ARTICLE

# Entangled States Generation of Two Atoms Interacting with a Cavity Field

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**Abstract**

We classify different classes of entangled states arise in a two-qubit system. Some of these classes are of Bell's state types, while others are of the Werner's state types. The degree of entanglement is quantified for different values of the atomic and the cavity parameters. We show that it is possible to generate entangled state with high degree of entanglement by controlling the detuning and the number of photon inside the cavity.

**Key Words:** Entangled state, Bell states, Werner-states

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## 1. Introduction

Although physicists have been able to entangle up to eight qubits at the same time -- whereas recently interest has been focused on just entangling two qubits, (Steffen et al., 2006) . They insist superconducting qubits are a viable approach towards quantum computing. Before this work, a number of applications have been studied experimentally and theoretically, see for example,( Phoneix and Knight 1988; Narozhny et al., 1981). The coupling of multi two-level atoms to a single quantized radiation mode has been solved. Finite algorithms for the eigenfrequencies were given for the case of resonance (Jaynes, 1963). The coupling of a single two-level atom to a number equal-frequency modes has been used as a simple model of decay processes (Compagno and Persico, 1977). However, as a result of the development of low-temperature high-$Q$ cavities, and the use of Rydberg atoms, the idealized situation of a single two-level atom interacting with a single-mode,quantized radiation field in a lossless cavity has been experimentally realized, (Rempe et al., 1987) . These advances in turn have certainly provided incentive for extending and generalizing this model. Thus many authors have been encouraged and stimulated to modify the original model and to generalize it into different directions.

Nowadays, new applications of the atomic systems have been appeared e.g. in quantum information, (You 2003; Yi, et al., 2003; Metwally et al., 2004a; Metwally et al.,2005b) and computations, Beige, 2005. The most important phenomena in the atomic systems which is needed in quantum information tasks is the entanglement. So it is of a great importance to investigate how one can generate entanglement between atoms (Shang, 2007; Jaehak et al., 2008).

In this contribution, we consider an atomic system interacts with a cavity mode field in a Fock state. This atomic system is prepared in excited or ground state. Due to this interaction, there are different classes of entangled states between the two atoms have been generated.

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Then we quantify the degree of entanglement contained in these entangled states. This article is organized as follows: In Sec.2, we introduce an analytical solution of the suggested model. The possibility of generating different classes of entangled states is shown in in Sec.3. The degree of entanglement between the two entangled atoms is quantified, where we consider different classes of initial states of the atomic system and different values of the field and atomics parameter. These results are depicted in Sec.4, where we consider the negativity as a measure of the degree of entanglement. Finally, we give our conclusion in Sec.5.

## 2 The Model and its Solution

The Hamiltonian which describes a system of two two-level atoms, each consisting of states $|e\rangle$ and $|g\rangle$ coupled to a single mode radiation field, in the rotating wave approximation is given by

$$H=\omega(a^{\dagger}a+\sum_{i=1}^{2}\sigma_z^i)+\sum_{i=1}^{2}[\lambda_i(a^{\dagger}\sigma_-^i+\sigma_+^i a)] \quad (1)$$

where $a(a^{\dagger})$ is the annihilation (creation) operator of the field mode, $\sigma_{\pm}^i$ and $\sigma_z^i$ are the atomic raising, lowering and inversion operators of the two atoms. The parameter $\lambda_i$ is the atom-field coupling constant and $\omega$, is the atomic transitions and the field mode frequency. The first term in Eq. $(1)$ represents the free-field and the non-interacting atoms, while the second term stands for the interaction Hamiltonian $H_{int}$. This model has been solved analytically for a general case (Obada et al., 1991). For simplicity, we consider the case of identical atoms i.e. $\lambda_1=\lambda_2$. Assume that the cavity field is initially prepared in a coherent state, $|\psi(0)\rangle_f=|n\rangle$ and the two atoms are in a superposition states of their ground and exited states $|\psi(0)\rangle_{1,2}=a_i|g_i\rangle+b_i|e_i\rangle, i=1,2$, where $1$ stands for the first atom and $2$ for the second atom. The initial state of the system is assumed to be,

