CRITICAL STUDY ON PROPERTIES OF ANNIHILATOR GRAPH OF A COMMUTATIVE RING

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ABSTRACT :
Let R be a commutative ring. In this paper, We introduced a zero-divisor based graph structure of a commutative ring R, and we denote this graph by $\Gamma_Z(R)$. In our discussion, we study the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma_Z(R)$. In our investigation, we characterize some basic properties of $\Gamma_Z(R)$ related to connectedness, diameter and girth. We show that $\Gamma_Z(R)$ is connected and diameter of $\Gamma_Z(R)$ is at most 2. If $\Gamma_Z(R)$ contains a cycle, we show that girth of $\Gamma_Z(R)$ is at most 3. We examine the diameter of $\Gamma_Z(R)$ for a direct product $R = R_1 \times R_2$ of two commutative rings R_1 and R_2 with respect to the zero-divisors and regular elements of R_1 and R_2. Then we study the diameter of $\Gamma_Z(R)$ for a finite direct product $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$ ($n > 2$) of commutative rings $R_1, R_2, R_3,\ldots,R_n$ ($n > 2$) with respect to the zero-divisors and regular elements of $R_1, R_2, R_3,\ldots,R_n$ ($n > 2$).

KEYWORDS : annihilator graph , commutative ring , structure , zero-divisor
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PROPERTIES OF ANNIHILATOR GRAPH OF A COMMUTATIVE RING :

Throughout this chapter, all rings are commutative ring not necessarily with unity unless otherwise stated. We denote X – {0} by X* for any subset X of a commutative ring R. The sets Z(R) and N(R) respectively denote the set of all the zero- divisors and set of all the nilpotent elements of a commutative ring R. We use 0 = (0, 0) and 0 = (0, 0, 0,…,0) when necessary.

1. Definitions and Preliminaries:

Here, A Zero-divisor based graph structure are as follows:

Definition 1. Let R be a commutative ring. Let $\mathcal{Z}_Z(R) = \{x \in R \mid xy \in Z(R) \text{ for some } y \in R^*\}$. A zero-divisor based graph structure of R can be defined as the undirected graph $\Gamma_Z(R)$ whose vertex set is $\mathcal{Z}_Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy \in Z(R)$, where Z(R) is the set of all zero-divisors of R.

D. F. Anderson and P. S. Livingston defined the zero-divisor graph of a commutative ring.

Definition 2. Let R be a commutative ring with unity. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$.

Ai- Hua Li and Qi - Sheng Li defined the nilpotent graph of a commutative ring.

Definition 3. Let R be a commutative ring. The nilpotent graph of R, denoted by $\Gamma_N(R)$, is the undirected graph whose vertex set is $\mathcal{Z}_N(R)^*$, and two distinct vertices x and y in $\mathcal{Z}_N(R)^*$ are adjacent if and only if $xy \in N(R)$, where $\mathcal{Z}_N(R) = \{x \in R \mid xy \in N(R) \text{ for some } y \in R^*\}$ and N(R) is the set of all nilpotent elements of R.

Since {0} and N(R) are subsets of Z(R), from the definitions it follows that $\Gamma(R)$ and $\Gamma_N(R)$ are subgraphs of $\Gamma_Z(R)$.

The proof of the Lemma 1, Lemma 2 and Lemma 3 are available in standard literature. But here we give the independent proof of them.

Lemma 1. Let R be a commutative ring and x and y be two elements of R. Then $xy \in Z(R)$ if and only if $x \in Z(R)$ or $y \in Z(R)$.

Proof. Let R be a commutative ring and x and y be two elements of R. Without loss of generality assume that $x \in$ Z(R). Then we have xz = 0 for some nonzero $z \in$ R. Then $(xy)\ z$ = (xz) = 0. Hence $xy \in$ Z(R). Conversely, suppose that $xy \in$ Z(R) and $x \notin$ Z(R). Then $(xy)\ z$ = 0 for some nonzero $z \in$R. This implies (yz) = 0. Since $x \notin$ Z(R), we have yz = 0. Hence $y \in$ Z(R).

Remark 1. Let R be a commutative ring. Suppose that x and y be two elements of R such that $x \in$ Z(R)* and $y \in$ R – Z(R). Then xy ≠ 0. Thus $xy \in$ Z(R)* by Lemma 1.

Lemma 2. Let R_1, R_2, R_3,…,R_n ($n \geq 2$) be commutative rings such that R = $R_1\times R_2\times R_3\times\ldots\times R_n$ ($n \geq 2$). Then R is not an integral domain.

Proof. Let $x_i \in R_i^*$ and $y_j \in R_j^*$ for some i, j ∈ {1,2,3 … , n } , where i ≠ j. Suppose that x is an element of R all of whose coordinates are zero expect x_i and y is an element of R all of whose coordinates are zero except y_j. Then xy = 0. Thus Z(R)* is non-empty, and hence R is not an integral domain.

Lemma 3. Let R_1, R_2, R_3,…,R_n ($n \geq 2$) be commutative rings such that R = $R_1\times R_2\times R_3\times\ldots\times R_n$ ($n \geq 2$). Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be any non-zero element of R. Then $x \in$ Z(R)* if and only if $xi \in Z(R_i)$ for some i ∈ {1, 2,3, … , n }.

Proof. Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be any non-zero element of R. First suppose that $x_i \in Z(R_i)$ for some i ∈ {1, 2, 3, … , n }. Then there is some nonzero $z_i \in R_i$ such that x_iz_l = 0. Let z = (0, 0, 0, … , z_i, … , 0). Then $xz = (x_1, x_2, x_3 \ldots x_i, \ldots, x_n)$ (0, 0, 0, … , z_i, … , 0) = (0, 0, 0 … , x_iz_i, … , 0) = (0, 0,0, … ,0) = 0. Thus $x \in$ Z(R)*. Conversely, suppose that $x_i \notin Z(R_i)$ for each i ∈{1, 2, 3, … , n}. Then for each i ∈{1, 2, … , n}, we have $x_iy_i \neq 0$ for each nonzero $y_l \in R_i$. Let $z = (z_1, z_2, z_3, \ldots, z_n)$ be any non-zero element of R. Then $z_i \neq 0$ for at least one i ∈{1, 2, 3,…, n}. Therefore $xz = (x_1z_1, x_2z_2, \ldots, x_nz_n)$. Since $z_i \neq 0$ for at least one i ∈ {1, 2, 3, … , n}, it follows that $xz \neq$ (0, 0, 0, … ,0) = 0. This implies $x \notin$ Z(R)*. Thus $x \in$ Z(R)* implies $x_i \in Z(R_i)$ for some i ∈ {1, 2, 3, … , n }.

