Regular Spaces Associated with p*gb-Open Sets

I. ARUNA GLORY SUDHA[1]
Research Scholar, Registration No:21212152092002,
Department of Mathematics, Nazareth Margoschis College at Pillaiyanmanai, Nazareth,
Affliated to Manonmaniam Sundaranar University,Abishekapatti, Tirunelveli-627 012, Tamilnadu, India
S. ZION CHELLARUTH[2]
Assistant Professor, Department of Mathematics,
Nazareth Margoschis College at Pillaiyanmanai,Nazareth,
Affliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamilnadu, India.

Abstract
Using the paradigm of pre star generalized b-closed and pre star generalized b-open sets, we present and investigate pre star generalized b-regular spaces .

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1.Introduction

Pre*-closed sets were given and some of their characteristics were explored by T. Selvi and A. PunithaDharani [3] in 2012. Pre*-generalized b-closed and pre*-generalized b-open sets are characterised in [4]. Pre* generalised b-regular and strongly p*gb-regular and weakly p*gb-regular spaces, respectively, are described and investigated in this study using p*gb-open and p*gb-closed sets.

2.Preliminaries

Definition 2.1.[1] In X, A subset M is called

(i)b-open if $M\subseteq Int(Cl(M))\cup Cl(Int(M))$

(ii)b-closed if $Int(Cl(M))\cap Cl(Int(M)) \subseteq M$.

Definition 2.2.[1] b-closure of A, denoted by $bCl(A)=\bigcap\{H:A\subseteq H$ and H is b-closed$\}$.

Definition 2.3. [3] A subset M of the space X is called

(i)pre*-open if $M\subseteq int^*(Cl(M))$

(ii) pre*-closed if $Cl^*(Int(M))\subseteq M$.

Definition 2.4.[4] A pre* generalized b-closed set (briefly, p*gb-closed) is a subset A of a Space (X, τ) if $bCl(A)\subseteq U$, whenever $A\subseteq U$, U is pre*-open in (X, τ).

Lemma 2.5.[4]For a topological space (X, τ), Every open set is p*gb-open.

Lemma 2.6. [4]

(a) Arbitrary intersection of p*gb-closed sets is p*gb-closed.
(b) Arbitrary union of p*gb-open sets is p*gb-open.

Remark 2.7.[4]

(a) The union of p*gb-closed sets need not be p*gb-closed.

(b) The intersection of p*gb-open sets is p*gb-open.

Definition 2.8.[5] Let X be a topological space and let x∈X. A subset N of X is said to be a p*gb-neighbourhood (shortly, p*gb-nbhd) of x if there exists a p*gb-open set U such that x∈U⊆N.

Theorem 2.9.[5] Every nbhd N of x∈X is a p*gb-nbhd of x.

Definition 2.10.[5] Let A be a subset of a topological space (X, τ). Then the union of all p*gb-open sets contained in A is called the p*gb-interior of A and it is denoted by p*gbInt(A). That is, p*gbInt(A)=∪{V:V⊆A and V∈p*gb-O(X)}.

Theorem 2.11.[5] Let A be a subset of a topological space (X, τ). Then

(a) p*gbInt(A) is the largest p*gb-open set contained in A.

(b) A is p*gb-open if and only if p*gbInt(A)=A.

(c) p*gbInt(φ)= φ and p*gbInt(X)=X.

(d) If A⊆B, then p*gbInt(A)⊆p*gbInt(B).

(e) p*gbInt(p*gbInt(A))=p*gbInt(A).

Definition 2.12.[5] Let A be a subset of a topological space (X, τ). Then the intersection of all p*gb-closed sets in X containing A is called the p*gb-closure of A and it is denoted by p*gbCl(A). That is, p*gbCl(A)=∩{F:A⊆F and F∈p*gb-C(X)}. The intersection of p*gb-closed set is p*gb-closed, then p*gbCl(A) is p*gb-closed.

Theorem 2.13.[5] Let A be a subset of a topological space (X, τ). Then

(a) p*gbCl(A) is the smallest p*gb-closed set containing A.

(b) A is p*gb-closed if and only if p*gbCl(A)=A.

(c) p*gbCl(φ)= φ and p*gbCl(X)=X.

(d) If A⊆B, then p*gbCl(A)⊆p*gbCl(B).

(e) p*gbCl(p*gbCl(A))=p*gbCl(A).

Definition 2.14[6]. A topological space X is quasi H-closed if every open cover has a finite proximate subcover. That is, every open cover has a finite subfamily whose closures cover the space.

Definition 2.15[6].A topological space (X,τ) is said to be **regular** if for each closed set A and a point x∉A, there exist disjoint open sets U and V such that A⊆U, x∈V.

3. p*gb-Regular Spaces

Definition 3.1.A Space X is called **p*gb-regular** if for each p*gb-closed set G and a point t∉G, ∃ separate p*gb-open sets A and B ∋: G⊆A, t∈B.

Example 3.2. Let X ={1, 2, 3} and τ = {φ, {1}, {2}, {1, 2}, X}. In this space X, the p*gb-open sets are φ, {1}, {2}, {1, 2}, {1, 3}, {2, 3}, X and the p*gb-closed sets are φ, {1}, {2}, {3}, {1, 3}, {2, 3}, X. Then X is p*gb-regular space.