$$\begin{aligned}|\psi_0\rangle&=(c_0^{(1)}|g_1g_2\rangle+c_0^{(2)}|e_1g_2\rangle\\&+c_0^{(3)}|g_1e_2\rangle+c_0^{(4)}|e_1e_2\rangle)\otimes|n\rangle\end{aligned} \quad (2)$$

where

$$\begin{aligned}&c_0^{(1)}=a_1a_2,\quad c_0^{(2)}=a_2b_1,\\&c_0^{(3)}=a_1b_2,\quad c_0^{(4)}=b_1b_2\end{aligned} \quad (3)$$

with $|a_i|^2+|b_i|^2=1$ for $i=1,2$. In the invariant sub-space of the global system, we can consider a set of complete basis of the qubit-field system as $|ee,n\rangle,|eg,n+1\rangle,|ge,n+1\rangle$ and $|gg,n+2\rangle$. The time evolution of the density operator of the system is given by

$$\rho_{cf}(t)=\mathrm{U}(t)\{\rho_a(0)\otimes\rho_f(0)\}\mathrm{U}^{\dagger}(t), \quad (4)$$

where $\mathrm{U}(t)=\exp\left(-i\hat{H}t/\bar{h}\right)$ is the unitary operator, its components are given by,

$$\mathrm{U}_{11}(t)=\sum_{i=1}^{3}(-1)^{i+1}\alpha_i e^{-i\mu_1 t}[\mu_i(\Delta+\mu_i)-2\beta^2],$$

$$\mathrm{U}_{12}(t)=\gamma\sum_{i=1}^{3}(-1)^{i+1}e^{-i\mu_i t}(\Delta+\mu_i),$$

$$\mathrm{U}_{13}(t)=\mathrm{U}_{12}(t),\quad \mathrm{U}_{14}(t)=2\beta\gamma\sum_{i=1}^{3}(-1)^{i+1}\alpha_i e^{-i\mu_i t},$$

$$\mathrm{U}_{21}(t)=\mathrm{U}_{12}(t)$$

$$\mathrm{U}_{22}(t)=\sum_{i=1}^{3}(-1)^{i+1}\frac{\alpha_i}{\mu_i}e^{-i\mu_i t}[(\beta^2(\Delta-\mu_i)-(\Delta+\mu_i))(\gamma^2+\mu_i(\Delta-\mu_i))]-\frac{\Delta(\beta^2-\gamma^2)}{\mu_1\mu_2\mu_3},$$

$$\mathrm{U}_{23}(t)=-\sum_{i=1}^{3}(-1)^{i+1}\frac{\alpha_i}{\mu_i}e^{-i\mu_i t}[\beta^2(\Delta-\mu_i)-\gamma^2(\Delta+\mu_i)]+\frac{\Delta(\beta^2-\gamma^2)}{\mu_1\mu_2\mu_3},$$

$$U_{24}(t) = -\beta \sum_{i=1}^{3} (-1)^{i+1} \alpha_i e^{-i\mu_i t} (\Delta - \mu_i),$$
$$U_{34}(t) = -\beta \sum_{i=1}^{3} (-1)^{i+1} \alpha_i e^{-i\mu_i t} (\Delta - \mu_i),$$
$$U_{31}(t) = U_{13}(t), \quad U_{32}(t) = U_{23}(t), \quad U_{33}(t) = U_{22}(t),$$
$$U_{41}(t) = U_{14}(t), \quad U_{42}(t) = U_{24}(t), \quad U_{43}(t) = U_{34}(t),$$
$$U_{44}(t) = -\sum_{i=1}^{3} (-1)^{i+1} \alpha_i e^{-i\mu_i t} [2\gamma^2 + \mu_i(\Delta - \mu_i)], \quad (5)$$

where

$\gamma = \sqrt{n+1}, \beta = \sqrt{n+2}$ , $\alpha_1 = (\mu_{12}\mu_{13})^{-1}, \alpha_2 = (\mu_{12}\mu_{23})^{-1}, \alpha_3 = (\mu_{13}\mu_{23})^{-1}$ , $\mu_{kj} = \mu_k - \mu_j$ and $\mu_i = \frac{2}{3}\kappa \cos\theta_i$ with $\kappa = \sqrt{3(\Delta^2 + 2(\beta^2 + \gamma^2))}$ and $\theta_1 = \frac{1}{3}\cos^{-1}\left(-\frac{27\Delta}{\kappa^3}\right)$,

$$\theta_2 = \frac{2\pi}{3} + \theta_1, \theta_3 = \frac{2\pi}{3} + \theta_2 .$$

Since we are interested in discussing some properties of the charge qubits system, we calculate the density matrix of the charged qubit by tracing out the field i.e $\rho_{ab} = tr_f\{\rho_{cf}\}$ .

