The Study of The Order of Fractional Operators Involving Incomplete Elliptic Integrals Generalized Multi-index Bessel-Maitland Function And Incomplete Aleph Functions

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ABSTRACT
In this article we introduce a new fractional integration operators associated with the incomplete Elliptic integral, Generalized Bessel's Maitland function and incomplete Aleph function in its kernels. Further we obtain two new Integrals and also we develop the Parseval-Goldstein theorem related to these operators.

KEYWORDS: Fractional operators, Incomplete Elliptic Integrals, Generalized Bessel's Maitland function, Incomplete Aleph function, Integral transforms.
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INTRODUCTION

INCOMPLETE ELLIPTIC INTEGRAL

The family of **incomplete elliptic integrals** used in this paper is defined and represented in the following form [10, pp.1178-1179, eqs. (1.12&2.6)]:

$$H(\phi,k;\gamma)=\int_0^{\phi}(1-k^2\sin^2\theta)^{\gamma-\frac{1}{2}}d\theta=\sin\phi\, F_1\left[\frac{1}{2},\frac{1}{2}-\gamma,\frac{1}{2};\frac{3}{2};k^2\sin^2\phi,\sin^2\phi\right]$$

$$=\frac{\sin\phi}{2\sqrt{\pi}\,\Gamma(1/2-\gamma)}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\frac{\Gamma(1/2+\xi_1+\xi_2)\Gamma(1/2-\gamma+\xi_1)\Gamma(1/2+\xi_2)\Gamma(-\xi_1)\Gamma(-\xi_2)}{\Gamma(3/2+\xi_1+\xi_2)}$$

$$(-k^2\sin^2\phi)^{\xi_1}(-\sin^2\phi)^{\xi_2}\,d\xi_1 d\xi_2 \qquad (1)$$

where

$$\left(|k^2|<1;0\leq\phi\leq\frac{\pi}{2};\left(\frac{1}{2}-\gamma\right)\neq 0,-1,-2\quad;\gamma\in C\right).$$

We mention below three special cases of this integral:

1. If in the above result we put $\varphi=\frac{\pi}{2}$, then we get

$$H\left(\frac{\pi}{2},k;\gamma\right)=H(k;\gamma)=\int_0^1\frac{(1-k^2t^2)^{\gamma-\frac{1}{2}}}{\sqrt{(1-t^2)}}dt \quad ,$$

$$=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-\gamma;1;k^2\right),\quad \left(\left|k^2\right|<1;\gamma\in C\right) \qquad (2)$$

2. By putting $\varphi=\frac{\pi}{2},\gamma=0$ in eq. (3.1.1), we arrive at the following result after making some obvious changes in parameters:

$$H\left(\frac{\pi}{2},k;0\right)=K(k)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right),\ \left(\left|k^2\right|<1\right) \qquad (3)$$

3. In eq.(3.1.1), we take $\varphi=\frac{\pi}{2},\gamma=1$ we get the following form:

$$H\left(\frac{\pi}{2},k;1\right)=E(k)=\int_0^1\sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt$$

$$=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;k^2\right),\left(\left|k^2\right|<1\right) \qquad (4)$$

The Incomplete Aleph function

Recently Bansal et. al [2] introduced and investigate the incomplete Aleph function *,,,Γ.-ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n.,z.*and *,,,γ.-ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n.(z)*

,,,,Γ.-ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n.,z.=,-,Γ.-,ℵ..-,p-i.,,q-i.,,ρ-i.,r-m,n.,z,,,,a-1.,,A-1.,y.,,,,a-j.,,A-j..-2,n.,,[,ρ-j.,,a-ji.,,A-ji..]-n+1,,p-1..-,,,b-j.,,B-j..-1,m.,,[,ρ-j.,,b-ji.,,B-ji..]-m+1,,q-1.....

