The trace extension property and cyclic Module Amenability of Banach Algebras

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Abstract.
In this paper, the relationship between the closed ideal I equipped the trace extension property and its source Banach algebra A with cyclic module amenability, has been investigated.

Keywords: Trace extension property, Cyclic derivation, Cyclic module derivation, Cyclic module amenability, Weak module amenability.
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1.Introduction and preliminaries

The concept of homology was considered one of the most important topics that studied in the middle of the 20th century in algebra trends, including algebraic geometry, algebraic topology, and displacement algebra, until Grothendieck, Kamowitz, and then Johnson raised this topic and used it in Banach algebras and related the concept of cohomology with the concept of homology by an analogy. Moreover Amini in [1] developed the concept of module amenability fora class of Banach algebras which is in fact a generalization ofthe Johnson's amenability.

Let Abe aBanach algebra and$X$ be a Banach A-bimodule. A bounded (continuous) linear map $D: A \longrightarrow X$ is called a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

Every $x \in X$ defines a derivation $ad_x(a) = a \cdot x - x \cdot a$ $(a \in A)$, which is called an inner derivation. We use the notations $\mathcal{Z}^1(A, X)$ and $\mathcal{B}^1(A, X)$ for the set of all derivationsfrom A into X, and the set of all inner derivationsfrom A into X, respectively. The first cohomology group of A with coefficients in X is defined to be

$$\mathcal{H}^1(A, X) := \mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X).$$

Let Abe a Banach algebra. We know that dual of A, which is displayed by A^*, is a Banach A-bimodule, with the following action

$$(a \cdot f)(b) = f(b \cdot a) \text{and} (f \cdot a)(b) = f(a \cdot b) (a, b \in A, f \in A^*).$$

A derivation $D: A \to A^*$ is called cyclic if $[D(a)](b) + [D(b)](a) = 0$, for every $a, b \in A$. We use the notation $\mathcal{ZC}^1(A, A^*)$ for the set of all cyclic derivations from A into A^*, and the first cyclic cohomology group of A is defined to be

$$\mathcal{HC}^1(A, A^*) := \mathcal{ZC}^1(A, A^*)/\mathcal{B}^1(A, A^*).$$

Definition 1.1 *Let Abe a Banach algebra. A is called; amenable, weak amenable and cyclic amenable, if for every Banach A-bimodule X, $\mathcal{H}^1(A, X^*) = 0$, $\mathcal{H}^1(A, A^*) = 0$ and $\mathcal{HC}^1(A, A^*) = \{0\}$, respectively.*

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, and let X be a left Banach A-module and a Banach $\mathfrak{A}$-bimodule if the actions are compatible, i.e.,

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x,\ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,\ a \cdot (x \cdot \alpha) = (a \cdot x) \cdot \alpha,$$

for all $a \in A, \alpha \in \mathfrak{A}$ and $x \in X$, then X is called a left Banach A-$\mathfrak{A}$-module. The right Banach A-$\mathfrak{A}$-module is defined similarly. If X be two-side Banach A-$\mathfrak{A}$-module, it is called Banach A-$\mathfrak{A}$-module. Also, X is called a commutative (bi-commutative) Banach A-$\mathfrak{A}$-module, if $\alpha \cdot x = x \cdot \alpha (a.x = x.a)$ for all $\alpha \in \mathfrak{A}, a \in A$ and $x \in X$.

A bounded map $D: A \longrightarrow X$ is called an $\mathfrak{A}$-module derivation if for all $a, b \in A$ and $\alpha \in \mathfrak{A}$:

$$D(a \pm b) = D(a) \pm D(b), D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha,$$

and

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

While X is commutative, each $x \in X$ defines a module derivation $ad_x(a) = a \cdot x - x \cdot a \quad (a \in A)$. It is called inner module derivation. Moreover, $\mathfrak{A}$-module derivation $D: A \longrightarrow A^*$ is called cyclic if it satisfy the following condition $[D(a)](b) + [D(b)](a) = 0$.