$$\rho_{atoms} = \begin{pmatrix} |A_n^{(1)}|^2 & A_n^{(1)}A_{n+1}^{*(2)} & A_n^{(1)}A_{n+1}^{*(3)} & A_n^{(1)}A_{n+2}^{*(4)} \\ A_{n+1}^{(2)}A_n^{*(1)} & |A_n^{(2)}|^2 & A_n^{(2)}A_n^{*(3)} & A_n^{(2)}A_{n+1}^{*(4)} \\ A_{n+1}^{(3)}A_n^{*(1)} & A_n^{(3)}A_n^{*(2)} & |A_n^{(3)}|^2 & A_n^{(3)}A_{n+1}^{*(4)} \\ A_{n+2}^{(4)}A_n^{*(1)} & A_{n+1}^{(4)}A_n^{*(2)} & A_{n+1}^{(4)}A_n^{*(3)} & |A_n^{(4)}|^2 \end{pmatrix} \quad (6)$$

with,

$$A_n^{(1)} = \sum_{j=1}^{4} U_{1j}(n)A^{(j)}(0), \quad A_n^{(2)} = \sum_{j=1}^{4} U_{2j}(n)A^{(j)}(0),$$
$$A_n^{(3)} = \sum_{j=1}^{4} U_{3j}(n)A^{(j)}(0), \quad A_n^{(4)} = \sum_{j=1}^{4} U_{4j}(n)A^{(j)}(0), \quad (7)$$

where, $A^{(1)}(0) = b_1b_2, A^{(2)}(0) = b_1a_2,$
$A^{(3)}(0) = a_1b_2, A^{(4)}(0) = a_1a_2.$

**3. Entangled state**

Due to the interaction, the dynamical system behaves as an entangled state for some time and as a product in another interval of time. In this section, we show the effect of the field and the atomic parameters on the possibility of generating entangled state with high degree of entanglement. For this purpose, we plot the populations. In Fig.(1), we plot the populations for atomic system is prepared initially in excited state. In this figure, we investigate the dynamics of this phenomena for different values of the detuning parameter. In Fig.(1a), we assume that $\Delta = 0.1$ and $\bar{n} = 0$, it is clear that the there are some entangled state are generated at different time. As an example for $\tau > 0$, the first partially entangled state has the form

$$|\psi_1\rangle = \mu(|eg\rangle + |ge\rangle), \quad (8)$$

where $0 < \mu \le 2.9$. This class of state is similar to the class of the Bell state, Nielsen, 2000

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle) .$$

This class of entangled state appears along the time expect at some time where it turns to a separable states at $\tau;1.5,2.5,3.75,...$ Then as the time increases more, there is a different class of partially entangled state is generated. This class could be written as,