Theorem 1. Let R be a commutative ring. Then R is an integral domain if and only if Γ_Z(R) is empty graph.

Proof. Suppose that R is an integral domain. Then Z(R)* is empty. Thus there is no two nonzero elements x and y in R such that $xy \in$ Z(R). Therefore $\mathcal{Z}_Z$(R)* is empty and hence Γ_Z(R) is empty graph. Conversely, suppose that Γ_Z(R) is empty graph. Then $\mathcal{Z}_Z$(R)* is empty. Thus there is no two nonzero elements x and y in R such that $xy \in$ Z(R). This implies that no nonzero element of R belongs to Z(R)* by Lemma 1. Therefore Z(R)* is empty, and hence R is an integral domain.

Remark 2. Suppose that R is a commutative ring that is not an integral domain. Let $r \in$ R*. Then for some $x \in$ Z(R)*, we have $rx \in$ Z(R) by Lemma 4.1.1. Thus R* = $\mathcal{Z}_Z$(R)* and hence the vertices of Γ_Z(R) are the nonzero elements of R.

Theorem 2. Let R_1, R_2, R_3,…,R_n ($n \geq 2$) be commutative rings such that R = $R_1 \times R_2 \times R_3 \times\ldots\times R_n$ ($n \geq 2$). Then Γ_Z(R) has at least $2^n - 1$ vertices.

Proof. Since $|R_i| \geq 2$ for each i ∈ {1, 2, 3, … , n }, we have R has at least 2^n elements. Thus $|R^*| \geq 2^n - 1$. Also R is not an integral domain by Lemma 4.1.2. Thus R* = ZZ(R)*, and hence Γ_Z(R) has at least $2^n - 1$ vertices by Remark 2.

Theorem 3. Let R = $\mathbb{Z}p^n$, where p is a prime and $n \in \mathbb{N}$. Then $\Gamma_Z(R) = \Gamma_N(R)$.

Proof. Let R = $\mathbb{Z}p^n$. To prove that $\Gamma_Z(R) = \Gamma_N(R)$, we have to show Z(R) = N(R). Since N(R) = Z(R), it is enough to show that Z(R) ⊆ N(R). Let $x \in$ Z(R). Then $p|x$. If x = 0, then x ∈ N(R). So assume that $x \in$ R − N(R). Therefore $x^m \not\equiv 0$ (mod p^n) for each $m \in \mathbb{N}$. This implies $(yp)^m \not\equiv 0$ (mod p^n), where $y \in$ R and $y \neq 0$. In particular, $(yp)^n \not\equiv 0$ (mod p^n). This implies $p^n \nmid y^n p^n$. This implies $p^n \nmid p^n$, a contradiction. Thus we have $x \in$ N (R). Therefore Z(R) ⊆ N (R). Thus Z(R) = N(R) and hence $\Gamma_Z(R) = \Gamma_N(R)$.

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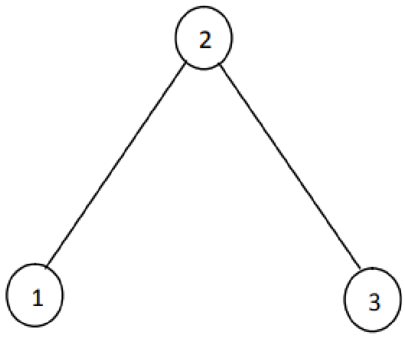
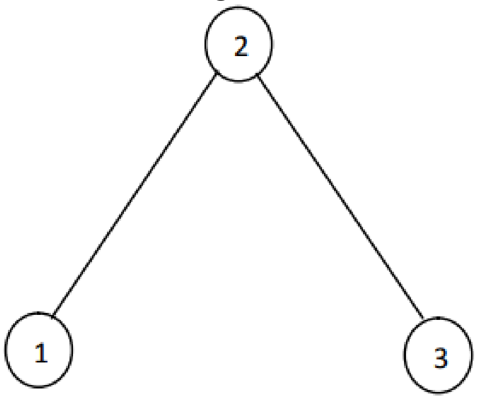
Example 1. Consider the ring R = $\mathbb{Z}_4 = \mathbb{Z}_2^2$. Then the graphs Γ_Z(R) and Γ_N(R) are as follows:



$\Gamma_Z(R)$

Figure 1



$\Gamma_N(R)$

Figure 2

Here Z(R) = N(R) = {0, 2} and $\Gamma_Z(R) = \Gamma_N(R) = K^{1,2}$.

2. Some Basic Properties of $\Gamma_Z(R)$:

The main aim of this section is to characterize the connectedness, diameter and girth of $\Gamma_Z(R)$ for a commutative ring R.

Theorem 1. Let R be a commutative ring that is not an integral domain. Then the following statements hold:

1. $\Gamma_Z(R)$ is connected;
2. $diam\ (\Gamma_Z(R)) \leq 2$;
3. If $\Gamma_Z(R)$ contains a cycle, then $(\Gamma_Z(R)) = 3$.

Proof. (1) and (2). Let x and y be any two distinct vertices of $\Gamma_Z(R)$.

Case 1. If $xy \in Z(R)$, then x and y are adjacent. So $(x, y) = 1$.

Case 2. Suppose that $xy \notin Z(R)$. Then neither x nor y belongs to Z(R) by Lemma 1. Then there is some $z \in Z(R)^*$ such that $xz \in Z(R)^*$ and $yz \in Z(R)^*$ by Remark 1. Hence $x - z - y$ is a path from x to y. So d(x, y) = 2.