We use the notation $\mathcal{ZC}^1_{\mathfrak{A}}(A, X)$ for the set of all cyclic $\mathfrak{A}$-module derivations $D: A \to X$, and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$ for those which are inner. The first cyclic $\mathfrak{A}$-module cohomology group with coefficients in X is denoted by $\mathcal{HC}^1_{\mathfrak{A}}(A, X)$ which is the quotient group $\mathcal{ZC}^1_{\mathfrak{A}}(A, X)/\mathcal{B}^1_{\mathfrak{A}}(A, X)$. 115

Definition 1.2 *Let A be a Banach algebra. A is called module amenable, if for every commutative Banach A-bimodule X, $\mathcal{H}^1_{\mathfrak{A}}(A, X^*) = 0$ and A is called weak module amenable, if $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$. Also A is called cyclic module amenable, if $\mathcal{HC}^1_{\mathfrak{A}}(A, A^*) = \{0\}$.*

2.The Main Results

Definition 2.1 *Let I be a closed ideal in A. We say that I has the trace extension property if for each $\lambda \in I^*$ with $a.\lambda = \lambda.a\ (a \in A)$ there is $\Lambda \in A^*$ such that $\Lambda_I = \lambda$ and $a \cdot \Lambda = \Lambda \cdot a$ for every $a \in A$.*

In the following, the conditions which extended the derivation D to $\widetilde{D}$ have been presented. Grønbæk and others have proved, we also mentioned it. We say that a bounded approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ for a Banach algebra A is quasi-central if for each $a \in A$, $\lim \| ae_\gamma - e_\gamma a \| = 0$.

Theorem 2.2 *Suppose I has a bounded approximate identity quasi-central for A. Then I has the trace extension property and every derivation $D: I \longrightarrow I^*$ can be lifted to a derivation $\widetilde{D}: A \longrightarrow A^*$.*

Proof. Suppose that $(e_\gamma)_{\gamma\in\Gamma}$ is a quasi-central bounded approximate identity for I. We define a family of bounded linear maps $\{J_\gamma: I^* \longrightarrow A^*\}_{(\gamma\in\Gamma)}$ by

$$\langle a, J_\gamma(m)\rangle = \langle e_\gamma a, m\rangle \quad (a \in A, m \in I^*).$$

By identifying $B(I^*, A^*) = (I^* \widehat{\otimes} A)^*$ and passing to a subnet if necessary, we may assume that the net $(J_\gamma)_{\gamma\in\Gamma}$ is convergent in the weak operator topology to a bounded linear map $J: I^* \longrightarrow A^*$. Using the quasi-central property of $(e_\gamma)_{\gamma\in\Gamma}$ one easily checks that J is a module map with the property that $J(m)$ is an extension of m for each $m \in I^*$.

Obviously, if $m \in Z^0(A, I^*)$ then, $J(m)$ is a trace. If $D: I \longrightarrow I^*$ is a derivation we first extend to a derivation $\overline{D}: A \longrightarrow I^*$, using the standard construction

$$\overline{D}(a) = w^* - \lim_\gamma D(e_\gamma a).$$

Then $\widetilde{D} = J\overline{D}$ is the desired derivation.

Remark 2.3 *Let A be a Banach algebra and $A^{\#} = A \oplus \mathbb{C}$ its unitization. Any $f \in (A^{\#})^*$ has the form (f_0, λ), where, $f_0 \in A^*$ and $\lambda = f(1) \in \mathbb{C}^* = \mathbb{C}$.*

Because,

$$\forall (a,\alpha) \in A^{\sharp}, \quad f(a,\alpha) = f(a,0) + f(0,\alpha) = f(a,0) + \alpha f(0,1).$$

If $f_0(a)$ and λ are defined by $f_0(a) = f(a,0)$ and $\lambda = f(1) = f(0,1)$, then

$$f = (f_0,\lambda)(\lambda \in \mathbb{C}, f_0 \in A^*)$$

Also, since

$$\begin{aligned}(f_0,\lambda)(a,\alpha) &= f_0(a) + \lambda\alpha \\ &= f(a,0) + \alpha f(0,1) \\ &= f(a,0) + f(0,\alpha) \\ &= f(a,\alpha),\end{aligned}$$

then $(A \oplus \mathbb{C})^* = A^* \oplus \mathbb{C}$.