Theorem 3.3. The bellows are similar in a Space X:

(a) X is p*gb-regular.

(b) Let p∈X and G be a p*gb-open set p∈X. Then ∃ a p*gb-open set M ∋: p∈M⊆p*gbCl(M)⊆G.

(c) Assume F is a p*gb-closed set. Therefore the intersection of all F's p*gb-closed, p*gb-neighborhoods is F.

(d) Given any set C and any p*gb-open set D, ∃ a p*gb-open set M ∋: C∩M≠φ and p*gbCl(M)⊆D.

(e) Assume C is a non-empty set and D is a p*gb-closed set, and C∩D=φ. Then there are separate p*gb-open sets M and N ∋: C∩M≠φ and D⊆N exist.

Proof: (a)⇒(b): Suppose X is p*gb-regular. Let p∈X and G be a p*gb-open set that includes p. Then p∉X\G and X\G is p*gb-closed. Since X is p*gb-regular, ∃ p*gb-open sets M and N ∋: M∩N=φ and p∈M, X\G⊆N. As a result, M⊆X\N⊆G and therefore p*gbCl(M)⊆p*gbCl(X\N)=X\N⊆G. That is, p∈M⊆p*gbCl(M)⊆G.

(b)⇒(c): Let F to be any p*gb-closed set and p∉F. X\F is then p*gb-open and p∈X\F. According to (b), ∃ a p*gb-open set M with the condition p∈M⊆p*gbCl(M)⊆X\F. Thus F⊆X\p*gbCl(M)⊆X\M. Now X\M is p*gb-closed, F's p*gb-nghd that does not contain p. Hence, the intersection of all p*gb-closed, p*gb-nhds of F is F.

(c)⇒(d):Suppose C∩D≠φ and D is p*gb-open. Let x∈C∩D. D is p*gb-open, therefore X\D is p*gb-closed and p∉X\D. ∃ a p*gb-open set G ∋: X\D⊆G⊆N for the p*gb-neighborhood N of X\D. Consider M= X\N. As a result, M is p*gb-open p∈M. Furthermore, C∩M≠φ and p*gbCl(M)⊆X\G⊆D.

(d)⇒(e): Let C be any non-empty set and D be p*gb-closed with C∩D=φ. Then X\D is p*gb-open and C∩(X\D)≠φ. According to our assumptions, ∃ a p*gb-open set M ∋: C∩M≠φ, p*gbCl(M)⊆X\D. Take N=X\p*gbCl(M). Since p*gbCl(M) is p*gb-closed, N is p*gb-open. Also D⊆N and M∩N⊆p*gbCl(M)∩(X\p*gbCl(M))=φ.

(e)⇒(a): Let S be p*gb-closed and p∉S. Then S∩{x}=φ. By (e), separate p*gb-open sets M and N exist ∋: M∩{p} ≠φ and S⊆N. That is, M and N are separate p*gb-open sets, each containing p and S. This demonstrates that X is p*gb-regular.

Corollary 3.4. The bellows are similar in a Space X:

(a) X is p*gb-regular.

(b) Let p∈X and G be a open set p∈X. Then ∃ a p*gb-open set M ∋: p∈M⊆p*gbCl(M)⊆G.

(c) Assume F is a closed set. Therefore the intersection of all F's p*gb-closed, p*gb-neighborhoods is F.

(d) Given any set C and any open set D, ∃ a p*gb-open set M ∋: C∩M≠φ and p*gbCl(M)⊆D.

(e) Assume C is a non-empty set and D is a closed set, and C∩D=φ.. Then there are separate p*gb-open sets M and N ∋: C∩M≠φ and D⊆N exist.

Proof. Since every open set is p*gb-open and follows from above theorem.

Theorem 3.5.A Space X is p*gb-regular ⟺∃ a p*gb-open set M ∋: p∈M⊆p*gbCl(M)⊆N for every p∈X and every p*gb-nbhd N containing p.

Proof. Let X be a p*gb-regular space. Let N be any p*gb-nbhd of x. then there exists p*gb-open set G such that x∈G⊆N. Since X\G is p*gb-closed and x∉X\G, by definition there exists p*gb-open sets L and M such that X\G⊆L and x∈M and L∩M=φ so that M⊆X\L. It follows that p*gbCl(M)⊆p*gbCl(X\L)=X\L. Also X\G⊆L implies X\L⊆G⊆N. Hence x∈M⊆p*gbCl(M)⊆N. Conversely, suppose for every x∈X and every p*gb-nbhd N containing x, there exists a p*gb-open set M such that x∈M⊆p*gbCl(M)⊆N. Let F be any p*gb-closed and x∉F. Then x∈X\F. Since X\F is p*gb-open set, X\F is p*gb-nbhd containing x. By hypothesis there exists a p*gb-open set M such that x∈M and p*gbCl(M)⊆X\F. This implies that, F⊆X\p*gbCl(M). Then X\p*gbCl(M) is p*gb-open set containing F. Also M∩(X\p*gbCl(M))= φ. Hence the space is p*gb-regular.