Therefore we conclude that $\Gamma_Z(R)$ is connected and $diam\ (\Gamma_Z(R)) \leq 2$. This completes the proof of (1) and (2).

(3) Suppose that $\Gamma_Z(R)$ contains a cycle of length n, where $n \geq 4$ and $x_0 - x_1 - x_2 - x_3 - x_4 - \ldots - x_{n-1} - x_0$ is such a cycle. In the cycle $x_0 - x_1$ is an edge. So we have $x_0x_1 \in Z(R)$. This implies $x_0 \in Z(R)^*$ or $x_1 \in Z(R)^*$ by Lemma 1. Without loss of generality assume that $x_0 \in Z(R)^*$. Then $x_0x_i \in Z(R)$ for each i ∈ {1, 2, 3, … , $n - 1$} by Lemma 4.1.1. Thus $x_0 - x_i - x_{i+1} - x_0$ is a cycle of length 3, where i ∈ {1, 2, 3, … , $n - 2$}. Hence $gr\ (\Gamma_Z(R)) = 3$.

Example 1. (1) Consider the commutative ring $R = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_2\right\}$. Then $\Gamma_Z(R)$ is the trivial graph with vertex $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and hence $diam$ (Γ_Z(R)) = 0 and gr(Γ_Z(R)) = ∞.

(2) Consider the commutative ring $R = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_4 \right\}$ Then Γ_Z(R) = K^3, and hence $diam$ (Γ_Z(R)) = 1 and (Γ_Z(R)) = 3.

(3) Consider the commutative ring R = $\mathbb{Z}_4$. Then Γ_Z(R) = $K^{1,2}$, and hence $diam$ (Γ_Z(R)) = 2 and gr(Γ_Z(R)) = ∞.

Theorem 2. Let R be a commutative ring and Γ_Z(R) be non-trivial. Then the following statements are equivalent:

gr (Γ_Z(R)) = ∞;

Γ_Z(R) is a star graph;

R is either a ring of order 3 with R^2 = {0} or Z(R) is a prime ideal of R with |Z (R) | = 2.

Proof. (1) ⟺ (2). Suppose that (Γ_Z(R)) = ∞. Since Γ_Z(R) is non-trivial, then there exists a vertex of ΓZ (R) which is adjacent to all the other vertices. So Γ_Z(R) is a star graph.

Conversely, suppose that Γ_Z(R) is a star graph. Then (Γ_Z(R)) = ∞.

(1) ⟺ (3). Let gr(Γ_Z(R)) = ∞. If |Z (R)*|≥ 3, assume that x, y, z are three distinct elements of Z(R)*. Then xy, xz, yz ∈ Z(R) by Lemma 4.1.1 Thus Γ_Z(R) contains a triangle $x - y - z - x$, a contradiction. Hence,|Z (R)* |≤ 2.

Case 1. If |Z (R)*|= 2, then it is clear that R = Z(R), otherwise we shall get a triangle which contradicts our supposition. Assume that R = {0, x, y} where $x \neq y$ and x, y ≠ 0. Suppose that x^2 ≠ 0 or y^2 ≠ 0. Then xy = 0. Clearly, $x + y \neq x$, y. Also $x + y$ ≠ 0, otherwise x^2 = 0 and y^2 = 0. So x, y, $x + y$ are pair wise distinct elements of R*, a contradiction. Thus x^2 = 0 and y^2 = 0. If xy ≠ 0, without loss of generality assume that $xy = x$. Then 0 = xy^2 = (xy) = xy = x, a contradiction. Thus xy = 0. Therefore R is ring of order 3 with R^2 = {0}.

Case 2. If|Z (R)*|= 1, then we assume that Z(R) = {0, z}, where z ≠ 0. Then z^2 = 0. Now z – 0 = z ∈ Z (R) and 0 – z = – z. Since (– z) z = – (zz) = – z^2 = 0, we have –z ∈ Z (R); (here z = – z). Let r be any element of R. Then, br ∈ Z(R) for each b ∈ Z(R) by Lemma 4.1.1. Therefore Z(R) = {0, z} is an ideal of R. For any r, s ∈ R, suppose that rs ∈ Z(R). Then r ∈ Z(R) or s ∈ Z(R) by Lemma 4.1.1. Thus Z(R) is a prime ideal of R with |Z (R) |= 2.

Conversely, first suppose that R is ring of order 3 with R^2 = {0}. Assume that R = {0, x, y}. Then we have xy = 0 ∈ Z (R). So $\mathcal{Z}_Z$(R)* = {x, y} and x and y are adjacent. Thus gr (Γ_Z(R)) = ∞. Next suppose that Z(R) is a prime ideal of R with |Z (R) |= 2. Assume that Z(R) = {0, z}. Since Γ_Z(R)) is non-trivial, so we have |$\mathcal{Z}_Z$(R)*| = |R*|≥ 2. Thus, zt ∈ Z(R)* for each t ∈ R* – {z} by Remark 4.1.1. Thus z is adjacent to each element of R* – {z}. Let x, y ∈ R* – {z}. Then xy ∉ Z(R), as Z(R) is prime. Therefore no elements of R* – {z} are adjacent to each other. Hence gr (Γ_Z(R)) = ∞.

Example 2. (1) Consider the commutative ring R = $\mathbb{Z}_4$. Then Z(R) = {0, 2} is a prime ideal of R, Γ_Z(R) = $K^{1,2}$, and hence and gr(Γ_Z(R)) = ∞.

(2) Consider the commutative ring $R = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_3 \right\}$ Then $\mathrm{R} = \mathrm{Z(R)} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \right\}$ with R^2 = {0}, Γ_Z(R) = $K^{1,1}$, and hence and g(Γ_Z(R)) = ∞.

Theorem 3. Let R be a commutative ring that is not an integral domain. Then $diam$ (Γ_Z(R)) = 0 if and only if R = Z(R) with |R*| = 1.