Moreover, the $A^{\sharp}$-module action on $(A^{\sharp})^*$ is

$$(a,\alpha)(f_0,\lambda) = (a.f_0 + \alpha f_0, f_0(a) + \alpha\lambda), (a,\alpha) \in A^{\sharp}, f = (f_0,\lambda) \in (A^{\sharp})^*$$

Because, if $(b,\beta) \in A \oplus \mathbb{C}$, then

$$\begin{aligned}\langle (b,\beta),(a,\alpha)(f_0,\lambda)\rangle &= \langle (b,\beta)(a,\alpha),(f_0,\lambda)\rangle \\ &= \langle (ba + \beta a + \alpha b, \alpha\beta),(f_0,\lambda)\rangle \\ &= f_0(ba + \alpha b + \beta a) + \lambda\alpha\beta \\ &= f_0(ba) + \alpha f_0(b) + \beta f_0(a) + \beta(\lambda\alpha) \\ &= ((a.f_0) + \alpha f_0)(b) + (f_0(a) + \alpha\lambda)\beta \\ &= (a.f_0 + \alpha f_0, f_0(a) + \alpha\lambda)(b,\beta) \\ &= \langle (b,\beta),(a.f_0 + \alpha f_0, f_0(a) + \alpha\lambda)\rangle.\end{aligned}$$

Theorem 2.4 *Let A be an non-unital Banach algebra, then A is cyclic amenable if and only if $A^{\#}$ is too.*

Proof. Suppose that A is cyclic amenable, and $D: A^{\sharp} \longrightarrow (A^{\sharp})^*$ is a cyclic derivation. We must to show that D is inner.

We know that, any functional in $(A^{\sharp})^*$ has following from (f,λ) where, $f \in A^*$ and $\lambda \in \mathbb{C}$ and this functional acts on any element $(a,\alpha) \in A^{\sharp}$ as follows;

$$(f,\lambda)(a,\alpha) = f(a) + \lambda\alpha.$$

Now, $D: A^{\sharp} \longrightarrow (A^{\sharp})^*$ is cylic derivation. In particular, for any $a \in A$‹the functional $D(a,0)$ in $(A^{\sharp})^*$ is of the form $D(a,0) = (f,\lambda)$, where, $f \in A^*$ and $\lambda \in \mathbb{C}$ are depend on a. Give; $f_a = D_0(a)$ and λ_a‹so we have

$$D(a,0) = (D_0(a),\lambda_a) = (f_a,\lambda_a).$$

and the map

$$\begin{aligned}&D_0: A \longrightarrow A^* \\ &a \longmapsto D_0(a)\end{aligned}$$

is a derivation on A. Note that, $D(0,1) = 0$, so

$$D(a,\alpha) = D(a,0) + \alpha D(0,1) = D(a,0) = (D_0(a),\lambda(a))$$

At the sequel, we use a instance $(a,0)$. Since,

$$\begin{aligned}&D: A^{\sharp} \longrightarrow (A^{\sharp})^* \\ &(a,\alpha) \longrightarrow D(a,\alpha)\end{aligned}$$

is cyclic, then

$$(D(a,\alpha))(b,\beta) + (D(b,\beta))(a,\alpha) = 0$$

but by the definition of the derivation $(A^{\sharp})^*$ on (b,β) we have,

$$(D_0(a))(b) + \lambda_a\beta + (D_0(b))(a) + \lambda_b\alpha = 0,$$

thus, $\lambda_a\beta + \lambda_b\alpha = 0(\alpha, \beta \in \mathbb{C})$.

Now, let $\alpha = 0$ and $\beta = 1$. Put $\lambda_a = 0$ we conclude that

$$D(a) = (D_0(a), \lambda_a)$$

Hence,

$$D(a) = (D_0(a), 0)$$

that is

$$D(a) = D_0(a)$$

In fact, $D_0: A \longrightarrow A^*$ is cyclic and cyclic amenability of A deduced that D_0 is an inner derivation. Therefore, D is inner too.

Conversely, suppose that $A^\sharp$ is cyclic amenable and $D: A \longrightarrow A^*$ is an cyclic derivation. We define

$\widetilde{D}: A^\sharp \longrightarrow (A^\sharp)^*$ by $\widetilde{D}(a, \alpha) := (D(a), 0)$.