Corollary 3.6.A topological space X is p*gb-regularif and only if every x∈X and every nbhd N containing x, there exists a p*gb-open set M such that x∈M⊆p*gbCl(M)⊆N.

Proof. Since every nbhd is p*gb-nbhd and follows from above theorem.

Corollary 3.7.A topological space X is p*gb-regularif and only if every x∈X and every p*gb-nbhd N containing x, there exists a p*gb-open set M such that x∈M⊆cl(M)⊆N.

Proof. Since p*gbCl(M)⊆cl(M) and follows from above theorem.

Theorem 3.8.A Space X is p*gb-regular ⟺ there are p*gb-open sets U and V of X for any p*gb-closed set F of X and each p∈X\F ∋: p∈U and F⊆V and p*gbCl(U)∩p*gbCl(V)= φ.

Proof: Suppose X is p*gb-regular. Let F be a p*gb-closed set in X and x∉F. Then there exist p*gb-open sets $U_x$and V such that x∈U_x, F⊆V and U_x∩V=φ. This implies that U_x∩p*gbCl(V)=φ. Also p*gbCl(V) is a p*gb-closed set and x∉p*gbCl(V). Since X is p*gb-regular, there exist p*gb-open sets G and H of X such that x∈G, p*gbCl(V)⊆H and G∩H=φ. This implies p*gbCl(G)∩H ⊆p*gbCl(X\H)∩H =(X\H)∩H=φ. Take U=G. Now U and V are p*gb-open sets in X such that x∈U and F⊆V. Also p*gbCl(U)∩p*gbCl(V) ⊆p*gbCl(G)∩H=φ. Conversely, suppose for each p*gb-closed set F ofX and each x∈X\F, there exist p*gb-open sets U and V of X such that

$x\in U$ and $F\subseteq V$ and $p^*gbCl(U)\cap p^*gbCl(V)=\varphi$. Now $U\cap V\subseteq p^*gbCl(U)\cap p^*gbCl(V)=\varphi$. Therefore $U\cap V=\varphi$. This proves that X is p*gb-regular.

Corollary 3.9.A topological space X is p*gb-regular if and only if for each closed set F of X and each $x\in X\setminus F$, there exist p*gb-open sets U and V of X such that $x\in U$ and $F\subseteq V$ and $p^*gbCl(U)\cap p^*gbCl(V)= \varphi$.

Theorem 3.10. Let X be a p*gb-regular space.

(i) In X, each p*gb-open set is a union of p*gb-closed sets.

(ii) In X, each p*gb-closed set is an intersection of p*gb-open sets.

Proof: (i) Suppose X is p*gb-regular. Assume G is a p*gb-open set and $p\in G$. $F=X\setminus G$ is thus p*gb-closed and $p\notin F$. Since X is p*gb-regular, separate p*gb-open sets U_p and V exist in X $\ni$: $p\in U_p$and $F\subseteq V$. Since $U_p\cap F\subseteq U_p\cap V=\varphi$, we have $U_p\subseteq X\setminus F=G$. Take $V_p=p^*gbCl(U_p)$. Then $V_p$is a p*gb-closed set and $V_x\cap V=\varphi$. Now $F\subseteq V$ implies that $V_p\cap F\subseteq V_p\cap V=\varphi$. Therefore $p\in V_p\subseteq X\setminus F=G$ is the result. This demonstrates that $G=\cup\{V_p: p\in G\}$. As a result, G is a union of p*gb-closed sets.

(ii) Follows from (i) and set theoretic properties.

Definition 3.11. A function $f:X\longrightarrow Y$ is called a p*gb-continuous if the inverse image of each open set in Y is p*gb-open in X.

Theorem 3.12. If f is a p*gb-continuous and closed injection of a topological space X into a regular space Y and if every p*gb-closed set in X is closed, then X is p*gb-regular.

Proof: Let $x\in X$ and A be a p*gb-closed set in X not containing x. Then by assumption, A is closed in X. Since f is closed, f(A) is a closed set in Y not containing f(x). Since Y is regular, there exist disjoint open sets V_1 and V_2 in Y such that $f(x)\in V_1$ and $f(A)\subseteq V_2$. Since f is p*gb-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint p*gb-open sets in X containing x and A respectively. Hence X is p*gb-regular.

Theorem 3.13. If f is a continuous p*gb-open bijection of a regular space X into a space Y and if every p*gb-closed set in Y is closed, then Y is p*gb-regular.

Proof: Let $y\in Y$ and B be a p*gb-closed set in Y not containing y. Then by assumption, B is closed in Y. Since f is a continuous bijection, $f^{-1}(B)$ is a closed set in X not containing the point $f^{-1}(y)$. Since X is regular, there exist disjoint open sets U_1 and U_2 in X such that $f^{-1}(y)\in U_1$ and $f^{-1}(B)\subseteq U_2$. Since f is p*gb-open, f(U_1) and f (U_2) are disjoint p*gb-open sets in Y containing x and B respectively. Hence Y is p*gb-regular.

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