Proof. Suppose that $diam$ (Γ_Z(R)) = 0. Since Z(R)* is non-empty, Γ_Z(R) is a trivial graph. Thus R = Z(R) with |R*| = 1. Conversely, suppose that R = Z(R) with |R*|= 1. Then Γ_Z(R) is a trivial graph, and hence $diam$ (Γ_Z(R)) = 0.

Corollary 1. Let R be a commutative ring that is not an integral domain. If R has unity, then $diam$ (Γ_Z(R)) ∈ {1, 2}.

Proof. Since R is commutative ring with unity that is not an integral domain, we have R ≠ Z(R) with |R*|≥ 2. Since Γ_Z(R) is connected and diameter of Γ_Z(R) is at most 2, we have $diam$ (Γ_Z(R)) ∈ {1, 2} by Theorem 3.

Example 3. (1) Consider the commutative ring $R = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_2\right\}$ Then R = Z(R) with $R* = \left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ Hence Γz(R) is the trivial graph, and hence $diam$ (Γz(R)) = 0.

(2) Consider the commutative ring $R = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}$. Then R is a ring with unity$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We have Γz(R) = K^3, and hence $diam$ (Γz(R)) = 1.

(3) Consider the commutative ring R = $\mathbb{Z}_4$. Then R is a ring with unity 1. We have Γz(R) = $K^{1,2}$, and hence $diam$ (Γz(R)) = 2.

Theorem 4. Let R be a commutative ring that is not an integral domain. Then $diam$ (Γz(R)) = 1 if and only if either R = Z(R) with | R*| ≥ 2 or |R - Z (R)|= 1.

Proof. Suppose that $diam$ (Γz(R)) = 1. So, every pair of distinct vertices of ΓZ (R) is adjacent and |R*|≥ 2. If possible, suppose |R - Z (R) |≥ 2. Then there exists distinct x, ? ∈ R – Z(R) such that ?? ∉ Z(R) by Lemma 1. Since Z(R)* ≠ ∅, there exists some ? ∈ Z(R)* such that ??, ?? ∈ Z(R)* by Remark 1. Thus d(?, ?) = 2, which contradicts that ?iam (Γz(R)) = 1. Therefore, we have either R = Z(R) with |R*|≥ 2 or |R - Z (R) |= 1.

Conversely, first suppose that R = Z(R) with |R*|≥ 2. Then Γz(R) has at least two vertices. Let x and y be any two distinct elements of $\mathcal{Z}$z(R)*. Since R = Z(R), we have xy ∈ Z(R). So, every pair of distinct vertices of Γz(R) is adjacent. Thus $diam$ (Γz(R)) = 1. Next, suppose that |R - Z (R) |= 1. Then Γz(R) has at least two vertices. Let $\mathcal{A}$ = {u ∈ R*|u ∈ Z(R)*} and $\mathcal{B}$ = {u ∈ R* |u ∉ Z(R)*}. From given conditions we have $\mathcal{A}$ ≠ ∅ and $\mathcal{B}$ ≠ ∅ . Then the vertex set of Γz(R) is $\mathcal{Z}$z(R)* = $\mathcal{A} \cup \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B}$ = ∅. Let u, v ∈ $\mathcal{A}$. Then uv ∈ Z(R) by Lemma 1. So, every pair of distinct vertices of $\mathcal{A}$ is adjacent. Since |R - Z (R) |= 1, $\mathcal{B}$ is a singleton. 2746

Let $\mathcal{B}$ = {u}. Then uv ∈ Z(R)* for every v ∈ $\mathcal{A}$ by Remark 1. So, u is adjacent to every vertices of $\mathcal{A}$. Thus $diam$ (Γz(R)) = 1.

Corollary 2. Let R be a commutative ring with unity that is not an integral domain. Then $diam$ (Γz(R)) =1 if and only if |R - Z (R) |= 1.

Proof. Since R is commutative ring with unity that is not an integral domain, we have R ≠ Z(R) with |R*|≥ 2. Then $diam$ (Γz(R)) = 1 if and only if |R - Z (R) |= 1 by Theorem 4.

Remark 1. Let R be a commutative ring with unity that is not an integral domain. Then Γz(R) complete if and only if |R - Z (R) |= 1 by Corollary 4.2.1 and Corollary 2.

Example 4. (1) Consider the commutative ring $R = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_3\right\}$. Then $R = Z(R) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right\}$. We have Γz(R) = $K^{1,1}$, and hence and $diam$(Γz(R)) = 1.

(2) Consider the commutative ring $R = \left\{\begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}$. Then R is a ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$and R – Z(R) =$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. We have Γz(R) = K^3, and hence $diam$ (Γz(R)) = 1.

Corollary 3. Let R be a commutative ring that is not an integral domain. Then $diam$ (Γz(R)) = 2 if and only if |R - Z (R) |≥ 2

Proof. Suppose that $diam$(Γz(R)) = 2. Then |R - Z (R) |≥ 2 by Theorem 3 and Theorem 4.

Conversely, suppose that |R - Z (R) |≥ 2. Then Γz (R) has at least three vertices. Since Γz(R) is connected and diameter of Γz(R) is at most two, we have $diam$(Γz(R)) = 2 by Theorem 4.

Example 5. Consider the commutative ring R = $\mathbb{Z}_4$. Then R – Z(R) = {1, 3}. We have Γz(R) = $K^{1,2}$, and hence $diam$ (Γz(R)) = 2.

Theorem 5. Let R be a commutative ring that is not an integral domain. Then Γz(R) is complete if and only if either R = Z(R) or |R - Z (R)|= 1.

Proof. Suppose that Γz(R) is complete. If Γz(R) is trivial, then $diam$(Γz(R)) = 0. Hence R = Z(R) with |R*|= 1 by Theorem 3. If Γz(R) is non-trivial, then $diam$(Γz(R)) = 1. Hence either R = Z(R) with |R*|≥ 2 or |R - Z (R) |= 1 by Theorem 4.

