, 1 \leq \alpha \leq m$, $h(v_\beta) = m + 3\beta - 2, 1 \leq \beta \leq n$, $h(w_\beta) = m + 3\beta - 1, 1 \leq \beta \leq n$.The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = \alpha + 1, 1 \leq \alpha \leq m-1$, $h^*(u_m u_1) = 1$, $h^*(v_\beta v_{\beta+1}) = m + 3\beta - 1, 1 \leq \beta \leq n-1$, $h^*(w_\beta w_{\beta+1}) = m + 3\beta, 1 \leq \beta \leq n-1$, $h^*(v_\beta w_\beta) = m + 3\beta - 2, 1 \leq \beta \leq n$. Thus h^* is bijective. Therefore, $C_m \cup L_n$ admits a square harmonic mean graph if $m \geq 3$ and $n \geq 2$.

Illustration 2.2. The image below displays asquare harmonic mean labeling of $C_4 \cup L_5$.

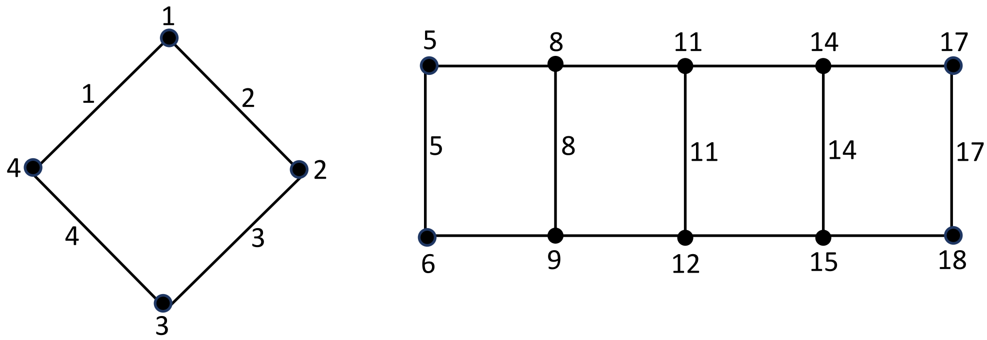

Figure I. $C_4 \cup L_5$

Theorem 2.3. $(C_m \odot K_1) \cup P_n$ admits a square harmonic mean graph for $m \geq 3$and $n \geq 2$.

Proof. Let $C_m \odot K_1$ be a graph obtained from a cycle $u_1, u_2, \ldots, u_m, u_1$ by joining the vertex u_α to pendant vertices v_α and let P_n be a path $w_1, w_2, \ldots, w_n$. Let $G = (C_m \odot K_1) \cup P_n$ with$V(G) = \{u_\alpha, v_\alpha, w_\beta : 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, w_\beta w_{\beta+1} : 1 \leq \alpha \leq m, 1 \leq \beta \leq n-1\}$. Then $|V(G)| = 2m + n$ and $|E(G)| = 2m + n - 1$.A function $h : V(G) \rightarrow \{1,2, \ldots, q+1\}$ is defined by $h(u_\alpha) = 2\alpha, 1 \leq \alpha \leq m$, $h(v_\alpha) = 2\alpha - 1, 1 \leq \alpha \leq m$, $h(w_\beta) = 2m + \beta$, $1 \leq \beta \leq n$.The corresponding induced edge labels are$h^*(u_\alpha u_{\alpha+1}) = 2\alpha + 2, 1 \leq \alpha \leq m-1$, $h^*(u_m u_1) = 2$, $h^*(u_\alpha v_\alpha) = 2\alpha - 1, 1 \leq \alpha \leq m$, $h^*(w_\beta w_{\beta+1}) = 2m + \beta, 1 \leq \beta \leq n-1$.Thus h^* is bijective. Therefore, $(C_m \odot K_1) \cup P_n$ admits a square harmonic mean graph if $m \geq 3$ and $n \geq 2$.