$$|\psi_2\rangle = \mu_1(|ee\rangle + |eg\rangle + |ge\rangle) \quad (9)$$

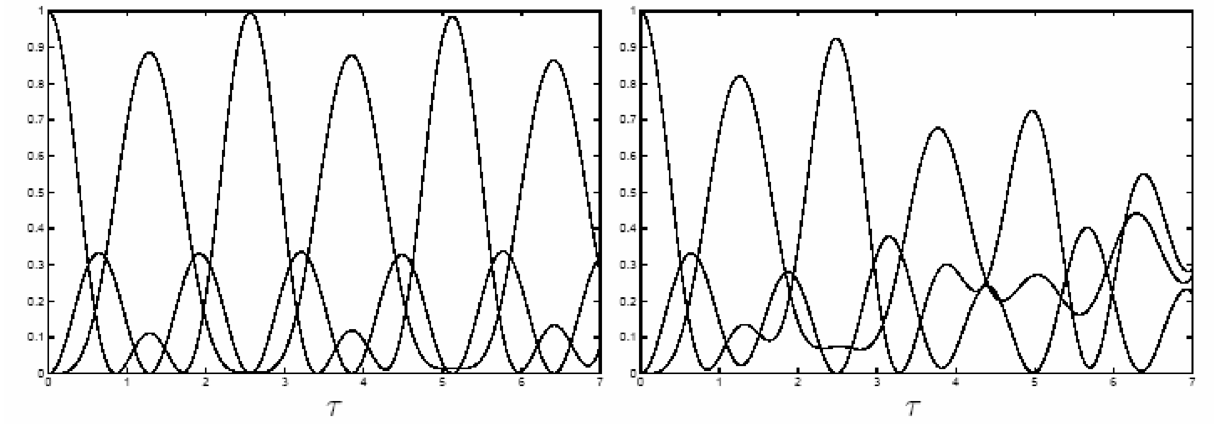


**Figure 1.** The population for the atomic system initially in excited state with $n = 0$ and (a) $\Delta = 0.1$ (b) $\Delta = 0.5$.

As the time goes on another partially entangled state is generated around $\tau;0.5$. This state could be written as superposition of the ground and excited state in additional to a separable part, this classes is similar to Werner states (Werener, 1989; Englert, 2000). In a mathematical form one writes this class as

$$|\psi_3\rangle = \eta(|ee\rangle + |gg\rangle) + \zeta\,|eg\rangle \qquad (10)$$

This state is periodically appears round $\tau;2,3.5,...$. Also, this class of states is similar to what is called Werner state, which consists of fraction of maximum entangled state in additional to completely mixed state On the other hand, there are some cases where the entangled system turns into a product system.

In Fig.(1b), we set $\Delta = 0.5$, in this case there are different classes of entangled state are generated in additional to the classes which has been appeared in Fig.(1a). As an example round $\tau;4.5$, there is a class of entangled state can be written as

$$|\psi_4\rangle; \mu_2(|ee\rangle + |gg\rangle) + \nu(|eg\rangle + |ge\rangle) \qquad \textbf{(11)}$$

It is clear as one increases the detuning the atomic system remains entangled for any $\tau > 0$.

To see the effect of the initial state setting, we consider the case where the atomic system is prepared initially in ground state i.e $|\psi(0)_a\rangle = |gg\rangle$. The behavior of this phenomena is seen in Fig.(2). In this figure the most interesting remark, that the generated entangled state between the two qubits is almost entangled at any time $\tau > 0$. Also, there are some different classes of entangled states appear at different $\tau$. As an example, round $\tau;1.5$ the generated entangled state takes the form,

$$|\psi_5\rangle = \chi_1\,|ee\rangle + \chi_2\,|gg\rangle + \chi_3(|eg\rangle + |ge\rangle). \qquad \textbf{(12)}$$

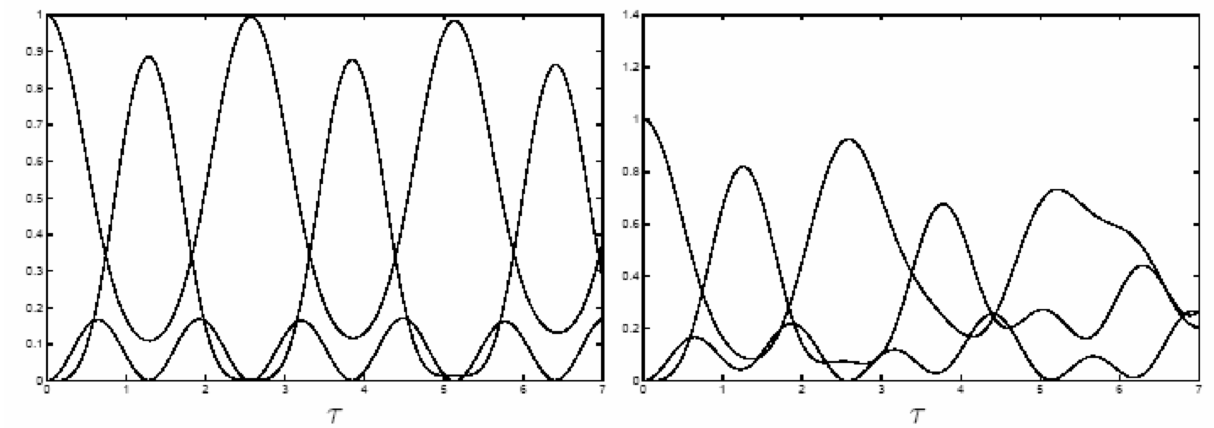


**Figure 2.** The same as Fig.(1) but the atomic system is prepared initially in ground state, $|\psi(0)_a\rangle = |gg\rangle$.