Conversely, first suppose that R = Z(R). If |R*|= 1, then $diam$(Γz(R)) = 0 by Theorem 4.2.3. Thus Γz(R) is trivial, and hence $\Gamma_Z(R)$ is complete. If $|R^*| \geq 2$, then $diam(\Gamma_Z(R)) = 1$ by Theorem 4, and hence $\Gamma_Z(R)$ is complete. Next suppose that $|R - Z(R)| = 1$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 4, and hence $\Gamma_Z(R)$ is complete.

Theorem 6. Let $f: R_1 \rightarrow R_2$ be a ring-monomorphism, where R_1 and R_2 are two commutative rings. If x and y are adjacent in $\Gamma_Z(R_1)$, then $f(x)$ and $f(y)$ are adjacent in $\Gamma_Z(f(R_1))$.

Proof. Suppose that x and y are adjacent in $\Gamma_Z(R_1)$. Then $xy \in Z(R_1)$. So, there exists some nonzero $z \in R_1$ such that $(xy)\ z = 0$. Therefore we have $f\ [(xy)\ z] = f\ (0)$. Since $f: R_1 \rightarrow R_2$ is a ring-monomorphism, we have $f\ (xy)\ f\ (z) = 0$ with $f(z) \neq 0 \in R_2$. This implies $f(xy) \in Z(R_2)$. This implies $f(x)\ f(y) \in Z(R_2)$, as $f: R_1 \rightarrow R_2$ is a ring- homomorphism. Hence $f(x)$ and $f(y)$ are adjacent in $\Gamma_Z(f(R_1))$.

Corollary 4. Let $f: R_1 \rightarrow R_2$ be a ring-monomorphism, where R_1 and R_2 are two commutative rings that are not integral domain. If $\Gamma_Z(R_1)$ is complete, then $\Gamma_Z(f(R_1))$ is also complete.

Proof. If the graph $\Gamma_Z(f(R_1))$ is trivial, then $\Gamma_Z(f(R_1))$ is complete. Let y_1 and y_2 be two distinct vertices of $\Gamma_Z(f(R_1))$. Then there exists two distinct $x_1, x_2 \in R_1^*$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $\Gamma_Z(R_1)$ is complete, x_1 and x_2 are adjacent in $\Gamma_Z(R_1)$. Then y_1 and y_2 are adjacent in $\Gamma_Z(f\ (R_1))$ by Theorem 6, and hence $\Gamma_Z(f\ (R_1))$ is complete.

Corollary 5. Let $f: R_1 \in R_2$ be a ring-isomorphism, where R_1 and R_2 are two commutative rings that are not integral domain. Then f is an isomorphism from $\Gamma_Z(R_1)$ onto $\Gamma_Z(R_2)$.

Proof. To prove the claim, we need only to show that the adjacency relation is preserved. If x is the only vertex of $\Gamma_Z(R_1)$ then $xx = 0$. This implies $f(xx) = 0$. Since f is an isomorphism, we have $f(x)\ f(x) = 0$. Suppose that x and y be any two vertices of $\Gamma_Z(R_1)$ such that x and y are adjacent. Then $f\ (x)$ and $f\ (y)$ are adjacent in $\Gamma_Z(R_2)$ by Theorem 4.2.6. Hence f is an isomorphism from $\Gamma_Z(R_1)$ onto $\Gamma_Z\ (R_2)$.

3. Diameter of $\Gamma_Z(R)$ for a direct product $R = R_1 \times R_2$

Let R_1 and R_2 be two commutative rings. In this section we investigate of diameters of $\Gamma_Z(R)$ for a finite direct product $R = R_1 \times R_2$ with respect to the zero- divisors and regular elements of R_1 and R_2. We have $\Gamma_Z(R)$ contains at least 3 vertices, by Theorem 2. Thus $diam(\Gamma_Z(R)) \in \{1, 2\}$ by Theorem 1. 2747

Theorem 1. Let R_1 and R_2 be two commutative rings and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ if and only if

either (1) $R_i = Z(R_i)$ for some $i \in \{1, 2\}$;

or (2) $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| = 1$.

Proof. We have $\Gamma_Z(R)$ contains at least 3 vertices, by Theorem 2.

First, suppose that (1) $R_i = Z(R_i)$ for some $i \in \{1, 2\}$. Without loss of generality assume that $R_1 = Z(R_1)$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two distinct vertices of $\Gamma_Z(R)$. Since $R_1 = Z(R_1)$, we have $x_1, y_1 \in Z(R_1)$, and hence $x, y \in Z(R)^*$ by Lemma 3. Therefore $xy = (x_1y_1, x_2y_2) \in Z(R)$ by Lemma 1. Thus x and y are adjacent and $d\ (x, y) = 1$. Hence $diam(\Gamma_Z(R)) = 1$.

Next, suppose that (2) $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| = 1$. Let $\mathcal{A} = \{(x_1, x_2) \in R^* \mid x_i \in Z(R_i) \text{ for some } i \in \{1,2\}\}$ and $\mathcal{B} = \{(u_1, u_2) \in R^* \mid u_i \notin Z(R_i) \text{ for each } i \in \{1,2\}\}$. Then $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. Thus each element of $\mathcal{A}$ belongs to $Z(R)^*$ and no element of $\mathcal{B}$ belongs to $Z(R)^*$ by Lemma 4.1.3. So the vertex set of $\Gamma_Z(R)$ is given by $\mathcal{Z}_Z(R)^* = \mathcal{A} \cup \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two elements of $\mathcal{A}$. Then $xy = (x_1y_1, x_2y_2) \in Z(R)$ by Lemma 4.1.1. Hence, every pair of distinct vertices of $\mathcal{A}$ is adjacent. Since $|R_1 - Z\ (R_1)| = 1$ and $|R_2 - Z\ (R_2)| = 1$, we have $\mathcal{B}$ is a singleton. Let $\mathcal{B} = \{u = (u_1, u_2)\}$. Then $ux = (u_1x_1, u_2x_2) \in Z(R)^*$ for every $x = (x_1, x_2) \in \mathcal{A}$ by Remark 4.1.1. So u is adjacent to every vertices of $\mathcal{A}$. Thus $diam\ (\Gamma_Z(R)) = 1$.