Illustration 2.4. The image below displays a square harmonic mean labeling of $(C_4 \odot K_1) \cup P_5$.

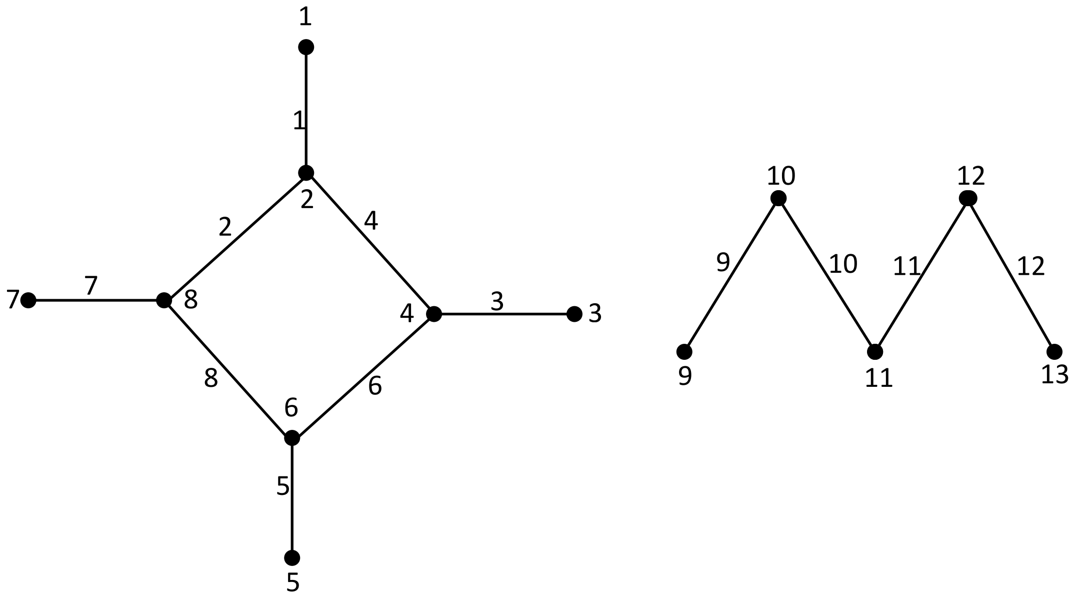

Figure II. $(\mathbf{C_4} \odot \mathbf{K_1}) \cup \mathbf{P_5}$

Theorem 2.5. $(\mathrm{C_m} \odot \mathrm{K_1}) \cup \mathrm{C_n}$ admits a square harmonic mean graph for $\mathrm{m} \geq 3$and $\mathrm{n} \geq 2$.

Proof. Let $C_m \odot K_1$ be a graph obtained from a cycle $u_1, u_2, \dots, u_m, u_1$ by joining the vertex u_α to pendant vertices v_α and let C_n be a cycle $w_1, w_2, \dots, w_n$. Let $G = (C_m \odot K_1) \cup C_n$ with$V(G) = \{u_\alpha, v_\alpha, w_\beta : 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, w_\beta w_{\beta+1} : 1 \leq \alpha \leq m,\ 1 \leq \beta \leq n\}$. Then $|V(G)| = 2m + n$ and $|E(G)| = 2m + n$.A function $h : V(G) \to \{1,2, \dots, q + 1\}$ is defined by $h(u_\alpha) = 2\alpha, 1 \leq \alpha \leq m,\ \ h(v_\alpha) = 2\alpha - 1, 1 \leq \alpha \leq m,\ \ h(w_\beta) = 2m + \beta + 1,\ 1 \leq \beta \leq n$.The corresponding induced edge labels are$h^*(u_\alpha u_{\alpha+1}) = 2\alpha + 2, 1 \leq \alpha \leq m - 1,\ \ h^*(u_m u_1) = 2$, $h^*(u_\alpha v_\alpha) = 2\alpha - 1, 1 \leq \alpha \leq m,\quad h^*(w_\beta w_{\beta+1}) = 2m + \beta + 1, 1 \leq \beta \leq n - 1, h^*(w_n w_1) = 2m + 1$.Thus h^* is bijective. Therefore, $(C_m \odot K_1) \cup C_n$ admits a square harmonic mean graph if $m \geq 3$ and $n \geq 2$.