=,1-2πi.,ℂ--K.,ξ,y.,,z.-−ξ.dξ,

(5)

where *z≠0,* and

K,ξ,y.=,Γ,1−,a-1.−,A-1.ξ,y.,j=1-m-Γ,,b-j.+,B-j.ξ.,j=2-n-Γ,1−,b-1.−,B-1.ξ,y...-,i=1-r-,ρ-i..,,j=m+1-,q-i.-Γ,1−,b-ji.+,B-ji.ξ.,j=n+1-,p-i.-Γ,,a-ji.−,A-ji.ξ..... *(6)*

and

,,,,γ.-ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n.,z.=,-,γ.-,ℵ..-,p-i.,,q-i.,,ρ-i.,r-m,n.,z,,,,a-1.,,A-1.,y.,,,,a-j.,,A-j..-2,n.,,[,ρ-j.,,a-ji.,,A-ji..]-n+1,,p-1..-,,,b-j.,,B-j..-1,m.,,[,ρ-j.,,b-ji.,,B-ji..]-m+1,,q-1.....

=,1-2πi.,ℂ--L.,ξ,y.,,z.-−ξ.dξ,

(7)

where *z≠0,* and

L,ξ,y.=,γ,1−,a-1.−,A-1.ξ,y.,j=1-m-Γ,,b-j.+,B-j.ξ.,j=2-n-Γ,1−,b-1.−,B-1.ξ,y...-,i=1-r-,ρ-i..,,j=m+1-,q-i.-Γ,1−,b-ji.+,B-ji.ξ.,j=n+1-,p-i.-Γ,,a-ji.−,A-ji.ξ..... *(8)*

The incomplete Aleph function given in (8) and (10) exist for all y>=0 under the same contour and the same set of conditions as stated in [2](see also [10]).A complete details of (5) and (7) can be found in [2].

Generalized Multiindex Bessel-Maitland function

Khan [ ] explored and invented generalized Multiindex Bessel-Maitland function as follows:

, J-,,β-j..,q-,,α-i..,γ.,x.=,m=0-∞-,,,γ.-qm.,,−x.-m.-,j=1-n-Γ.,,α-j.m+,β-j.+1.m!.. (9)

where j=1,2,…,n; ℜ,γ.>0,ℜ,,β-j..>1,ℜ[,j=1-n-,α-j.]>,max-,0,q−1..,qϵ(0,1). Uℕ and nϵℕ

and ,,γ.-qm. is the Pochhammer symbol, which can be written in terms of gamma function as

,,γ.-qm.=,Γ,γ+mq.-Γ,γ.. (10)

a number of researchers have looked at the applications in the various field of physics, engineering mathematics and sciences.

FRACTIONAL INTEGRAL OPERATORS

Now we present the following two unified fractional integral operators involving incomplete elliptic integral, Incomplete Aleph function and genralized Multi-index Bessel's Maitland function having general arguments defined and represented in the following manner:

$$I_x^{\nu,\lambda}\left[f(t)\right] = I_{x\ :z_0,\phi,\gamma:z_1,P\ :z_2,\rho,\alpha,\beta}^{\nu,\lambda:\nu_0,\lambda_0:\nu_1,\lambda_1:\nu_2,\lambda_2}[f(t)] = x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} H\left(\phi, z_0\left(\frac{t}{x}\right)^{\nu_0}\left(1-\frac{t}{x}\right)^{\lambda_0};\gamma\right)$$

,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,t-x..-,ν-2..,,1−,t-x..-,λ-2...f(t)dt (11)

where $f(t) \in A$ and A denotes the class of functions for which

$$f(t) = \begin{cases} O\left\{|t|^{\zeta}\right\} & , \max\left\{|t|\right\} \to 0 \\ O\left\{|t|^{w_1} e^{-w_2|t|}\right\} & , \min\left\{|t|\right\} \to \infty \end{cases} \quad (12)$$

provided that

$$\left.\begin{array}{l} \mathrm{Re}\left[\nu+\zeta+\nu_1\right] > -1, \gamma \in \quad, \alpha > 0, \mathrm{Re}\left[\lambda+\lambda_1\right] > -1 \\ \min\left[(\lambda_0,\lambda_2,\nu_0,\nu_2)\right] \geq 0, (\text{not all simultaneously zero}) \end{array}\right\} \quad (13)$$

$$J_x^{\nu,\lambda}\left[f(t)\right] = J_{x\ :z_0,\phi,\gamma:z_1,P\ :z_2,\rho,\alpha,\beta}^{\nu,\lambda:\nu_0,\lambda_0:\nu_1,\lambda_1:\nu_2,\lambda_2}[f(t)] = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} H\left(\phi, z_0\left(\frac{x}{t}\right)^{\nu_0}\left(1-\frac{x}{t}\right)^{\lambda_0};\gamma\right)$$