Since $\widetilde{D}(a, \alpha) \in (A^\sharp)^*$ so has the form (f_0, λ), where $f_0 \in A^*$ and $\lambda \in \mathbb{C}$.

Due to $D(a) \in A^*$, so we give f_0 and $D(a)$ as the same and $\lambda = 0$.

Now, at the first $\widetilde{D}$ is a derivation, because for any $(b, \beta), (a, \alpha) \in A^\sharp$ we have

$$\begin{aligned}
&\widetilde{D}((a, \alpha).(b, \beta)) = \widetilde{D}(ab + a\beta + \alpha b, \alpha\beta) \\
&= (D(ab + a\beta + \alpha b), 0) \\
&= a.(Db) + (Da).b + \beta D(a) + \alpha D(b) + (D(a))(b) + (D(b))(a) \\
&= (Da).b + \beta D(a) + (D(a))(b) + a.(Db) + \alpha D(b) + (D(b))(a) \\
&= (D(a), 0).(b, \beta) + (a, \alpha).(D(b), 0) \\
&= \widetilde{D}(a, \alpha).(b, \beta) + (a, \alpha).\widetilde{D}(b, \beta).
\end{aligned}$$

At the second, based on the definition of $\widetilde{D}$ it is clear that $\widetilde{D}$ is cyclic derivation on $A^\sharp$. Since, $A^\sharp$ cyclic amenable, then $\widetilde{D}$ must be inner. So there exists $(f, \alpha) \in (A^\sharp)^* = A^* + \mathbb{C}$ that, $\widetilde{D} = ad_{(f,\lambda)}$.

Now,

$$\begin{aligned}
&D(a) = \widetilde{D}(a, \alpha) = (a, \alpha).(f, \lambda) - (f, \lambda).(a, \alpha) \\
&= (a.f + \alpha f, f(a) + \alpha\lambda) - (f.\alpha + \alpha f, f(a) + \lambda\alpha) \\
&= a.f - f.a
\end{aligned}$$

but $D(a) = \widetilde{D}(a, \alpha)$ so $D(a) = a.f - f.a$. Therefore D is inner.

Proposition 2.5 *Let A/I be cyclic $\mathfrak{A}$-module amenable. Then I has the trace extension property.*

Proof. Let $\lambda \in I^*$, such that $a \cdot \lambda = \lambda \cdot a$ for every $a \in A$. By the Hahn-Banach extension theorem take $\tau \in A^*$ with $\tau|_I = \lambda$. Define

$$\begin{aligned}
&D: A/I \to A^* \\
&[a] \mapsto a \cdot \tau - \tau \cdot a.
\end{aligned}$$

We see immediately that D is a derivation. Furthermore for every $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned}
&D(\alpha \cdot [a]) = D([\alpha \cdot a]) = (\alpha \cdot a) \cdot \tau - \tau \cdot (\alpha \cdot a) \\
&= \alpha \cdot (a \cdot \tau) - (\tau \cdot \alpha) \cdot a = \alpha \cdot (a \cdot \tau) - (\alpha \cdot \tau) \cdot a \\
&= \alpha \cdot (a \cdot \tau - \tau \cdot a) = \alpha \cdot D([a]).
\end{aligned}$$

Similarly we can show that $D([a] \cdot \alpha) = D([a]) \cdot \alpha$. This means that D is $\mathfrak{A}$-module map. Suppose that $i \in I$ and $a \in A$, since $i \cdot a - a \cdot i \in I$,

$$\begin{aligned}
&D([a])(i) = (a \cdot \tau - \tau \cdot a)(i) = \tau(i \cdot a - a \cdot i) \\
&= \lambda(i \cdot a - a \cdot i) = (a \cdot \lambda - \lambda \cdot a)(i) \\
&= 0.
\end{aligned}$$

This shows that $D([a])|_I = 0$, for every $a \in A$. So $\operatorname{Im} D \subseteq (A/I)^* = I^\perp$. Therefore $D \in \mathcal{Z}^1_{\mathfrak{A}}(A/I, (A/$