Illustration 2.6. The image below displays a square harmonic mean labeling of $(\mathrm{C_6} \odot \mathrm{K_1}) \cup \mathrm{C_5}$.

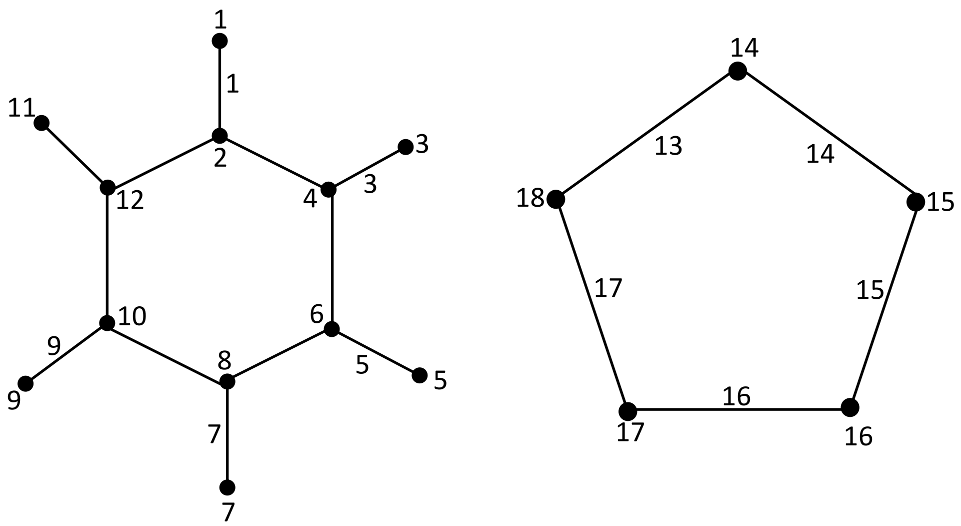

Figure III. $(\mathbf{C_6} \odot \mathbf{K_1}) \cup \mathbf{C_5}$

Theorem 2.7. $(\mathrm{C_m} \odot \mathrm{K_1}) \cup (\mathrm{P_n} \odot \mathrm{K_1})$ admits a square harmonic mean graph for $\mathrm{m} \geq 3$ and $\mathrm{n} \geq 2$.

Proof. Let $u_1, u_2, \dots, u_m, u_1$ be the cycle C_m and let v_α be the pendant vertex joined to the vertex u_α of C_m. The resultant graph is $C_m \odot K_1$.Let $w_1, w_2, \dots, w_n$ be a path P_n and t_β be the vertex joined to the vertex w_j of path P_n. The resultant graph is $P_n \odot K_1$. Let $G = (C_m \odot$

$K_1) \cup (P_n \odot K_1)$with$V(G) = \{u_\alpha, v_\alpha, w_\beta, t_\beta : 1 \le \alpha \le m, 1 \le \beta \le n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, w_\beta w_{\beta+1}, w_\beta t_\beta, w_n t_n : 1 \le \alpha \le m, 1 \le \beta \le n-1\}$.Then $|V(G)| = 2m + 2n$ and $|E(G)| = 2m + 2n - 1$. A function$h : V(G) \to \{1,2, \dots, q+1\}$ is defined by $h(u_\alpha) = 2\alpha, 1 \le \alpha \le m$, $h(v_\alpha) = 2\alpha - 1, 1 \le \alpha \le m$, $h(w_\beta) = 2m + 2\beta - 1, 1 \le \beta \le n$, $h(t_\beta) = 2m + 2\beta, 1 \le \beta \le n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 2\alpha + 2, 1 \le \alpha \le m-1$, $h^*(u_m u_1) = 2$, $h^*(u_\alpha v_\alpha) = 2\alpha - 1, 1 \le \alpha \le m$, $h^*(w_\beta w_{\beta+1}) = 2m + 2\beta, 1 \le \beta \le n-1$, $h^*(w_\beta t_\beta) = 2m + 2\beta - 1, 1 \le \beta \le n$.Thus h^* is bijective.Therefore, $(C_m \odot K_1) \cup (P_n \odot K_1)$ admits a square harmonic mean graph if $m \ge 3$ and $n \ge 2$.