,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,x-t..-,ν-1..,,1−,x-t..-,λ-1...f(t)dt (14)

where

$$\left.\begin{array}{l} \mathrm{Re}(w_2) > 0 \text{ or } \mathrm{Re}(w_2) = 0 \text{ and } \mathrm{Re}(\nu - w_1 + \nu_1) > 0, \gamma \in \quad, \alpha > 0 \\ \mathrm{Re}(\lambda+\lambda_1) > -1, \min(\lambda_0,\lambda_2,\nu_0,\nu_2) \geq 0, (\text{not all simultaneously zero}) \end{array}\right\} \quad (15)$$

2. IMAGES

In this section, we willfind the images of some usefulfunctions in our operators defined by (11) and (14). We have:

Image-1:

, I-x-ν,λ.,,,t-μ.,Γ-,,ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n..-.,,z-1.,,t.-,U-1..,,x−t.-,V-1....=,,,x-μ.-.sinϕ-2,-π.Γ,,1-2.−γ..,m=0-∞-,,,γ.-qm.,,−z.-m.-,j=1-n-Γ,,α-j.m+,β-j.m+1.m!...
,a=0-∞-,b=0-∞-,,,,1-2..-a+b.,,,,1-2.−γ.-a.,,,1-2..-b.,,z-0.,k-2.,sin-2.φ.-a.,,,sin-2.φ.-b.-,,,3-2..-a+b.a!b!...
,,-,Γ.-,ℵ..-,p-i.+2,,q-i.+1,,ρ-i.,r-m,n+2.,,z-,1..,x-,U-1.+,V-1..,,,,a-1.,,A-1.,y.,,,,a-j.,,A-j..-2,n.,,−ν−μ−,ν-1.m,−2ν-0.a,,U-1..-,,,b-j.,,B-j..-1,m.,,−1−ν−μ−λ−,,ν-1.+,λ-1..m−,, 2ν-0.,+2λ-0..a,,U-1.+,V-1.....
,,,,−λ−,λ-1.m,−2λ-0.a,,V-1..,-,,[,ρ-j.,,b-ji.,,B-ji..]-m+1,,q-1.... (16)

where Re,ν+μ.>−1,γϵℂ,α>0; Re,λ.>−1,min(,λ-0.,,ν-0.,,λ-1.,,ν-1.)≥0 also let $A_1>0$, arg$z| < \pi/2, A_1$ (i = 1.....r). Then the fractional integral formula holds for the image of an arbitrary incomplete Aleph function.

Image-2:

,J-x-ν,λ.,,,t-μ.,Γ-,,ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n..-.,,z-1.,,t.-,−U-1..,,1−x/t.-,V-1....=,,,x-μ.-.sinϕ-2,-π.Γ,,1-2.−γ..,m=0-∞-,,,γ.-qm.,,−z.-m.-,j=1-n-Γ,,α-j.m+,β-j.m+1.m!...

,a=0-∞-,b=0-∞-,,,,1-2..-a+b.,,,,1-2.−γ.-a.,,,1-2..-b.,,z-0.,k-2.,sin-2.φ.-a.,,,sin-2.φ.-b.-,,,3-2..-a+b.a!b!...

,,-,Γ.-,ℵ..-,p-i.+2,,q-i.+1,,ρ-i.,r-m,n+2.,,,z-,1..,x−,,U-1.+,V-1....,,,a-1.,,A-1.,y.,,,,a-j.,,A-j..-2,n.,,1−ν+μ−,ν-1.m,−2ν-0.a,,U-1..,-,,,b-j.,,B-j..-1,m.,,−ν+μ−λ−,,ν-1.+,λ-1..m−,, 2ν-0.,+2λ-0..a,,U-1.+,V-1....
,,,−λ−,λ-1.m,−2λ-0.a,,V-1..,-,,[,ρ-j.,,b-ji.,,B-ji..]-m+1,,q-1.... (17)

provided that the conditions easily obtainable from those mentioned in above result and [8] are satisfied.