This behavior is clear from Fig.(2a). For large values of the detuning the same behavior is seen as shown in Fig.(1b). Also, the same classes of the entangled states which has been depicted in Fig.(1a) appear clearly in Fig.(2b) at different values of $\tau$ and weight.

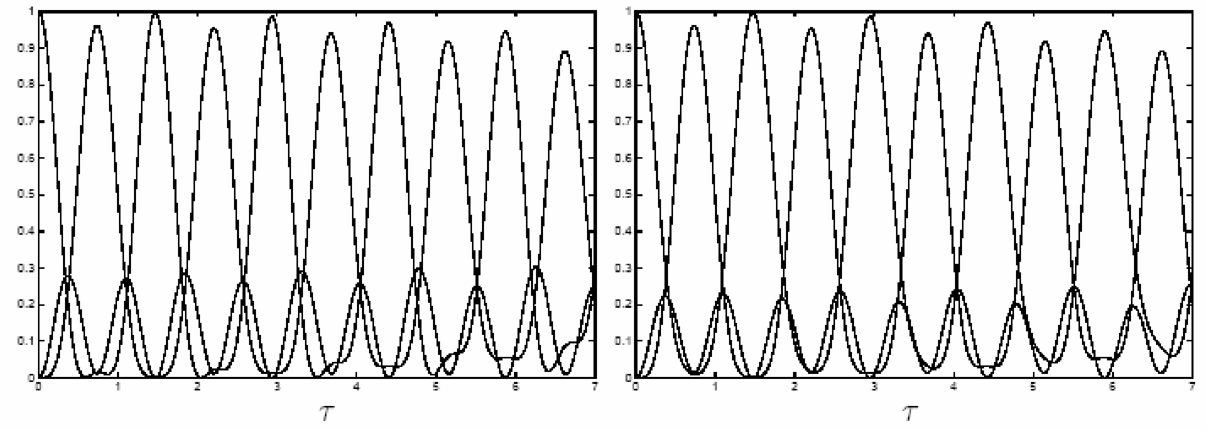

**Figure 3.** The population for $\Delta = 0.5$ and $n = 3$. (a) The system is initially prepared in excited state $|\psi(0)_a\rangle = |ee\rangle$ (b) The atomic system is prepared in ground state $|\psi(0)_a\rangle = |gg\rangle$.

Finally, we investigate the effect of the number of photon inside the cavity on the populations dynamics. Fig.(3), shows this behavior for different initial states setting, excited or ground state. The general behavior as expected is that the number of oscillations is increased. This is clearly seen by comparing Fig.(1b) and Fig(3), for a system prepared in excited state and Fig.(2b) and (3b), for atomic system prepared in the ground state. Also, the same classes of the generated entangled states are seen in this case at different values of $\tau$. But these entangled state has a large weight, so the degree of entanglement as we shall see in the next section is much greater.

So, one can conclude that, to generate different classes of entangled states between two qubits, one has to increase the number of photon inside the cavity. Also, by increasing the detuning the generated states are almost entangled for any time $\tau > 0$.

**4. Entanglement**

We have seen in the previous section, there are some entangled state generated in some interval of time. In this section, we quantify the amount of entanglement contained in theses states. To quantify the amount of entanglement contained in the entangled states, we shall use a measurement introduced by (Zyczkowski, 1998). This measure states that if the eigenvalues of the partial transpose are given by $\mu_i, i = 1,2,3,4$, then the degree of entanglement is defined by

$$\mathsf{N} = \sum_{i=1}^{4} |\mu_i| - 1 \qquad \textbf{(13)}$$

It is clear that the $\mathsf{N}$ is equal to zero for separable states and it is equal to one for the maximal entangled states. Figure 2, shows the behavior of the $\mathsf{N}$ as a function of the scaled time $\tau$.