$I)^*)$. On the other hand, for every $[a],[b]\in A/I$,

$$\begin{aligned}\langle[a],D([b])\rangle+\langle[b],D([a]\rangle &= \langle[a],b\cdot\tau-\tau\cdot b)\rangle+\langle[b],a\cdot\tau-\tau\cdot a\rangle\\ &=\langle[a\cdot b-b\cdot a],\tau\rangle+\langle[b\cdot a-a\cdot b],\tau\rangle\\ &=\langle[a\cdot b-b\cdot a+b\cdot a-a\cdot b],\tau\rangle\\ &=0.\end{aligned}$$

Hence D is cyclic and so $D\in\mathcal{ZC}^1_{\mathfrak{A}}(A/I,(A/I)^*)$. Since A/I is cyclic $\mathfrak{A}$-module amenable, there exists a $\lambda^{'}\in(A/I)^*=I^{\perp}$such that $D=ad_{\lambda^{'}}$. Set $\Lambda=\tau-\lambda^{'}\in A^*$. We have

$$\Lambda|_I=(\tau-\lambda^{'})|_I=\tau|_I-\lambda^{'}|_I=\tau|_I-0=\lambda$$

and for every $a\in A$,

$$\begin{aligned}a\cdot\Lambda-\Lambda\cdot a&=(\tau-\lambda^{'})\cdot a-a\cdot(\tau-\lambda^{'})\\ &=(\tau\cdot a-a\cdot\tau)+(a\cdot\lambda^{'}-\lambda^{'}\cdot a)\\ &=(\tau\cdot a-a\cdot\tau)+([a]\cdot\lambda^{'}-\lambda^{'}\cdot[a])\\ &=-D([a])+ad_{\lambda^{'}}([a])\\ &=-D([a])+D([a])=0.\end{aligned}$$

This shows that I has the trace extension property.

Proposition 2.6 *Let A be cyclic $\mathfrak{A}$-module amenable and I has the trace extension property. Then A/I is cyclic $\mathfrak{A}$-module amenable.*

Proof. Suppose $\pi: A\longrightarrow A/I$ is the quotient map and $D\in\mathcal{ZC}^1_{\mathfrak{A}}(A/I,(A/I)^*)$. Set $\widetilde{D}=\pi^*\circ D\circ\pi: A\to A^*$. Clearly $\widetilde{D}$ is a derivation. Furthermore for every $\alpha\in\mathfrak{A}$ and $a,b\in A$,

$$\begin{aligned}\langle b,\widetilde{D}(\alpha\cdot a)\rangle&=\langle b,(\pi^*\circ D\circ\pi)(\alpha\cdot a)\rangle=\langle[b],D([\alpha\cdot a])\rangle\\ &=\langle[b],\alpha\cdot D([a])\rangle=\langle[b]\cdot\alpha,D([a])\rangle\\ &=\langle[b\cdot\alpha],D([a])\rangle=\langle b\cdot\alpha,(\pi^*\circ D\circ\pi)(a)\rangle\\ &=\langle b\cdot\alpha,\widetilde{D}(a)\rangle=\langle b,\alpha\cdot\widetilde{D}(a)\rangle.\end{aligned}$$

So $\widetilde{D}(\alpha\cdot a)=\alpha\cdot\widetilde{D}(a)$ and similarly we can show that $\widetilde{D}(a\cdot\alpha)=\widetilde{D}(a)\cdot\alpha$. This means that $\widetilde{D}$ is $\mathfrak{A}$-module map. Since D is cyclic, for every $a,b\in A$, we have

$$\begin{aligned}\langle b,\widetilde{D}(a)\rangle+\langle a,\widetilde{D}(b)\rangle&=\langle b,(\pi^*\circ D\circ\pi)(a)\rangle+\langle a,(\pi^*\circ D\circ\pi)(b)\rangle\\ &=\langle[b],D([a])\rangle+\langle[a],D([b])\rangle\\ &=0.\end{aligned}$$

So $\widetilde{D}$ is cyclic and thus $\widetilde{D}\in\mathcal{ZC}^1_{\mathfrak{A}}(A,(A^*)$. Since A is cyclic amenable, there exists $\lambda\in A^*$ with

$$\widetilde{D}(a)=ad_\lambda(a)=a\cdot\lambda-\lambda\cdot a\qquad(a\in A).$$

Clearly $\widetilde{D}(a)|_I=0$. Set $\lambda'=\lambda|_I$. For every $a\in A$, we have

$$\begin{aligned}a\cdot\lambda'-\lambda'\cdot a&=a\cdot(\lambda|_I)-(\lambda|_I)\cdot a=(a\cdot\lambda)|_I-(\lambda\cdot a)|_I\\ &=(a\cdot\lambda-\lambda\cdot a)|_I=\widetilde{D}(a)|_I\\ &=0.\end{aligned}$$