Illustration 2.8. The image below displays a square harmonic mean labeling of $(C_6 \odot K_1) \cup (P_5 \odot K_1)$.

5376

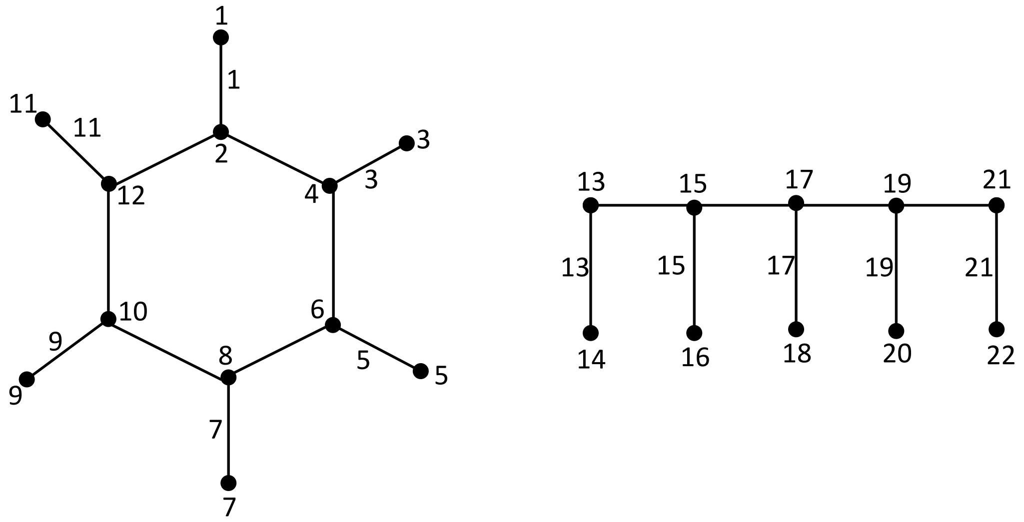

Figure IV.$(\mathbf{C_6} \odot \mathbf{K_1}) \cup (\mathbf{P_5} \odot \mathbf{K_1})$

Theorem 2.9. $(P_m \odot K_3) \cup P_n$ admits a square harmonic mean graph for $m \ge 3$ and $n > 1$.

Proof. Let $P_m \odot K_3$ be a graph obtained from a path $u_1, u_2, \dots, u_m$ by joining the vertices v_α, w_α of K_3 respectively and P_n be the path $r_1, r_2, \dots, r_n$. Let $G = (P_m \odot K_3) \cup P_n$ with$V(G) = \{u_\alpha, v_\alpha, w_\alpha, r_\beta : 1 \le \alpha \le m, 1 \le \beta \le n\}$ and $E(G) = \{u_\alpha v_\alpha, u_\alpha w_\alpha, v_\alpha w_\alpha : 1 \le \alpha \le m\} \cup \{u_\alpha u_{\alpha+1}, r_\beta r_{\beta+1} : 1 \le \alpha \le m-1, 1 \le \beta \le n-1\}$Then $|V(G)| = 3m + n$ and $|E(G)| = 4m + n - 2$.A function $h : V(G) \to \{1,2, \dots, q+1\}$ is defined by $h(u_\alpha) = 4\alpha - 3, 1 \le \alpha \le m$,$h(v_\alpha) = 4\alpha - 2$, $1 \le \alpha \le m$,$h(w_\alpha) = 4\alpha - 1, 1 \le \alpha \le m$, $h(r_\beta) = 4m + \beta - 1, 1 \le \beta \le n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 4\alpha, 1 \le \alpha \le m-1$, $h^*(u_\alpha v_\alpha) = 4\alpha - 3, 1 \le \alpha \le m$, $h^*(u_\alpha w_\alpha) = 4\alpha - 2{,}1 \le \alpha \le m$, $h^*(v_\alpha w_\alpha) = 4\alpha - 1, 1 \le \alpha \le m$, $h^*(r_\beta r_{\beta+1}) = 4m + \beta - 1, 1 \le \beta \le n$.Thus h^* is bijective. Therefore, $(P_m \odot K_3) \cup P_n$ admits a square harmonic mean graph if $m \ge 3$ and $n > 1$.