Fig.(4) shows the dynamics of the degree of entanglement for atomic system is prepared initially in excited state. Fig.(4a), is plotted for $n = 0$ and $\Delta = 0.1$. It is clear as soon as the interaction goes on an entangled state is generated. At $\tau = 0$, the degree of entanglement $\mathsf{N} = 0$ and then increase until it reaches its maximum values then reduces to zero, where the atomic system converted into a product state. As time increases, this behavior is repeated, but the average amount of entanglement is increased. In Fig.(4a),we consider the detuning parameter $\Delta = 0.5$, we can see that in this case, the generated entangled state is much better and does not converted into a separable state as the time increases. So, one can say that by increasing the value of the detuning parameter, one can get an entangled state with long-lived entanglement.

To see the effect of the initial state setting, we consider the atomic system is prepared initially in the ground state. The dynamics of entanglement for this case is depicted in Fig.(5). It is clear the same behavior is seen as that depicted in Fig.(4), but the generated entangled state is survival for a long time. This is clear if one compares Fig.(4a) with Fig.(5a), where the degree entanglement vanishes round $\tau = 0.8$, for the first time while it vanishes round $\tau = 1.6$ for a system is initially prepared in the ground state. Also, as one increases the detuning parameter the possibility of long-lived entanglement is much better. So, one of the most better choice to generate en entangled state with high degree of entanglement, is that the atomic system is prepared in ground state and a large values of the detuning parameter (Abdel-Aty, 2007).

Finally, we consider the effect of the number of photon inside the cavity $n$, on the degree of entanglement. This behavior is depicted in Fig.(6), where we consider $\Delta = 0.5$ and $n = 3$. In Fig. (6a), we assume that the atomic system is prepared in excited state $|ee\rangle\langle ee|$. From this figure, although the generated entangled state is long-lived entanglement, but the degree of entanglement N, is smaller than that has been shown in Fig,(4a), where the number of photon inside the cavity $n = 0$. Also, as a remark, we can see that the maximum entangled state is shifted to the right as the time increases.

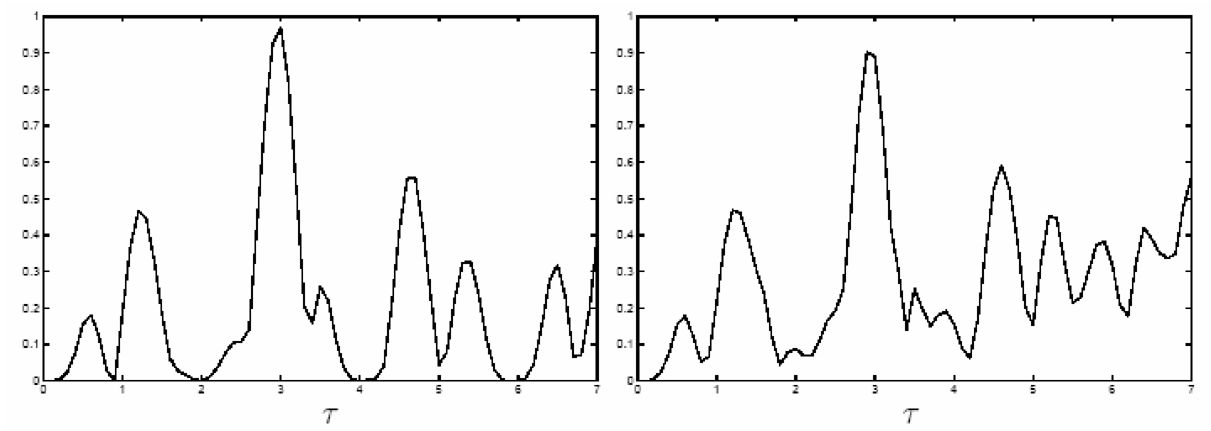


**Figure 4**. The degree of entanglement for the atomic system is prepared initially in excited state with $n = 0$ and (a) $\Delta = 1$ (b) $\Delta = 0.5$.