Thus by assumption there exists a $\Lambda\in A^*$ such that $\Lambda|_I=\lambda'$ and $a\cdot\Lambda-\Lambda\cdot a=0$ for every $a\in A$. But $\lambda-\Lambda\in I^{\perp}$ and for every $a,b\in A$,

$$\begin{aligned}\langle[b],D([a])\rangle&=\langle b,(\pi^*\circ D\circ\pi)(a)\rangle\\ &=\langle b,\widetilde{D}(a)\rangle\\ &=\langle b,a\cdot\lambda-\lambda\cdot a\rangle\\ &=\langle b,a\cdot(\lambda-\Lambda)-(\lambda-\Lambda)\cdot a\rangle\\ &=\langle[□],[□]\cdot(□-\Lambda)-(□-\Lambda)\cdot[□]\rangle.\end{aligned}$$

This shows that □([□]) = □□$_□$([□]), where □ = □ − Λ. Thus □ is inner and it follows that □/□ is cyclic □-module amenable.

Proposition 2.7 *Let □ and □/□ be cyclic □-module amenable and $\overline{□^2}$ = □. Then □ is cyclic*

▯-module amenable.

Proof. Suppose ▯: ▯ ⟶ ▯ is the natural embedding and ▯ ∈ ▯▯$^1_▯$(▯, ▯*). Define ▯′: = ▯* ∘ ▯ ∘ ▯. It is easy to prove that ▯′ is cyclic derivation. We show that ▯′ is ▯-module map. Let ▯ ∈ ▯ and ▯, ▯ ∈ ▯,

⟨▯, ▯′(▯ · ▯)⟩ = ⟨▯, (▯* ∘ ▯ ∘ ▯)(▯ · ▯)⟩ = ⟨▯, ▯(▯ · ▯)⟩
= ⟨▯, ▯ · ▯(▯)⟩ = ⟨▯ · ▯, ▯(▯)⟩
= ⟨▯ · ▯, (▯* ∘ ▯ ∘ ▯)(▯)⟩ = ⟨▯ · ▯, ▯′(▯)⟩
= ⟨▯, ▯ · ▯′(▯)⟩.

So ▯′ ∈ ▯▯$^1_▯$(▯, ▯*) and since ▯ is cyclic ▯-module amenable, there exists ▯ ∈ ▯* with

(▯* ∘ ▯)(▯) = ▯▯$_▯$(▯)(▯ ∈ ▯).

Now define $\widetilde{D} = D - \mathrm{ad}_{\tilde{\lambda}}$, where $\tilde{\lambda} \in A^*$ is extended of λ. Therefore $(\boldsymbol{\iota}^* \circ \widetilde{D})|_I = 0$ and for every $i, j \in I$ and $a \in A$,

$$\begin{aligned}\langle a, \widetilde{D}(ij)\rangle &= \langle a, i \cdot \widetilde{D}(j) + \widetilde{D}(i) \cdot j\rangle \\ &= \langle a, i \cdot \widetilde{D}(j)\rangle + \langle a, \widetilde{D}(i)j\rangle \\ &= \langle ai, \widetilde{D}(j)\rangle + \langle ja, \widetilde{D}(i)\rangle \\ &= \langle ai, (\boldsymbol{\iota}^* \circ \widetilde{D})(j)\rangle + \langle ja, (\boldsymbol{\iota}^* \circ \widetilde{D})(i)\rangle \\ &= 0.\end{aligned}$$