Conversely, suppose that $diam(\Gamma_Z(R)) = 1$. Assume that neither (1) $R_i = Z(Ri)$ for some $i \in \{1, 2\}$; nor (2) $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z\ (R_2)| = 1$. Then we have the following cases:

Case 1. Suppose that $|R_1 - Z\ (R_1)| \geq 2$ and $|R_2 - Z\ (R_2)| \geq 2$. Let $x_1, y_1 \in R_1 - Z(R_1)$ such that $x_1 \neq y_1$ and $x_2, y_2 \in R_2 - Z(R_2)$ such that $x_2 \neq y_2$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two distinct nonzero elements of R. We have $x, y \notin Z(R)^*$ by Lemma 3. Since x and y are nonzero elements of R, we have $x, y \notin Z(R)$. Thus $xy \notin Z(R)$ by Lemma 1. Since $Z(R)^*$ is non-empty, there exists some $z = (z_1, z_2) \in Z(R)^*$ such that $xz = (x_1z_1, x_2z_2) \in Z(R)^*$ and $yz = (y_1z_1, y_2z_2) \in Z(R)^*$ by Remark 1. Thus we have a path $x - z - y$ of length 2 in $\Gamma_Z(R)$ and hence $(x, y) = 2$, which contradicts $di(\Gamma_Z(R)) = 1$.

Case 2. Without loss of generality assume that $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| \geq 2$. Let $x_1 \in R_1 - Z(R_1)$ and $x_2, y_2 \in R_2 - Z(R_2)$ such that $x_2 \neq y_2$. Then $x = (x_1, x_2)$ and $y = (x_1, y_2)$ are two distinct nonzero elements of R. We have $x, y \notin Z(R)^*$ by Lemma 3. Since x and y are nonzero elements of R, we have $x, y \notin Z(R)$. Thus $xy \notin Z(R)$ by Lemma 4.1.1. Since $Z(R)^*$ is non-empty, there exists some $z = (z_1, z_2) \in Z(R)^*$ such that $xz = (x_1z_1, x_2z_2) \in Z(R)^*$ and $yz = (x_1z_1, y_2z_2) \in Z(R)^*$ by Remark 4.1.1. Thus we have a path $x - z - y$ of length 2 in $\Gamma_Z(R)$ and hence $(x, y) = 2$, which contradicts $di(\Gamma_Z(R)) = 1$.
Hence either (1) $R_i = Z(R_i)$ for some $i \in \{1, 2\}$;
Or (2) $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| = 1$.

Corollary 1. Let R_1 and R_2 be two commutative rings with unity and suppose that $R = R_1 \times R_2$. Then $di(\Gamma_Z(R)) = 1$ if and only if $|R_1 - Z(R_1) = 1$ and $|R_2 - Z(R_2)| = 1$.

Proof. Since R_1 and R_2 are commutative rings with unity, we have $R_1 \neq Z(R_1)$ and $R_2 \neq Z(R_2)$. Thus $diam(\Gamma_Z(R)) = 1$ if and only if $|R_1 - Z(R_1) = 1$ and $|R_2 - Z(R_2)| = 1$ by Theorem 1.

Example 1. Consider the commutative rings $S_1 = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_3\right\}, S_2 = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}, S_3 = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} | p \in \mathbb{Z}_4\right\}$ and $\mathbb{Z}_2$. Thus we have $S_1 = Z(S_1)$, $S_2 - Z(S_2) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, S_3 - Z(S_3) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ and $\mathbb{Z}_2 - Z(\mathbb{Z}_2) = \{1\}$.

(1) Let $R_1 = S_1$ and $R_2 = S_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.
(2) Let $R_1 = S_1$ and $R_2 = \mathbb{Z}_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 4.3.1.
(3) Let $R_1 = S_1$ and $R_2 = S_3$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.
(4) Let $R_1 = R_2 = S_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.
(5) Let $R_1 = R_2 = \mathbb{Z}_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.
(6) Let $R_1 = S_2$ and $R_2 = \mathbb{Z}_2$ and suppose that $R = R_1 \times R_2$. Then $di(\Gamma_Z(R)) = 1$ by Theorem 1.

Corollary 2. Let R_1 and R_2 be two commutative rings and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ if and only if
either (1) $|R_1 - Z(R_1)| \geq 2$ and $|R_2 - Z(R_2)| \geq 2$;
or (2) Without loss of generality $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| \geq 2$. 2748

Proof. We have $\Gamma_Z(R)$ contains at least 3 vertices by Theorem 2.
Suppose that either (1) $|R_1 - Z(R_1)| \geq 2$ and $|R_2 - Z(R_2)| \geq 2$; or (2) Without loss of generality $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| \geq 2$. Since $\Gamma_Z(R)$ is connected and diameter of $\Gamma_Z(R)$ is at most two, we have $di(\Gamma_Z(R)) = 2$ by Theorem 1.
Conversely, suppose that $diam(\Gamma_Z(R)) = 2$. Then either (1) $|R_1 - Z(R_1)| \geq 2$ and $|R_2 - Z(R_2)| \geq 2$; or (2) Without loss of generality $|R_1 - Z(R_1)| = 1$ and $|R_2 - Z(R_2)| \geq 2$ by Theorem 1.

Example 2. Consider the commutative rings $S_1 = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}$, $S_2 = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} | x \in \mathbb{Z}_4\right\}$ and $\mathbb{Z}3$. Thus we have $S_1 - Z(S_1) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, $S_2 - Z(S_2) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ and $\mathbb{Z}3 - Z(\mathbb{Z}3) = \{1, 2\}$.