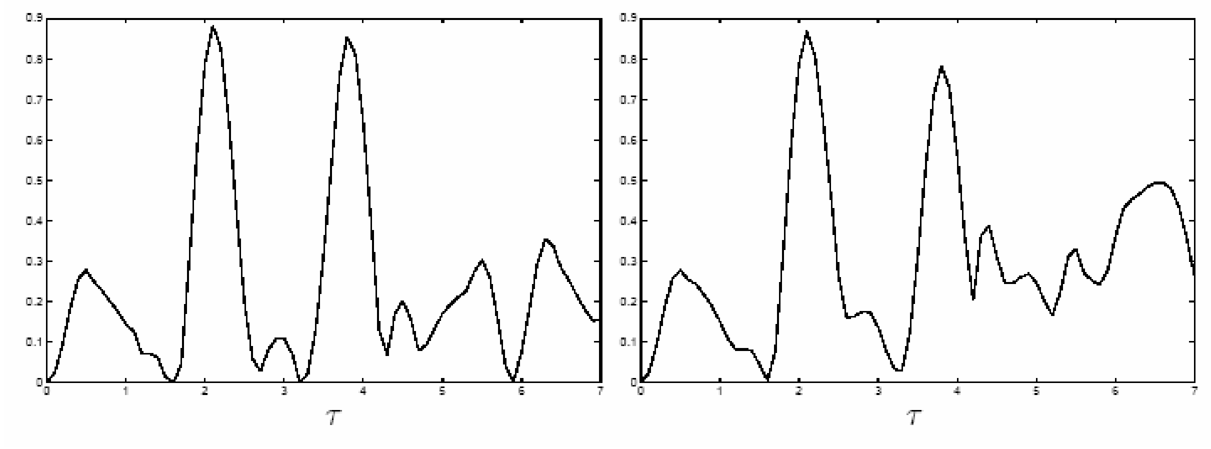


**Figure 5.** The same as Fig.(1), but the system is prepared initially in the ground state.

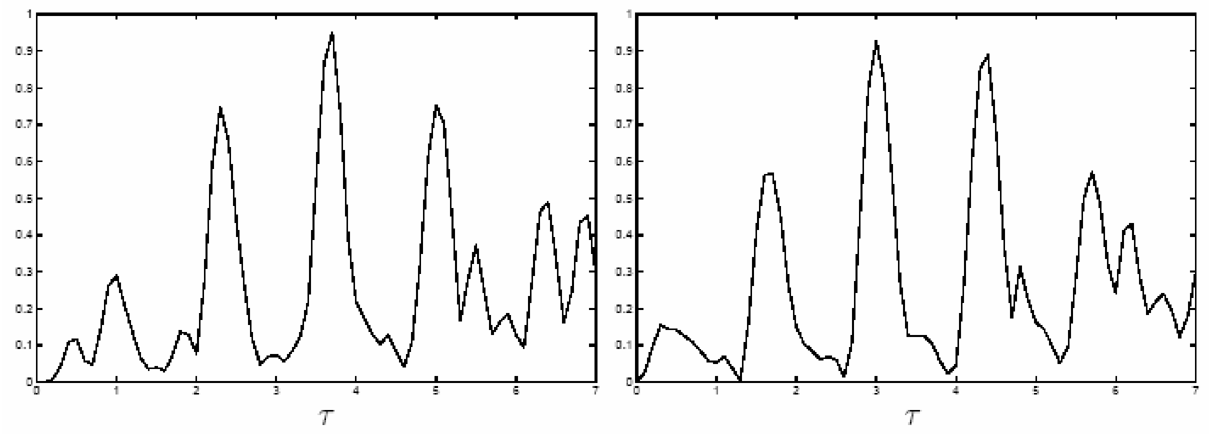

**Figure 6.** The degree of entanglement where $\Delta = 0.5, n = 3$ .(a) the system is initially prepared in the excited state(b) The system is prepared in ground state

## 5. Conclusion

In this contribution, we have investigated the dynamics of an atomic system interacting with a cavity field. We considered different classes of the atomic initial states and Fock state as an initial state of the field. The effect of the number of photon as well as the detuning parameter on generating entangled states is investigated in different regimes. We find that there are several classes of entangled states can be generated. Some of these classes are of Bell's state types, others are of Werner state types. Also, it is shown that, the generation of entangled states with long lived entanglement depends on the initial state setting. If the atomic system is prepared in the ground state, the generated entangled state survived longer than the case in which the system is prepared in an excited state. On the other hand as one increase the numbers of photon inside the cavity, the entangled state live longer although the amount of entanglement is smaller in average. Finally, it is shown that the detuning parameter plays a central role on controlling the entanglement, and the survivability of entanglement increases once the detuning is increased.

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