Hence $\widetilde{D}|_{I^2} = 0$. Since $\overline{I^2} = I$ so $\widetilde{D}|_I = 0$. If we set $F = \overline{IA + AI}$, then $F = \overline{I^2} = I$. Now for each $a \in A$ and $i \in I$, $\widetilde{D}(a) \cdot i = \widetilde{D}(ai) - a \cdot \widetilde{D}(i) = 0$, and so $\widetilde{D}(a) \cdot i = 0$. Taking $b \in A$, we get

$$\langle i \cdot b, \widetilde{D}(a)\rangle = \langle b, \widetilde{D}(a) \cdot i\rangle = 0,$$

and so $\widetilde{D}(a)|_{IA} = 0$. Similarly $\widetilde{D}(a)|_{AI} = 0$, and hence $\widetilde{D}(a)|_I = 0$. Thus $\mathrm{Im}\widetilde{D} \subseteq I^{\perp}$ and the map

$$\begin{aligned}\widehat{D}&: A/I \longrightarrow (A/I)^* = I^{\perp} \\ &[a] \longmapsto \widetilde{D}(a)\end{aligned}$$

is well define. Regard the $\widetilde{D}(a): A/I \longrightarrow \mathbb{C}$ by $\langle [b], \widetilde{D}(a)\rangle = \langle b, \widetilde{D}(a)\rangle$. Clearly $\widehat{D}$ is derivation. Let $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned}\widehat{D}([\alpha \cdot a]) &= \widetilde{D}(\alpha \cdot a) \\ &= (D - ad_{\tilde{\lambda}})(\alpha \cdot a) \\ &= D(\alpha \cdot a) - ad_{\tilde{\lambda}}(\alpha \cdot a) \\ &= \alpha \cdot D(a) - (\alpha \cdot a) \cdot \tilde{\lambda} + \tilde{\lambda} \cdot (\alpha \cdot a) \\ &= \alpha \cdot D(a) - \alpha \cdot (a \cdot \tilde{\lambda}) + (\tilde{\lambda} \cdot \alpha) \cdot a \\ &= \alpha \cdot D(a) - \alpha \cdot (a \cdot \tilde{\lambda}) + (\alpha \cdot \tilde{\lambda}) \cdot a \\ &= \alpha \cdot D(a) - \alpha \cdot (a \cdot \tilde{\lambda}) + \alpha \cdot (\tilde{\lambda} \cdot a) \\ &= \alpha \cdot (D(a) - a \cdot \tilde{\lambda} + \tilde{\lambda} \cdot a) \\ &= \alpha \cdot (D - ad_{\tilde{\lambda}})(a) \\ &= \alpha \cdot \widetilde{D}(a) = \alpha \cdot \widehat{D}([a])\end{aligned}$$

so $\widehat{D}([\alpha \cdot a]) = \alpha \cdot \widehat{D}([a])$ and similarly $\widehat{D}([a \cdot \alpha]) = \widehat{D}([a]) \cdot \alpha$. This means that $\widehat{D}$ is $\mathfrak{A}$-module map. Since D is cyclic, for every $a, b \in A$, we have

$$\begin{aligned}\langle [b], \widehat{D}([a])\rangle + \langle [a], \widehat{D}([b])\rangle &= \langle [b], \widetilde{D}(a)\rangle + \langle [a], \widetilde{D}(b)\rangle \\ &= \langle b, \widetilde{D}(a)\rangle + \langle a, \widetilde{D}(b)\rangle \\ &= \langle b, (D - ad_{\tilde{\lambda}})(a)\rangle + \langle a, (D - ad_{\tilde{\lambda}})(b)\rangle \\ &= \langle b, D(a)\rangle + \langle a, D(b)\rangle - \langle b, ad_{\tilde{\lambda}}(a)\rangle - \langle a, ad_{\tilde{\lambda}}(b)\rangle \\ &= 0 - \langle ba - ab, \tilde{\lambda}\rangle - \langle ab - ba, \tilde{\lambda}\rangle \\ &= 0.\end{aligned}$$

This shows that $\widehat{D}$ is cyclic. According to previous findings, $\widehat{D} \in \mathcal{ZC}^1_{\mathfrak{A}}(A/I, (A/I)^*)$ and since by hypothesis A/I is cyclic $\mathfrak{A}$-module amenable, so there exists a $\lambda' \subseteq I^{\perp}$ such that $\widehat{D} = ad_{\lambda'}$. With some simple calculations given to the reader, it can be shown that $D = ad_{\lambda+\lambda'}$. It now follows that A

is cyclic $\mathfrak{A}$-module amenable.

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