(1) Let $R_1 = S_1$ and $R_2 = S_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.
(2) Let $R_1 = S_1$ and $R_2 = \mathbb{Z}_3$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.
(3) Let $R_1 = R_2 = S_2$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.
(4) Let $R_1 = R_2 = \mathbb{Z}_3$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.
(5) Let $R_1 = S_2$ and $R_2 = \mathbb{Z}_3$ and suppose that $R = R_1 \times R_2$. Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

4. Diameter Of $\Gamma_Z(R)$ for a finite direct product $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$ ($n > 2$)

Let $R_1, R_2, R_3, \ldots, R_n$ ($n > 2$) be commutative rings. In this section we investigate the diameters of $\Gamma_Z(R)$ for a finite direct product $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$ ($n > 2$) with respect to the zero-divisors and regular elements of $R_1, R_2, R_3, \ldots, R_n$ ($n > 2$) .We have $\Gamma_Z(R)$ contains at least $2^n - 1$ vertices, by Theorem 2. Thus $diam(\Gamma_Z(R)) \in \{1, 2\}$ by Theorem 1.

Theorem 1. Let $R_1, R_2, R_3, \ldots, R_n$ ($n > 2$) be commutative rings and suppose that $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 1$ if and only if

either (1) $R_i = Z(R_i)$ for some i ∈ {1, 2, 3,…, n};
or (2) $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}.

Proof. We have $\Gamma_Z(R)$ contains at least $2^n - 1$ vertices, by Theorem 2.

First, suppose that (1) $R_i = Z(R_i)$ for some i ∈ {1, 2, 3,…, n}. Let $x = (x_1, x_2, x_3, \dots, x_i, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_i, \dots, y_n)$ be any two distinct vertices of $\Gamma_Z(R)$. Since $R_i = Z(R_i)$, we have $x_i, y_i \in Z(R_i)$, and hence $x, y \in Z(R)^*$ by Lemma 3. Therefore $xy = (x_1y_1, x_2y_2, x_3y_3, \dots, x_iy_i, \dots, x_ny_n) \in Z(R)$ by Lemma 4.1.1. Thus x and y are adjacent and $d(x, y) = 1$. Hence $diam(\Gamma_Z(R)) = 1$.

Next, suppose that (2) $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}. Let $\mathcal{A} = \{(x_1, x_2, x_3, \dots, x_n) \in R^* \mid x_i \in Z(R_i)$ for some i ∈ {1, 2, 3,…, n }} and $\mathcal{B} = \{ (u_1, u_2, u_3, \dots, u_n) \in R^* \mid u_i \notin Z(R_i)$ for every i ∈ {1, 2, 3,…,n }}. Then $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. Thus each element of $\mathcal{A}$ belongs to $Z(R)^*$ and no element of $\mathcal{B}$ belongs to $Z(R)^*$ by Lemma 4.1.3. So the vertex set of $\Gamma_Z(R)$ is given by $\mathcal{Z}_Z(R)^* = \mathcal{A} \cup B$ such that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Let $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$ be two elements of $\mathcal{A}$. Then $xy = (x_1 y_1, x_2y_2, x_3y_3, \dots, x_ny_n) \in Z(R)$ by Lemma 4.1.1. Hence, every pair of distinct vertices of $\mathcal{A}$ is adjacent. Since $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}, we have $\mathcal{B}$ is a singleton. Let B = {$u = (u_1, u_2, u_3, \dots, u_n)$}. Then $ux = (u_1x_1, u_2x_2, u_3x_3, \dots, u_nx_n) \in Z(R)^*$ for every $x = (x_1, x_2, x_3, \dots, x_n) \in \mathcal{A}$ by Remark 4.1.1. So u is adjacent to every vertices of $\mathcal{A}$. Thus $diam(\Gamma_Z(R)) = 1$.

Conversely, suppose that $diam(\Gamma_Z(R)) = 1$. Assume that neither (1) $R_i = Z(R_i)$ for some I ∈ {1, 2, 3,…, n}; nor (2) $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}. Then we have the following cases:

Case 1. Suppose that $|R_i - Z(R_i)| \geq 2$ for every i ∈ {1, 2, 3,…, n}. Let $x_i, y_i \in R_i - Z(R_i)$ such that $x_i \neq y_I$ for every i ∈ {1, 2, 3,…, n}. Then $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$ are two distinct nonzero elements of R. We have $x, y \notin Z(R)^*$ by Lemma 4.1.3. Since x and y are nonzero elements of R, we have $x, y \notin Z(R)$. Thus $xy \notin Z(R)$ by Lemma 4.1.1. Since $Z(R)^*$ is non-empty, there exists some $z = (z_1, z_2, z_3, \dots, z_n) \in Z(R)^*$ such that $xz = (x_1z_1, x_2z_2, x_3z_3, \dots, x_nz_n) \in Z(R)^*$ and $yz = (y_1z_1, y_2z_2, y_3z_3, \dots, y_nz_n) \in Z(R)^*$ by Remark 4.1.1. Thus we have a path $x - z - y$ of length 2 in $\Gamma_Z(R)$ and hence $(x, y) = 2$, which contradicts $di(\Gamma_Z(R)) = 1$. 2749

Case 2. Without loss of generality assume that $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, m} and $|R_j - Z(R_j)| \geq 2$ for every j ∈ { m + 1, m + 2, m + 3,…, n}, where m ∈ {1, 2, 3,…, n – 1 }. Let $x_i \in R_i - Z(R_i)$ for every i ∈ {1, 2, 3,…, m} and $x_j, y_j \in R_j - Z(R_j)$ such that $x_j \neq y_j$ for every j ∈{ m + 1, m + 2, m + 3,…, n}. Then $x = (x_1, x_2, x_3, \dots, x_m, x_{m+1}, x_{m+2}, x_{m+3}, \dots, x_n)$ and $y = (x_1, x_2, x_3, \dots, x_m, y_{m+1}, y_{m+2}, y_{m+3}, \dots, y_n)$ are two distinct nonzero elements of R. We have $x, y \notin Z(R)^*$ by Lemma 4.1.3. Since x and y are nonzero elements of R, we have $x, y \notin Z(R)$. Thus $xy \notin Z(R)$ by Lemma 4.1.1. Since $Z(R)^*$ is non-empty, there exists some $z = (z_1, z_2, z_3, \dots, z_m, z_{m+1}, z_{m+2}, z_{m+3}, \dots, z_n) \in Z(R)^*$ such that $xz = (x_1z_1, x_2z_2, x_3z_3, \dots, x_m z_m, x_{m+1} z_{m+1}, x_{m+2} z_{m+2}, x_{m+3}z_{m+3}, \dots, x_nz_n) \in Z(R)^*$ and $yz = (x_1z_1, x_2z_2, x_3z_3, \dots, x_m z_m, y_{m+1} z_{m+1}, y_{m+2} z_{m+2}, y_{m+3}z_{m+3}, \dots, y_nz_n) \in Z(R)^*$ by Remark 4.1.1. Thus we have a path $x - z - y$ of length 2 in $\Gamma_Z(R)$ and hence $(x, y) = 2$, which contradicts $di(\Gamma_Z(R)) = 1$.