Illustration 2.10. The image below displays a square harmonic mean labeling of $(P_4 \odot K_3) \cup P_5$.

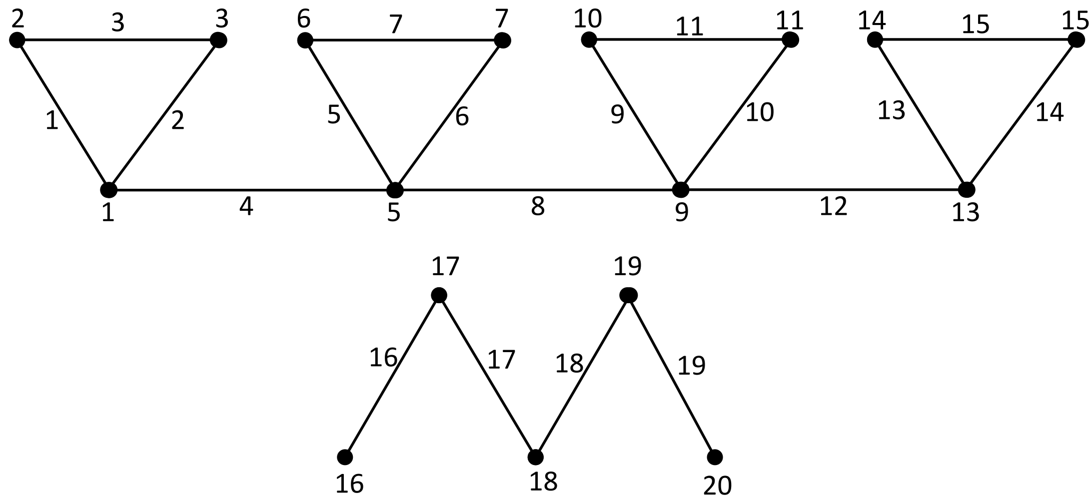

Figure V. $(\mathbf{P_4} \odot \mathbf{K_3}) \cup \boldsymbol{P_5}$

Theorem 2.11. $(\mathrm{P_m} \odot \mathrm{K}_{1,2}) \cup C_n$ admits a square harmonic mean graph for $\mathrm{m} \geq 2$ and $\mathrm{n} \geq 3$.