Proof: To prove Result 1, first of all we express the I-operator involved in its left hand side in the integral form with the help of (11):

,I-x-ν,λ.,,,t-μ.,Γ-,,ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n..-.,,z-1.,,t.-,U-1..,,x−t.-,V-1.... =

$$x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} H\left(\phi, z_0\left(\frac{t}{x}\right)^{\nu_0}\left(1-\frac{t}{x}\right)^{\lambda_0};\gamma\right)$$

,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,t-x..-,ν-2..,,1−,t-x..-,λ-2...,,t-μ.,Γ-,,ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n..-.,,z-1.,,t.-,U-1..,,x−t.-,V-1...dt (18)

Then,we express incomplete elliptic integral and generalized Bessel's Mait-land function in terms of series from with the help of (1)& (5) ,next the IAF express in contour integral with the help of (8).Now we interchange the order of summation and t-integral and the left hand side assumes the following form (say Δ) after a little simplification:

$$\Delta= x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} H\left(\phi, z_0\left(\frac{t}{x}\right)^{\nu_0}\left(1-\frac{t}{x}\right)^{\lambda_0};\gamma\right)$$

,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,t-x..-,ν-2..,,1−,t-x..-,λ-2...,,t-μ.,Γ-,,ℵ.-,p-i.,,q-i.,,ρ-i.,r-m,n..-.,,z-1.,,t.-,U-1..,,x−t.-,V-1...dt
,,,x-μ.-.sinϕ-2,-π.Γ,,1-2.−γ..,m=0-∞-,,,γ.-qm.,,−z.-m.-,j=1-n-Γ,,α-j.m+,β-j.m+1.m!...,a=0-∞-,b=0-∞-,,,,1-2..-a+b.,,,,1-2.−γ.-a.,,,1-2..-b.,,z-0.,k-2.,sin-2.φ.-a.,,,sin-2.φ.-b.-,,,3-2..-a+b.a!b!...
.

(19)

Finally, on evaluating the t-integral and re-interpreting the result thus obtained in terms of incomplete Aleph function (IAF),we easily arrive at the required result after a little simplification.

The proof of Image 2 can be obtained by proceeding on similar lines to those given above.

3. In this section we shall present a theorem analogous to the Parseval Goldstein theorem for the operator of our study.

Theorem

If $$\psi_1(x) = I_x^{\nu,\lambda}[f_1(t)] \quad (20)$$

and $$\psi_2(x) = J_x^{\nu,\lambda}[f_2(t)] \quad (21)$$

then $$\int_0^{\infty}\psi_1(x)f_2(x)dx = \int_0^{\infty}\psi_2(x)f_1(x)dx \quad (22)$$

provided that the integrals involved in (20), (21)and (22) are absolutelyconvergent.

Proof: To prove the above theorem, we substitute the value of $\psi_1(x)$ from (20) in the left hand side of (22) and obtain the following form, using the definition (11) of the I-operator:

$$\int_0^\infty \psi_1(x) f_2(x)dx = \int_0^\infty \left\{ x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda H\left(\phi, z_0\left(\frac{t}{x}\right)^{\nu_0}\left(1-\frac{t}{x}\right)^{\lambda_0};\gamma_0\right)\right.$$

,,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,t-x..-,ν-2..,,1−,t-x..-,λ-2...,f-1.,t.dt.,f-2.,x.dx (23)

Now we interchange the order of x- and t- integrals (which is permissible under given conditions) and get:

$$= \int_0^\infty t^\nu f_1(t) \left\{ \int_x^t x^{-\nu-\lambda-1} (x-t)^\lambda H\left(\phi, z_0\left(\frac{t}{x}\right)^{\nu_0}\left(1-\frac{t}{x}\right)^{\lambda_0};\gamma\right)\right.$$,,J-,,β-j..,q-,,α-i..,γ.,,z-1.,,,t-x..-,ν-2..,,1−,t-x..-,λ-2...,f-2.,x.dx.dt

(24)

Now on interpreting the expression thus obtained in the form of J-operator with the help of (14), we arrive at the desired result by (21) after a little simplification.

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