Hence either (1) $R_i = Z(R_i)$ for some i ∈ {1, 2, 3,…, n};
Or (2) $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}.

Corollary 1. Let $R_1, R_2, R_3, \dots, R_n$ ($n > 2$) be commutative rings with unity and suppose that $R = R_1 \times R_2 \times R_3 \times \dots \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 1$ if and only if $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n}.

Proof. Since $R_1, R_2, R_3, \dots, R_n$ ($n > 2$) be commutative rings with unity, we have $R_i \neq Z(R_i)$ for every i ∈ {1, 2, 3,…, n}.. Thus $di(\Gamma Z(R)) = 1$ if and only if $|R_i - Z(R_i)| = 1$ for every i ∈ {1, 2, 3,…, n} by Theorem 1.

Example 1. Consider the commutative rings $S_1 = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}_3\right\}, S_2 = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}, S_3 = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} | x \in \mathbb{Z}_4\right\}$ and $\mathbb{Z}_2$. Thus we have $S_1 = Z(S_1)$, $S_2 - Z(S_2) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, S_3 - Z(S_3) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ and $\mathbb{Z}_2 - Z(\mathbb{Z}_2) = \{1\}$.

(1) Let $R_1 = S_1$, $R_2 = S_2$ and $R_3 = R_4 = \dots = R_n = S_3$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times \dots \times R_n$ ($n > 2$). Then $diam(\Gamma Z(R)) = 1$ by Theorem 1.

(2) Let $R_1 = R_2 = R_3 = \dots = R_n = S_2$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times \dots \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.

(3) Let $R_1 = S_1$, $R_2 = \mathbb{Z}_2$ and $R_3 = R_4 = ...= R_n = S_3$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z (R)) = 1$ by Theorem 1.

(4) Let $R_1 = \mathbb{Z}_2$ and $R_2 = R_3 = R_4 = ...= R_n = S_2$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 1$ by Theorem 1.

(5) Let $R_1 = R_2 = R_3 = ...= R_n = \mathbb{Z}_2$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $di(\Gamma Z (R)) = 1$ by Theorem 1.

Corollary 2. Let $R_1, R_2, R_3,..., R_n$ ($n > 2$) be commutative rings and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ if and only if
either (1) $|R_i - Z(R_i)| \geq 2$ for every $i \in \{1, 2, 3,..., n\}$;
or (2) Without loss of generality $|R_i - Z(R_i)| = 1$ for every $i \in \{1, 2, 3,..., m\}$ and $|R_j - Z(R_j)| \geq 2$ for every $j \in \{ +1, m + 2, m + 3,..., n\}$, where $m \in \{1, 2, 3,..., n - 1\}$.

Proof. We have $\Gamma_Z(R)$ contains at least $2^n - 1$ vertices by Theorem 2.
Suppose that either (1) $|R_i - Z(R_i)| \geq 2$ for every $i \in \{1, 2, 3,..., n\}$; or (2) Without loss of generality $|R_i - Z(R_i)| = 1$ for every $i \in \{1, 2, 3,..., m\}$ and $|R_j - Z(R_j)| \geq 2$ for every $j \in \{ +1, m + 2, m + 3,..., n\}$, where $m \in \{1, 2, 3,..., n - 1\}$. Since $\Gamma_Z(R)$ is connected and diameter of $\Gamma_Z(R)$ is at most two, we have $diam(\Gamma_Z(R)) = 2$ by Theorem 1
Conversely, suppose that $diam(\Gamma_Z(R)) = 2$. Then either (1) $|R_i - Z(R_i)| \geq 2$ for every $i \in \{1, 2, 3,..., n\}$; or (2) Without loss of generality $|R_i - Z(R_i)| = 1$ for every $i \in \{1, 2, 3,..., m\}$ and $|R_j - Z(R_j)| \geq 2$ for every $j \in \{ +1, m + 2, m + 3,..., n\}$, where $m \in \{1, 2, 3,..., n - 1\}$ by Theorem 1.

Example 2. Consider the commutative rings $S_1 = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} | x, y \in \mathbb{Z}_2\right\}, S_2 = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} | x \in \mathbb{Z}_4\right\}$ and $\mathbb{Z}_3$. Thus we have $S_1 - Z(S_1) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}, S_2 - Z(S_2) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ and $\mathbb{Z}_3 - Z(\mathbb{Z}_3) = \{1, 2\}$. 2750

(1) Let $R_1 = S_1$, $R_2 = S_2$ and $R_3 = R_4 = ...= R_n = \mathbb{Z}_3$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

(2) Let $R_1 = R_2 = R_3 = ...= R_n = S_2$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

(3) Let $R_1 = S_1$, $R_2 = \mathbb{Z}_3$ and $R_3 = R_4 = ...= R_n = S_2$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

(4) Let $R_1 = R_2 = R_3 = ...= R_n = \mathbb{Z}_3$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

(5) Let $R_1 = R_2 = S_2$ and $R_3 = ...= R_n = \mathbb{Z}_3$ ($n > 2$) and suppose that $R = R_1 \times R_2 \times R_3 \times ... \times R_n$ ($n > 2$). Then $diam(\Gamma_Z(R)) = 2$ by Corollary 2.

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