Proof. Let $P_m \odot K_{1,2}$ be a graph obtained from a path $u_1, u_2, \ldots, u_m$ and let v_α, w_α be the vertices of $K_{1,2}$ which are joined the vertices u_α. Let C_n be the cycle $r_1, r_2, \ldots, r_n$ respectively. Let $G = (\mathrm{P_m} \odot \mathrm{K}_{1,2}) \cup C_n$ with$V(G) = \{u_\alpha, v_\alpha, w_\alpha, r_\beta : 1 \leq \alpha \leq m, 1 \leq \beta \leq n\}$ and $E(G) = \{u_\alpha u_{\alpha+1}, u_\alpha v_\alpha, u_m v_m, u_\alpha w_\alpha, u_m w_m, r_\beta r_{\beta+1} : 1 \leq \alpha \leq m-1,\ 1 \leq \beta \leq n\}$. Then $|V(G)| = 3m + n$ and $|E(G)| = 3m + n - 1$.A function $h : V(G) \to \{1,2, \ldots, q+1\}$ is defined by $h(u_\alpha) = 3\alpha - 2, 1 \leq \alpha \leq m, h(v_\alpha) = 3\alpha - 1,\ 1 \leq \alpha \leq m, h(w_\alpha) = 3\alpha, 1 \leq \alpha \leq m,\ \ h(r_\beta) = 3m + \beta, 1 \leq \beta \leq n$. The corresponding induced edge labels are $h^*(u_\alpha u_{\alpha+1}) = 3\alpha, 1 \leq \alpha \leq m-1$, $h^*(u_\alpha v_\alpha) = 3\alpha - 2, 1 \leq \alpha \leq m$, $h^*(u_\alpha w_\alpha) = 3\alpha - 1, 1 \leq \alpha \leq m$, $h^*(r_\beta r_{\beta+1}) = 3m + \beta, 1 \leq \beta \leq n-1, h^*(r_n r_1) = 3m$.Thus h^* is bijective. Therefore, $(\mathrm{P_m} \odot \mathrm{K}_{1,2}) \cup C_n$ admits a square harmonic mean graph if $m \geq 2$ and $n \geq 3$.

Illustration 2.12. The image below displays a square harmonic mean labeling of $(\mathrm{P_4} \odot \mathrm{K}_{1,2}) \cup C_5$.

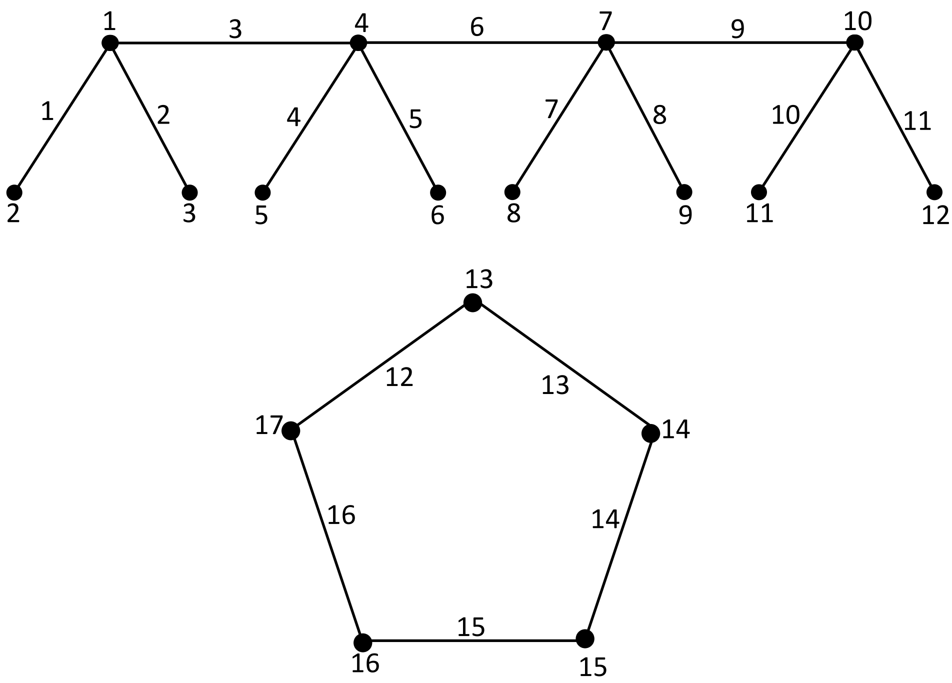

Figure VI. $(\mathbf{P_4} \odot \mathbf{K_{1,2}}) \cup \boldsymbol{C_5}$

Conclusion:

In this paper, we investigated the behavior of disconnected graphs are square harmonic mean labeling. The derived results are demonstrated by means of sufficient illustrations which provide better understanding.

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